Bowdoin College

Math 2603: Introduction to Analysis
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Homework 3

1. A map $f : \mathbb{Q} \to \mathbb{Q}$ is called an *additive homomorphism* if it satisfies

$$f(a+b) = f(a) + f(b).$$

   What are all the additive homomorphisms from $\mathbb{Q}$ to $\mathbb{Q}$?

2. Consider vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$. We can define a function $\rho_1 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by letting

$$\rho_1(\vec{v}, \vec{w}) = \sum_{i=1}^{n} |v_i - w_i|.$$

   In $\mathbb{R}^2$ this is called the "taxicab metric." Prove that when endowed with this metric, $\mathbb{R}^2$ becomes a metric space.

3. Recall the definitions of the metrics ρ_2, ρ_1, and ρ_∞ on the set $\mathbb{R}^n$ from class. They are a part of a larger family of metrics called the p-metrics.

   Definition. *Let $\vec{v} = (v_1, \ldots, v_n)$ and $\vec{w} = (w_1, \ldots, w_n) \in \mathbb{R}^n$. For $p \in \mathbb{N}$, we define a function*

$$\rho_p(\vec{v}, \vec{w}) = \sqrt[p]{|v_1 - w_1|^p + \ldots |v_n - w_n|^p}.$$

   It turns out that this is always a metric. I urge you to stay away from the proof of the triangle inequality. Armed with this knowledge, try do visualize how these metrics are interrelated by the following exercise. Let $\vec{0}$ be the origin in $\mathbb{R}^n$. Draw the unit sphere $S_1(\vec{0})$ in the metric spaces $(\mathbb{R}^2, \rho_2)$, $(\mathbb{R}^2, \rho_1)$, $(\mathbb{R}^2, \rho_\infty)$, $(\mathbb{R}^3, \rho_1)$, and $(\mathbb{R}^2, \rho_3)$.

4. Recall the field $\mathbb{Q}[\sqrt{2}]$ from the previous assignment. Since all of its elements are real numbers, it inherits a the usual ordering from $\mathbb{R}$ and it is easy to verify that $\mathbb{Q}[\sqrt{2}]$ is also an ordered field. Find another way of ordering the elements of $\mathbb{Q}[\sqrt{2}]$, with the new order denoted by $\prec$, so that $(\mathbb{Q}[\sqrt{2}], \prec)$ is still an ordered field.