

BOWDOIN COLLEGE

MATH 3603: ADVANCED ANALYSIS
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HOMEWORK 9

1. Our existence proof for Brownian motion requires a crucial inequality. As your instructor is too lazy to prove the inequality in class, you will have to do prove it for homework. Let f be a normally-distributed random variable with mean 0 and variance 1 defined on a probability space X . Let μ be probability measure on X . Show that whenever $a > 0$,

$$\mu\{x \mid f(x) > a\} \leq \frac{1}{\sqrt{2\pi}} \frac{e^{-\frac{a^2}{2}}}{a}$$

Hint: You can express the value of the left side precisely using an integral.

2. We aim to define the space of square-integrable functions, but the definition is a little subtle. The purpose of this exercise is to flush out this subtlety. Consider a measure space $(X, \mathcal{F}, \mu)$ and two functions $f, g : X \rightarrow \mathbb{R}$. We say that they are *equivalent* iff $f = g$ except perhaps on a set of measure zero. If we write $[g]$ for the equivalence class of functions containing g , then

$$\mathcal{L}^2(X) = \{[f] \mid \int_X f^2 d\mu < \infty\}.$$

When endowed with the metric ρ defined by $\rho(f, g) = \sqrt{\int_X (f - g)^2}$, the set $\mathcal{L}^2(X)$ becomes a complete metric space.

- (a) Verify that the relation $f \sim g$ iff $f = g$ a.e. is indeed an equivalence relation. Be precise.
 - (b) Prove that $\mathcal{L}^2(X)$ is well-defined.
 - (c) Prove that ρ is well-defined.
3. To compute the quadratic variation of Brownian motion (an upcoming attraction), we will need a fact about normally-distributed random variables. Show that if f is a normally-distributed random variable with expected value 0 and variance σ^2 , then

$$E(f^4) = 3(\sigma^2)^2.$$

Computing the expected value of a random variable with a particular distribution is harder than it seems, at least at first glance. If f is the random variable under consideration, the formula is simply

$$E(f) = \int_X f \, d\mu.$$

In many situations the probability space X is not clearly defined and actually carrying this computation out is tricky at best. We detail a less direct computation that is often easier to carry out. Suppose that F is the probability density function of f , or in other words, that

$$\mu(\{x \in X \mid f(x) \in (a, b)\}) = \int_a^b F(x) \, dx.$$

Let us also assume for simplicity that f is non-negative. Recall that the Lebesgue integral of f is defined as a supremum of integrals of simple functions. In fact, by a result from class, we can find an increasing sequence of simple functions $\{s_n\}$ which converges to f . Define I_i to be the interval $[\frac{i-1}{2^n}, \frac{i}{2^n})$ and let $E_i = f^{-1}(I_i)$. The value of the simple function s_n on E_i is $\frac{i-1}{2^n}$. Then, since $\mu(\{x \in X \mid f(x) \in (a, b)\}) = \int_a^b F(x) \, dx$, we have

$$\mu(E_i) = \int_{I_i} F(x) \, dx.$$

Furthermore:

$$\int_X f \, d\mu = \lim_{n \rightarrow \infty} \int_X s_n \, d\mu = \lim_{n \rightarrow \infty} \sum_{i=1}^{n2^n} \frac{i-1}{2^n} \mu(E_i)$$

We can now use our formula for $\mu(E_i)$ and continue:

$$\int_X f \, d\mu = \lim_{n \rightarrow \infty} \sum_{i=1}^{n2^n} \frac{i-1}{2^n} \int_{I_i} F(x) \, dx = \lim_{n \rightarrow \infty} \sum_{i=1}^{n2^n} \int_{I_i} \frac{i-1}{2^n} F(x) \, dx$$

Since $I_i = [\frac{i-1}{2^n}, \frac{i}{2^n})$, $\frac{i-1}{2^n} \leq x$ for all $x \in I_i$, hence

$$\int_X f \, d\mu \leq \lim_{n \rightarrow \infty} \sum_{i=1}^{n2^n} \int_{I_i} x F(x) \, dx \leq \lim_{n \rightarrow \infty} \sum_{i=1}^{n2^n} \int_{I_i} \frac{i}{2^n} F(x) \, dx = \int_X f \, d\mu.$$

By the Sandwich Theorem, this means that

$$\int_X f \, d\mu = \lim_{n \rightarrow \infty} \sum_{i=1}^{n2^n} \int_{I_i} x F(x) \, dx = \lim_{n \rightarrow \infty} \int_{\cup I_i} x F(x) \, dx = \int_{\mathbb{R}} x F(x) \, dx.$$

This argument is easily adapted to a more general situation including f that may not be non-negative. Consequently, we have:

Theorem. *Suppose f is a random variable with probability density function F . Then*

$$E(f) = \int_X f \, d\mu = \int_{\mathbb{R}} x F(x) \, dx.$$

Note that whenever $F(x)$ is an even function, then $E(f) = 0$ since then $x F(x)$ is odd.