Bowdoin College

Math 2603: Introduction to Analysis
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Homework 10 Solutions

1. Consider a function $f : S_1 \to S_2$. Show that if x is not a limit point of S_1, then f must be continuous at x.

   Solution:

   There is a video solution available for this problem.

   If x_0 is not a limit point of S_1, we can negate the definition of a limit point and conclude that there must be a $\delta > 0$ such that $B_\delta(x_0)$ contains no points of S_1 except for x_0. We are ready to show that f is continuous at x_0. Given any $\epsilon > 0$, let δ be as above. Then $f(B_\delta(x_0)) = f(x_0)$ since $B_\delta(x_0)$ contains only x_0. But then $f(B_\delta(x_0)) \subset B_\epsilon(f(x_0))$! This is exactly the definition of continuity of a function at x_0.

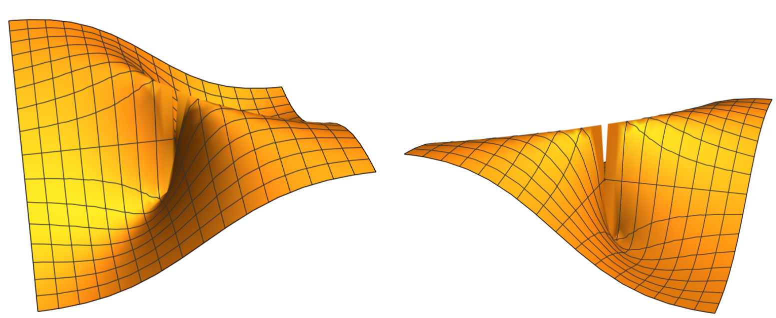
2. Show that every polynomial $p : \mathbb{R} \to \mathbb{R}$ is a continuous function.

   Solution: From a theorem from class, we know that the sums and products of continuous functions are themselves continuous. We also know that the identity function $f(x) = x$ is always continuous. But since any polynomial f is just a sum of products of f, we can conclude that itself, it must be continuous. The most careful proof of this uses induction to show that every x^n is continuous.

3. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

   The plot of this function is somewhat interesting. Below are two perspectives:



   Determine whether $f(x, y)$ is continuous at the origin $(0, 0)$ and justify your answer.

   Solution: Clearly something funky is going on the the origin. It looks like depending on the direction that you approach it from, the values of the function tend to different limits. It seems like we should use the sequential characterization of continuity and look for sequences of points which converge to $(0,0)$. There are a lot of choices, but consider the sequence $s_n = (\frac{1}{n}, \frac{1}{n})$. Then $\lim s_n = (0,0)$ and

$$\lim f(s_n) = \lim f(\tfrac{1}{n}, \tfrac{1}{n}) = \frac{\frac{1}{n} \cdot \frac{1}{n}}{\frac{1}{n}^2 + \frac{1}{n}^2} = \lim \tfrac{1}{2} = \tfrac{1}{2}.$$

But for a function to be continous at s, we need to know that for every sequence that if $s = \lim s_n$, we have that $\lim f(s_n) = f(s)$. For our particular example, we have $\lim s_n = (0,0)$ and

$$\lim f(s_n) = \tfrac{1}{2} \neq f((0,0)) = 0.$$

Thus f is not continous at the origin.

4. Consider a function $f : \mathbb{R} \to \mathbb{R}$ which satisfies $f(x) = 0$ for all $x \in \mathbb{Q}$. Show that if f is continuous, then in fact $f(x) = 0$ for all $x \in \mathbb{R}$!

Solution: Consider an irrational number $r \in \mathbb{R}$ and let r_n be a sequence of rational numbers which converge to r. Since f is continuous,

$$f(\lim r_n) = \lim f(r_n).$$

Since $\lim r_n = r$ and $f(r_n) = 0$ for all n, this equation means that $f(r) = 0$. Since r was an arbitrary irrational, f must equal to zero on all real numbers.

5. This problem revisits the notion of an additive homomorphism, this time with a slightly enlarged domain. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is an additive homomorphism; that is, it satisfies

$$f(x+y) = f(x) + f(y) \text{ for all } x, y \in \mathbb{R}.$$

Describe the set of all possibilities for f if we assume that it is continuous.

Extra Credit: What are the possibilities for f if we don't assume that it is continuous?

Solution: From a previous exercise, we know that the only functions $f : \mathbb{Q} \to \mathbb{R}$ which are additive homomorphisms are of the from $f(x) = mx$. Consider an additive homomorphism $f : \mathbb{R} \to \mathbb{R}$; on the rationals, it must be $f(x) = mx$, but we know nothing about it for irrational values of x.

Define a new function $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = f(x) - mx$. Note that since $f(x)$ and mx are both continuous functions on $\mathbb{R}$, so is $g(x)$. Furthermore, if x is rational, $g(x) = 0$. Hence by the previous exercise, $g(x) = 0$ for all $x \in \mathbb{R}$. In other words, $f(x) = mx$ for all $x \in \mathbb{R}$.

When f is not continuous, things are *much* more interesting. Suppose that $f(x) = mx$ when x is rational. This says nothing about what $f(\sqrt{2})$ should be , so let $f(\sqrt{2}) = l$. Using the additive homomorphism property of f, we can deduce that

$$f(x\sqrt{2}) = lx \text{ if } x \in \mathbb{Q}$$

Recall the notation $\mathbb{Q}[\sqrt{2}]$. This set consists of all numbers of the form $a + b\sqrt{2}$ where $a, b \in \mathbb{Q}$. It is easy to check that

$$f(x + y\sqrt{2}) = mx + ly$$

when $x, y \in \mathbb{Q}$ is an additive homomorphism. Of course our work says nothing about what $f(\sqrt{3})$ should be. So we can continue playing this game, each time adding irrational numbers not covered by our previous work. The result is a very interesting function! Please see the attached article on the Hamel basis for a more thorough treatment.

6. Let (S, ρ) be a metric space endowed with the discrete metric. Describe all of its connected subsets.

Solution: Suppose that $T \subset S$ contains at least two points. Choose a point $p \in T$ and let $A = \{p\}$ and $B = \{p\}^c$. Since all of the subsets of S are open, A and B are non-empty disjoint open sets which cover T and intersect T non-trivially. Hence T is not connected. However, if $T = \{p\}$, then no such choice of sets A and B is possible; since they are to be disjoint, both can't intersect T non-trivially! Hence singleton points are the connected subsets of S. There is one more susbset to consider: the empty set. Is it disconnected? Can we find two disjoint non-empty sets A and B each of which contain at least one point of T. The answer is no, so the empty set is in fact connected.