Bowdoin College

Math 2020: Introduction to Mathematical Reasoning
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Truth Value of Quantified Statements

For each of the quantified statements below, please do three things:

(1) Decide whether the statement is true or false.
(2) Justify the truth value.
(3) Write the negation, simplifying as much as possible, which means until there are no more negation signs in your answer.

To complete the problems below, you will need to watch the video on quantified statements, and the video on negating quantified statements.

1. Statement: $\forall x \in \mathbb{R},\ x^2 \in \mathbb{Q}$

Truth value: False. ← so I have to give one counterexample. I'll make a choice of x which does not satisfy $x^2 \in \mathbb{Q}$.

Justification: Choose $x = \sqrt{\pi}$. Then $x \in \mathbb{R}$ and $x^2 = \pi \notin \mathbb{Q}$.

Negation: $\exists x \in \mathbb{R}$ so that $x^2 \notin \mathbb{Q}$.

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2. Statement: $\exists x \in \mathbb{Q},\ x^2 < x$

This tells me that I will have to make a specific choice for x.

Truth value: True.

Justification: Choose $x = \frac{1}{2}$. Then $\frac{1}{2} \in \mathbb{Q}$ because $1, 2 \in \mathbb{Z}$ and $2 \neq 0$, and
$x^2 = \frac{1}{4} < \frac{1}{2} = x$.

You have to tell me why $\frac{1}{2}$ is a rational #.

Here you are showing that the property is satisfied.

Negation: $\forall x \in \mathbb{Q},\ x^2 \geq x$.

3. Statement: $\exists x \in \mathbb{R}$, so that $x^2 = 2$

Truth value: True.

Justification: $\underbrace{\text{Choose } x = \sqrt{2}}_{\text{Make your choice.}}$. Then $\underbrace{x \in \mathbb{R}}_{\text{State (or prove if } \mathbb{Q}\text{) that it's in the right set.}}$ and $\underbrace{x^2 = (\sqrt{2})^2 = 2}_{\text{Show that your choice satisfies the desired property.}}$.

Negation: $\forall x \in \mathbb{R}$, $x^2 \neq 2$.

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4. Statement: $\exists x \in \mathbb{R}$, so that $x^2 + 2x + 2 = 5$

Truth value: True.

Justification: Choose $x = 1$. Then $x \in \mathbb{R}$ and $x^2 + 2x + 2 = 1^2 + 2(1) + 2 = 5$.

Negation: $\forall x \in \mathbb{R}$, $x^2 + 2x + 2 \neq 5$.

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5. Statement: $\forall x \in \mathbb{Z}$, so that $x^2 = x$

Truth value: False.

Justification: Choose $x = 2$. Then $x \in \mathbb{Z}$ and $x^2 = 4 \neq 2 = x$.

Negation: $\exists x \in \mathbb{Z}$ so that $x^2 \neq x$.

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6. Statement: $\forall x \in \mathbb{Q}$, $x + 1 \in \mathbb{Q}$

Truth value: True.

Here I am using the definition of a rational #.

Justification: Let x be any rational number. So $x = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ with $b \neq 0$. Then $x + 1 = \frac{a}{b} + 1 = \frac{a}{b} + \frac{b}{b} = \frac{a+b}{b}$. Since $a, b \in \mathbb{Z}$, $a + b \in \mathbb{Z}$. We already know that $b \in \mathbb{Z}$ and $b \neq 0$. So $x + 1 = \frac{a+b}{b} \in \mathbb{Q}$.

Now I need to check that $\frac{a+b}{b}$ satisfies the definition of a rational #.

Negation:

$\exists x \in \mathbb{Q}$ so that $x + 1 \notin \mathbb{Q}$.