Bowdoin College

Math 2603: Introduction to Analysis
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Homework 12 Solutions

1. The goal of this sequence of exercises is to derive the power rule for differentiation. It begins with a formal definition of the natural logarithm.

   (a) Define a differentiable function $L : \mathbb{R}_{>0} \to \mathbb{R}$ by requiring that

   i. $L'(x) = \frac{1}{x}$, and

   ii. $L(1) = 0$.

   Prove that these conditions define L uniquely. That is, if M is another function satisfying both of the above, then $L(x) = M(x)$ for all $x \in \mathbb{R}_{>0}$. We will write $\ln(x)$ instead of $L(x)$.

   Solution: Assume that $M(x)$ is another function with $M'(x) = \frac{1}{x}$ and $M(1) = 0$. Consider the function $f = L - M$. It is also differentiable and the its derivative is the difference of the derivatives of L and M. The properties of L and M imply $f'(x) = 0$ for all $x \in \mathbb{R}_{>0}$ and $f(1) = 0$. The first statement implies that f is constant, and the second implies this constant is zero, i.e. $f(x) = L(x) - M(x) = 0$ for all $x \in \mathbb{R}_{>0}$, as desired.

   (b) Show that $\ln x$ is a bijection from $\mathbb{R}_{>0}$ to $\mathbb{R}$.

   Hint: To show that it is surjective, first show that it has neither an upper nor a lower bound and then use the intermediate value theorem. One way to show that $\ln x$ fails to have an upper bound is to first show that $\ln n \geq \sum_{k=2}^{n} \frac{1}{k}$. for all n.

   Solution: We first show that $\ln x$ is injective. Suppose not, and assume $\ln x = \ln y$ for some numbers x and y. Since $[x, y]$ is compact and $\ln x$ is continuous (it is, after all, differentiable), it attains a maximum and a minimum at some points, say a and b. If, as sets, $\{a, b\} = \{\ln x, \ln y\}$, then $\ln x$ must be constant, which cannot be true since it's derivative is not equal to zero anywhere.

   Otherwise, one of the points a or b must lie in the interior of the interval $[\ln x, \ln y]$, and by a theorem from class, the derivative of $\ln x$ must equal zero there. This is again a contradiction, since the derivative of $\ln x$ is always positive.

   To show that $\ln x$ is surjective, we use the fact that its image has no upper or lower bound. Knowing this, let $M \in \mathbb{R}$. Then there is a point x_1 for which $f(x_1) > M$ and another point at which $f(x_2) < M$ (otherwise, the image of f would be bounded, either above or below. Since $\ln x$ is continuous, the intermediate value theorem implies there is a point c between x_1 and x_2 for which $f(c) = M$. Since this can be done for every $M \in \mathbb{R}$, $\ln x$ is surjective.

   To finish, we must show that $\ln x$ is not bounded above or below. We focus on "above." We know that $\ln 1 = 0$ with slope at least $\frac{1}{2}$ on the interval $[1, 2]$. This means $\ln 2 \geq \frac{1}{2}$. [1] Similarly, its slope is at least $\frac{1}{3}$ on $[2, 3]$ so that $\ln 3 \geq \frac{1}{2} + \frac{1}{3}$. Continuing in this manner,

   $$\ln n \geq \sum_{2}^{n} \frac{1}{k}.$$

   Since the latter diverges to infinity, $\ln x$ has no upper bound. More precisely, $\sum_2^n \frac{1}{k}$ is an increasing sequence. We have shown in class that it is not Cauchy, so it does not converge.

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[1] If you are a stickler, this fact requires proof. If $\ln 2 < \frac{1}{2}$, the Mean Value Theorem would guarantee a point $c \in (1, 2)$ where $\ln'(c) = \frac{\ln(2) - \ln(1)}{2-1} = \ln(2) < \frac{1}{2}$. But we know that for all $c \in (1, 2)$, $\ln'(c) = \frac{1}{c} > \frac{1}{2}$. Hence $\ln 2 \geq \frac{1}{2}$.

Were it bounded, it would have to converge, by another fact from class. So it is not bounded, and so the same is true of $\ln n$ and hence of $\ln x$.

(c) Since $\ln x$ is a bijection, it has an inverse function, which we define to be e^x. Using the chain rule, prove that $(e^x)' = e^x$.

Solution: Note that $e^{\ln(x)} = x$. Using the chain rule:

$$(e^{\ln x})' = e'(\ln(x)) \cdot \frac{1}{x} = 1$$

Let $y = \ln x$, so that $x = e^y$. Our equation can be rewritten as

$$(e^y)' \cdot \frac{1}{e^y} = 1$$

or, in other words, $(e^y)' = e^y$, as desired.

(d) For a positive real number x and any real α, we can now define x^α as

$$x^\alpha = e^{\alpha \ln x}.$$

Armed with this definition, show that

$$(x^\alpha)' = \alpha x^{\alpha-1}.$$

Solution: Note that

$$(x^\alpha)' = (e^{\alpha \ln x})' = e^{\alpha \ln x} \cdot \alpha \frac{1}{x} = x^\alpha \cdot \alpha \cdot \frac{1}{x} = \alpha x^{\alpha-1}.$$