Bowdoin College

Math 2603: Introduction to Analysis
Prof. Thomas Pietraho

Homework 12

1. The goal of this sequence of exercises is to derive the power rule for differentiation. It begins with a formal definition of the natural logarithm.

   (a) Define a differentiable function $L : \mathbb{R}_{>0} \to \mathbb{R}$ by requiring that

      i. $L'(x) = \frac{1}{x}$, and

      ii. $L(1) = 0$.

   Prove that these conditions define L uniquely. That is, if M is another function satisfying both of the above, then $L(x) = M(x)$ for all $x \in \mathbb{R}_{>0}$. We will write $\ln(x)$ instead of $L(x)$.

   (b) Show that $\ln x$ is a bijection from $\mathbb{R}_{>0}$ to $\mathbb{R}$.

   Hint: To show that it is surjective, first show that it has neither an upper nor a lower bound and then use the intermediate value theorem. One way to show that $\ln x$ fails to have an upper bound is to first show that $\ln n \geq \sum_{k=2}^{n} \frac{1}{k}$. for all n.

   (c) Since $\ln x$ is a bijection, it has an inverse function, which we define to be e^x. Using the chain rule, prove that $(e^x)' = e^x$.

   (d) For a positive real number x and any real α, we can now define x^α as

   $$x^\alpha = e^{\alpha \ln x}.$$

   Armed with this definition, show that

   $$(x^\alpha)' = \alpha x^{\alpha - 1}.$$