Bowdoin College

Math 3603: Advanced analysis
Prof. Thomas Pietraho

Homework 6

1. In class, we briefly discussed the notions of the limit supremum and limit infimum of a sequence of functions. The purpose of this exercise is to verify the details exploited in the lecture, albeit in the context of sequence of real numbers. We begin with the appropriate definitions:

   Definition. *Consider a sequence of real numbers $\{a_i\}$ and define another sequence of reals by*

   $$b_i = \inf_{k \geq i} a_k.$$

   Note that the new sequence $\{b_i\}$ is increasing, hence it has a supremum within the extended real numbers. Let $\liminf a_i = \sup_{i \in \mathbb{N}} b_i$.

   (a) Following the above, carefully define the notion of a $\limsup$ of a sequence of real numbers.

   (b) Explain why for any sequence, the $\limsup$ and $\liminf$ exist in $\widetilde{\mathbb{R}}$. Be as precise as you can.

   (c) Find a sequence of real numbers for which the $\limsup$ and $\liminf$ do not equal each other.

   (d) Finally, show that a sequence of real numbers has a limit iff its $\limsup$ and $\liminf$ coincide.

2. Consider the following definition from class:

   Definition. *The characteristic, or indicator, function χ_A of a set A is defined by*

   $$\chi_A(x) = \begin{cases} 1 & \textit{if } x \in A, \textit{ and} \\ 0 & \textit{if } x \notin A \end{cases}$$

   The purpose of this exercise is to define a relationship between the definitions of $\liminf$ and $\limsup$ on sequences of sets versus sequences of real numbers. Consider a set X and a collection of subsets $\{A_i\}_{i=1}^{\infty}$. Write

   $$A_+ = \limsup A_i \qquad \text{and} \qquad A_- = \liminf A_i$$

   Let $f_+ = \chi_{A_+}$, $f_- = \chi_{A_-}$, and $f_n = \chi_{A_n}$, be the characteristic functions of the corresponding sets. Prove that $f_+ = \limsup f_n$ and $f_- = \liminf f_n$

3. Let $(X, \mathcal{F}, \mu)$ be a measure space with $\mu(X) < \infty$. A sequence $\{f_n\}$ of measurable functions is said to *converge to zero in measure* if for all $\epsilon > 0$

   $$\lim_{n \to \infty} \mu(\{x \mid |f_n(x)| > \epsilon\} = 0$$

   (a) Prove that if f_n converges to zero pointwise except on a set of measure zero, then f_n converges to zero in measure.

   (b) Show that the converse is not true.