Bowdoin College

Math 2020: Introduction to Mathematical Reasoning
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Homework 10

1. Show that the set of integers which are three more than a multiple of seven form a countable set.

2. **SPOILER** We will show in class that there are sets which do NOT have the same cardinality as the natural numbers. Such a set is called uncountable. We will show that $\mathbb{R}$ and the interval $(0, 1)$ are both uncountable sets.

Assuming the fact that $(0, 1)$ is an uncountable set, show that any open interval (a, b) has the same cardinality as $(0, 1)$ and thus is also an uncountable set.

The next part of your homework introduces to a new mathematical idea called an *equivalence relation*. I explain it below, and there is a section in the text as well, and then I ask you to do a few problems. Please come to office hours or LA hours to ask about equivalence relations if you have questions!

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An introduction to Equivalence Relations

A relation on a set S is a just a rule which pairs certain elements of S. For example, if $S = \mathbb{Z}$, then our rule might be to pair n and $2n$. We would write this as a set of ordered pairs:

$$\{(n, 2n) | n \in \mathbb{Z}\}.$$

Note that the order matters: $(1, 2)$ is a pair in the relation but $(4, 2)$ is not.

This means that a relation picks out a set of pairs of elements that satisfy a given property. For example, we might say that two integers x and y are related if one is the square of the other. We write this as follows:

$$x \sim y \text{ iff either } x^2 = y \text{ or } y^2 = x$$

So mathematically, $x \sim y$ means "x is related to y". So $2 \sim 4$ and $100 \sim 10$, for example, using the above relation. As a set of ordered pairs, the relation would include the pairs $(2, 4)$ and $(100, 10)$, among others.

We can ask whether a relation on a set satisfies certain extra properties. If a relation satisfies the three properties below, it is called an *equivalence relation*. We are interested in these properties because many of the relations we will consider for the rest of the semester (and in other math classes) will be equivalence relations, and then we know these extra properties hold.

A relation $\sim$ on a set S is called an equivalence relation if it satisfies three special properties:

(1) It is **reflexive**, which means that $x \sim x$ for all $x \in S$.
Note: this is not always true: suppose that $x \sim y$ is defined by $x = \sqrt{y}$. Then $1 \sim 1$ but $4 \nsim 4$.)

(2) It is **symmetric**, which means that for all $x, y \in S$, if x is related to y, then it is also true that y is related to x.
Again using our example $x \sim y$ defined by $x = \sqrt{y}$, we see that $2 \sim 4$ but $4 \nsim 2$ so this relation is not symmetric.

(3) It is **transitive**, which means that for all $x, y, z \in S$, if $x \sim y$ and $y \sim z$ then $x \sim z$.

So an equivalence relation is a very special type of relation since it has to satisfy these three properties. Here is an example of an equivalence relation on the set of integers:

> $x \sim y$ if and only if x and y are both even or odd. See whether you can reason out the three properties above!

Here is another example of a relation on the set of integers: $x \sim y$ iff $y = kx$ for some k in the set $S = \{x | x = 1 \text{ or x is even}\}$. So this means, in words, that two integers are related if the second one is either equal to the first one, or an even multiple of the first one. For example, $5 \sim 10$ because $10 = 2 \times 5$ but it is not true that $2 \sim 10$ because you need an *odd* multiple of 2 to create 10.

Here's how to check the three properties to see if this relation is an equivalence relation:

(1) Since $x = 1 \cdot x$ and $1 \in S$ we see that $x \sim x$ so this relation is reflexive.

(2) Now let's see whether this relation is symmetric. Suppose that $x \sim y$. We have to figure out whether $y \sim x$. Since we are assuming that $x \sim y$, we know that $y = kx$ for some $k \in S$. However, if we solve this for x, we see that $x = \frac{1}{k}y$. Now, if $k = 1$ then $\frac{1}{k} \in S$, but there will be lots of examples of $x \sim y$ when $k \neq 1$ (consider $2 \sim 4$). So we cannot conclude that $y \sim x$ which means the symmetric property does not hold.

If this was a homework question, you could stop here and say that this relation is NOT an equivalence relation. You don't even have to check the third property - once one of the properties is false, it's not an equivalence relation. However, I'll investigate the transitive property for this relation so you can see how that's done.

(3) To see whether the transitive property holds, we will assume that $x \sim y$ and $y \sim z$ for some $x, y, z \in \mathbb{Z}$. This means that $y = kx$ and $z = ly$ for some $k, l \in S$. Substituting, we see that $z = ly = l(kx) = (lk)x$. Since $k, l \in \mathbb{Z}$ and $\mathbb{Z}$ is closed under multiplication, $lk \in \mathbb{Z}$. Thus $x \sim z$ and the transitive property holds.

You can also define a relation on a set by just listing the ordered pairs which are related to each other.

Here is an example of a relation on the set $\{a, b, c\}$ which is NOT an equivalence relation:

$$R = \{(a, a), (b, b), (c, c), (a, b), (a, c)\}.$$

If x is related to y, then it is also true that y is related to x. You can also think of this relation as

$$a \sim a,\ b \sim b,\ c \sim c,\ a \sim b,\ a \sim c.$$

This relation is reflexive, because for each element $x \in \{a, b, c\}$ we can see that $x \sim x$. However, notice that $(a, b) \in R$ but $(b, a) \notin R$. Thus the relation is not symmetric, hence not an equivalence relation.

Problems on Equivalence Relations

3. Does $x \sim y$ defined by $(x, y) \in \{(1,1), (1,2), (2,1), (2,2)\}$ define an equivalence relation on the set $S = \{1, 2\}$? Please justify your answer.

4. Page 94 problem 6. The problem asks you to define several relations on the set. This means to pick out a set of ordered pairs whose entries come from the original set, satisfying the properties required in the problem.

5. Page 96, problem 20. For this problem you only need to determine whether the relation is an equivalence relation. You do NOT need to find the partition, since we have not discussed that.

If the relation is an equivalence relation, just write "Yes." If not, please give an example of which condition fails to be satisfied.

6. Define a relation on the set of all sets by saying, for sets S and S, that $S \sim T$ iff $S \approx T$, that is, S and T have the same cardinality. Prove that $\sim$ is an equivalence relation.

In your proof, you can quote theorems and lemmas which we proved when we were talking about functions, if that is helpful.

If you say that a function is a bijection as part of your proof, you will have to justify that using either a theorem from class or the definition of a bijection.