Bowdoin College

Math 2603: Introduction to Analysis
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Homework 13

1. Consider a function $f : [a, b] \to \mathbb{R}$ which may or may not be Riemann integrable. For any two partitions P and Q of the interval $[a, b]$, show that

$$L(f, P) \le U(f, Q),$$

   that is, *any* upper sum is greater than or equal to *any* lower sum for f.

   Hint: Use a common refinement, that is, a partition that is both a refinement of P and Q.

2. From class, we know that bounded continuous functions on a compact interval in $\mathbb{R}$ are Riemann integrable. The following exercise will show that a function can have one jump or removable discontinuity and still remain Riemann integrable. Consider an interval $[a, b] \in \mathbb{R}$ and a point $c \in [a, b]$. Define a function $f : [a, b] \to \mathbb{R}$ by $f(x) = 0$ unless $x = c$, when $f(c) = 1$. In other words, $f = \chi_{\{c\}}$, the indicator function of the set $\{c\}$.

   (a) Show that f is Riemann integrable on $[a, b]$.

   (b) Conclude that any function which is continuous except for possibly for a jump or a removable discontinuity at one point is Riemann integrable.

   In fact, by induction one can extend this exercise to show that a finite number of such discontinuities do not affect Riemann integrability. Consequently, things like step functions are Riemann integrable as well. Can this requirement be relaxed even further?

   The answer is "yes", and in fact, by quite a bit. The complete answer was found by Henri Lebesgue in his doctoral thesis.[1] The complete answer involves the measure of a set. We will say a subset of $\mathbb{R}$ has *measure zero* if for every $\epsilon > 0$, it can be covered by a countable number of open intervals whose total length is less than ϵ. It turns out that the rational numbers and the Cantor set both have measure zero. Here is Lebesgue's observation:

   Theorem (Riemann-Lebesgue Theorem)**.** *Suppose that $f : [a, b] \to \mathbb{R}$ is a bounded function and let D be the set of points where it is discontinuous. Then $f \in \mathcal{R}[a, b]$ if and only if D has measure zero.*

   We will not have a chance to prove this in this class, so you will have to refrain using this result in what follows!

3. Consider two Riemann-integrable functions $f, g : [a, b] \to \mathbb{R}$. Show that if for all $x \in [a, b]$ we have $f(x) \le g(x)$, then

$$\int_a^b f \le \int_a^b g.$$

4. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and $f(x) \ge 0$ for all $x \in [a, b]$. Prove that if $\int_a^b f = 0$, then f is the zero function.

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[1] Therein, he also constructed what is known today as the *Lebesgue integral* that has come to completely supercede the Riemann integral. But this is a story for Math 3603.

5. Suppose that f is a continuous real-valued function on the interval $[a, b]$. Prove that these exists a point $c \in [a, b]$ such that

$$\int_a^b f = f(c)(b - a).$$

This is the *Mean Value Theorem for Integrals.*