Bowdoin College

Math 2020: Introduction to Mathematical Reasoning
Prof. Jennifer Taback

Introduction to Logical Reasoning

These problems must be completed independently but are not turned in to Prof. Taback. You will be expected to do similar problems on a quiz or exam, or in class. These problems will be discussed during an extra class session.

It is important that you learn the terminology in this handout. If I use the word "contrapositive," for example, I don't want to get blank looks! These ideas will be used throughout the semester in a variety of different contexts. Please learn them!

As part of this assignment, you will read Sections 1.1, 1.2, 1.3 and (most of) 1.5 from the textbook. The notes below add some commentary to your reading, and tell you which parts are most important. I would read along with the outline below instead of just reading the sections independently.

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Our language of mathematics has the statement as its building block. A **statement** has a defined truth value – it can be true or false but not both; for example, "James is in College" is not a statement because its truth value varies depending on who James is. But 2+2=4 is a statement because it is always true. Similarly, 2+3=1 is a statement as well, because it is always false.

Look at problem 1 on page 5. Can you decide which are statements?

Just like in English, where we connect nouns and verbs with various connectives (and, or, but, if, etc) we have similar structures in logical reasoning. Read Section 1.2, which will introduce the negation of a statement, the conjunction of statements ("and") and the disjunction ("or"). We will begin to represent statements by letters like P and Q. When we write "P and Q" where P and Q represent statements, we don't immediately know the truth value of this compound statement, because it will depend on what P and Q actually are. So we call "P and Q" (mathematically written $P \bigwedge Q$) a **statement form**. This means that when you substitute actual words for P and Q you always get a statement, that is, something with a fixed truth value. The P and Q are like placeholders. To keep track of all the possible truth values of P and Q separately, as well as the truth value of the compound statement, we use a <u>truth table</u>. These are explained in Section 1.2. (Note that the book calls statement forms sentential forms.)

At the end of Section 1.2, you should be able to make truth tables for $P \bigwedge Q$, $P \bigvee Q$ and $\sim P$. You should understand the difference between a statement and a statement form.

Now try problem 1 on page 15. Also make truth tables for the following statement forms:

(1) $P \bigwedge \sim Q$
(2) $P \bigwedge \sim P$
(3) $P \bigvee (Q \bigvee \sim Q)$ This works just like order of operations – evaluate the truth value of expressions inside parentheses first.

Now read Section 1.3. This section introduces a very important structure both in English and in mathematics: the conditional statement (i.e. "if $\cdots$ then"). Here is the way you should think about the truth value of a statement form representing a conditional: it is true when it is not false. So figure out when it is false - when the first thing happens and the second one does not. In all other cases it must be true. This leads to things like "If Searles is made of chocolate then pigs fly" being a true *mathematical* statement.

After this section you should be able to make a truth table for $P \to Q$. Do problems 11 and 18 on page 17. Remember the word *tautology* from this problem. Some statement forms are called *contradictions*. What do you think the truth table for a contradiction looks like? Can you make up a statement form which is a contradiction?

Try making some more complicated truth tables, for the following:

(1) $(P \wedge Q) \to R$
(2) $(P \to Q) \vee R$

The key is to make columns for all the different expressions using the order of operations dictated by the parentheses. So, in the first example, my truth table will have columns corresponding to: $P, Q, R, P \wedge Q$ and finally $(P \wedge Q) \to R$. Always start with the single variables – first you will have to figure out how many different combinations of truth values 3 variables can have!

Variations on the conditional statement: If $P \to Q$ is your original conditional statement, then:

- $Q \to P$ is called its converse,
- $\sim P \to \sim Q$ is called its inverse, and
- $\sim Q \to \sim P$ is called its contrapositive.

Consider the conditional statement "If p is prime, then p^2 is not prime." and write its converse, inverse and contrapositive statements, in words.

Read Section 1.5, up to (but not including) the replacement principle. Here are the main ideas of this section.

- Two statement forms are said to be logically equivalent if they have the same truth tables. This means that they are true in exactly the same cases and can be used interchangeably.
- DeMorgan's Laws are rules about how to negate a compound statement like $P \bigwedge Q$ or $P \bigvee Q$. Make sure you can reason out in words why DeMorgan's Laws are true. (This will also be discussed in the session.)

Which of the following are logically equivalent? Please show your truth tables.

$$P \to Q \qquad Q \to P \qquad \sim P \to \sim Q \qquad \sim Q \to \sim P$$

Do problem 3(a,b) on Page 30.

Look again at De Morgan's laws on page 28. (Equation 1.28). Apply them repeatedly to negate the following expressions. Continue to apply them until there are no negated parenthetical expressions.

(1) $\sim ((P \wedge Q) \vee R)$
(2) $\sim ((P \vee Q) \wedge (Q \vee R))$

Figure out how to negate a conditional statement. **Surprisingly, the negation of a conditional is NOT a conditional!** Write out the truth table for $P \to Q$. From this, you can figure out what the final column in the truth table for $\sim (P \to Q)$ must be. Write this down.

Everyone always thinks that $\sim (P \to Q)$ must be $\sim P \to \sim Q$. Make a truth table for this statement to see that it is NOT the negation of $P \to Q$.

To figure out what $\sim (P \to Q)$ must be, ask yourself this: If $\sim (P \to Q)$ is true, how would I make it false? See if the answer to that question leads you to the negation you are looking for. You can always check your answer by making a truth table, because you KNOW what the truth table for

$\sim (P \rightarrow Q)$ must have as its final column! This will be discussed at length in one of the videos on the Logical Reasoning webpage.