Bowdoin College

Math 3603: Advanced analysis
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Homework 11

1. The Brownian motion model for stock prices is too simple for our purposes; in particular, it implies no drift in the long term behavior of equities. The notion of an Itô process, defined as

$$X_t = X_0 + \int_0^t f(x)\, dB_s + \int_0^t g(s)\, ds$$

accounts for a much larger family of random motion. We defined the integral of an Itô process as follows:

Definition. *Consider and Itô process X^t and an unanticipating random function $h(t)$ which satisfies*

$$\int_0^t (hf)^2 < \infty \text{ and } \int_0^t |hg| < \infty.$$

Then

$$\int_0^t h(s)\, dX_s = \int_0^t h(s)f(s)\, dB_s + \int_0^t h(s)g(s)\, ds.$$

If $h(t)$ is interpreted as the number of shares of a stock held at time t, explain how the above creates a model for capital gains if X_t is our model for the per-share price of the stock.

2. This problem is aimed to supplement the proof of the Itô formula from class. Under the conditions of the theorem, show that with probability one,

$$\lim_{||\Pi||\to 0} \sum_{i=0}^{n-1} f_{xx}(t_i, B_{t_i})(B_{t_{i+1}} - B_{t_i})^2 = \int_0^T f_{xx}(t, B_t)\, dt.$$

Hint: Use our formula for the quadratic variation of Brownian motion. Unlike our proof in class, be careful and make sure your manipulations are justified. One approach is to compute the expected value of both sides, and then the variance of their difference. You may assume the following are true in our current circumstances without proof:

- If $f(t_i, B_{t_i})$ and $(B_{t_{i+1}} - B_{t_i})$ are independent, then so are $f_{xx}(t_i, B_{t_i})$ and $(B_{t_{i+1}} - B_{t_i})$.
- Limit and integral operators commute.