

BOWDOIN COLLEGE

MATH 2603: INTRODUCTION TO ANALYSIS
PROF. THOMAS PIETRAHO

HOMEWORK 2

1. Define the set of real numbers

$$\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}.$$

Together with the usual addition and multiplication, $(\mathbb{Q}[\sqrt{2}], +, \cdot)$ forms a field. One could verify this fact directly by checking all the axioms of a field hold, but it turns out that is not necessary to work quite that hard. The key observation is that the set of real numbers $(\mathbb{R}, +, \cdot)$ is itself a field and every element of $\mathbb{Q}[\sqrt{2}]$ is a real number.

Question: Identify which field axioms must be explicitly checked to verify that $\mathbb{Q}[\sqrt{2}]$ is indeed a field, and which axioms already hold by viewing $\mathbb{Q}[\sqrt{2}]$ as a subset of $\mathbb{R}$. It is not necessary to carry out the proofs.

2. Suppose that $(F, +, \cdot)$ is a field and x, y , and z are elements of F . Show that

(a) If $x \neq 0$ and $xy = xz$, then $y = z$.

(b) If $x \neq 0$ and $xy = x$, then $y = 1$.

3. Suppose that $(F, +, \cdot)$ is an ordered field with $x, y \in F$. Show that

(a) If $x < y$ then $-y < -x$.

(b) $1 > 0$.

4. Can the set of complex numbers $\mathbb{C}$ form an ordered field?

Hint: Consider the relationship between i and 0 .

5. Consider a set $S \subset \mathbb{R}$ and suppose that it has a lower bound. Define the set $-S$ as $\{-r \mid r \in S\}$.

(a) Define precisely the notion of a greatest lower bound for a set S and explain why one exists if S has a lower bound. The greatest lower bound for a set S is often written as $\inf S$, the infimum of S .

(b) Show that $\inf S = -\sup(-S)$.