How (not?) to grade a true-or-false exam; or, Is teaching different than training a machine learning model?

Thomas Pietraho

Part 1. Rethinking true-or-false exams

A quick quiz

Q1: $\sin' 2x = 2\cos 2x$

A quick quiz

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Q2: $\arctan' x = \frac{1}{1+x^2}$ TRUE

Q3: The king of diamonds has a mustache

A quick quiz

Q1: $\sin' 2x = 2\cos 2x$ TRUE

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Question: Is there a better way? Partial credit? Answer with $p \in [0, 1]$?

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Source: Terry Tao's blog

Paul Erdös and Terry Tao.

A quick quiz: again

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A quick quiz: again

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Q1: $\sin' 2x = 2\cos 2x$ TRUE : 90%

Q2: $\arctan' x = \frac{1}{1+x^2}$ TRUE : 70%

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A quick quiz: again

Q1: $\sin' 2x = 2\cos 2x$	TRUE : 90%
Q2: $\arctan' x = \frac{1}{1+x^2}$	TRUE : 70%
Q3: The king of diamonds has a mustache	FALSE

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Q1: $\sin' 2x = 2\cos 2x$ TRUE : 90%

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A quick quiz: again

Q1: $\sin' 2x = 2\cos 2x$	TRUE : 90%
Q2: $\arctan' x = \frac{1}{1+x^2}$	TRUE : 70%
Q3: The king of diamonds has a mustache	FALSE: 55%

Assumption: There is some process by which a exam taker can determine their certainty $p \in [0, 1]$ when answering such questions.

A quick quiz: again

Q1: $\sin' 2x = 2\cos 2x$ — TRUE : 90%

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Q3: The king of diamonds has a mustache — FALSE: 55%

Assumption: There is some process by which a exam taker can determine their certainty $p \in [0, 1]$ when answering such questions.

Question: How should a certainty-based true-or-false exam be graded?

Mathematical setup

Exam taker believes statement is

- TRUE with confidence 60%
- FALSE with confidence 40%

Writes down TRUE: 60%.

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Let p be the confidence exam taker has in the correct answer. Score by assigning $S(p)$ points.

- $S : [0, 1] \to \mathbb{R}$
- S increases with p
- Anything else?

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- Anything else?

Q1: $\sin' 2x = 2\cos 2x$ TRUE: 90%
Q2: $\arctan' x = \frac{1}{1+x^2}$ TRUE: 70%

Both answers are correct.
Total score: $S(90\%) + S(70\%)$
Total confidence: 63%

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Both answers are correct.
Total score: $S(90\%) + S(70\%)$
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Definition: We will call a scoring function S is **reasonable** if

1. $S : [0,1] \to \mathbb{R}$
2. S increases with p
3. $S(pq) = S(p) + S(q)$

Goal: Find all reasonable scoring functions.

Part 2. A mathematical gem

Easier question: *Find all functions which satisfy*

$$f(x + y) = f(x) + f(y).$$

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$f(0+0) = f(0) + f(0)$

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Irrational numbers?

$f(3) = c \cdot 3$

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Irrational numbers?

$f(3.1) = c \cdot 3.1$

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Irrational numbers?

$f(3.14) = c \cdot 3.14$

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Irrational numbers?

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Irrational numbers?

$f(3.1415) = c \cdot 3.1415$

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Irrational numbers?

$f(\pi) = c \cdot \pi$

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Irrational numbers?

$f(\pi) = c \cdot \pi$

Theorem

Yes, but if f is continuous.

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Negative numbers?

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Negative numbers?

$f(x - x) = f(x) + f(-x)$

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Theorem

If f is continuous and $f(x+y) = f(x) + f(y)$, then $f(x) = c \cdot x$.

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Could try similar approach. But there is a better way.

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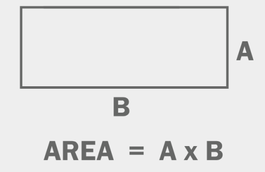
Theorem

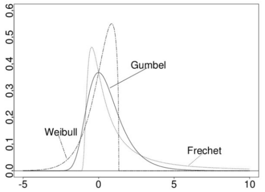
If S is continuous and satisfies $S(x \cdot y) = S(x) + S(y)$, then

$$S(x) = c \cdot \log x.$$

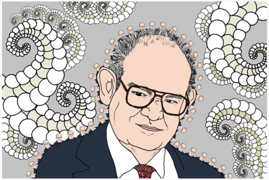
Aside: Cauchy functional equations



Notion of area Are there other ways area could be defined? Is the way we are asked to compute area a historical artifact? Work on this question requires you to find functions $f(x+y) = f(x) + f(y)$. Image from Archtoobox.com.



Extreme value distributions Estimate the cost the repair damages caused by a 100-year flood. Work on this question eventually requires you to find functions satisfying $f(x \cdot y) = f(x) + f(y)$. Image by Gennady Samorodnitsky.



Benoit Mandelbrot Known for his work on fractals, Mandelbrot first noticed that equity returns have "power-law" and not "normal" tails. He studied functions of the form $f(x \cdot y) = f(x) \cdot f(y)$. Image from from *Tablet Magazine*.

Solution and examples

Proposal: *Let's use* $S(p) = \log p$ *as the scoring function.*

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Scores		
p	correct	incorrect
50%	-0.30	-0.30
60%	-0.22	-0.40
70%	-0.15	-0.52
80%	-0.10	-0.70
90%	-0.04	-1.00
100%	0.00	$-\infty$

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Question: *Should I use this scoring method on an exam in one of my courses?*

Part 3. Context

Claude Shannon and information theory

Courtesy of Shannon family.

Question: Suppose we observe:

0000100000110000010100100001000000100101000

How surprised should be if the next digit is 1? And how surprising is this entire string of 0s and 1s?

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Definition: We will call S a **surprise function** if

1. $S : [0, 1] \to \mathbb{R}$
2. S decreases with p
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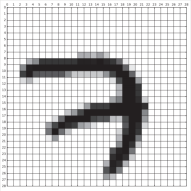
For marketing purposes, Shannon actually called S the **information function** and the field of Information Theory was born.

Classifiers and machine learning

Binary classification problem: Does an object belong to a given class?



Example: Is this digit a 7?

Machine learning pipeline:

1. Ask a model, say a neural network, to make a prediction. Require $p \in [0, 1]$.
2. Assign a score to that prediction
3. Adjust the model via some process that improves the score

Binary cross-entropy: The score, or "loss" function often used:

$$H(p) = \begin{cases} -\log p & \text{if TRUE} \\ -\log(1-p) & \text{if FALSE} \end{cases}$$

Scoring rules and decision theory

The broader context for all of this is known as **decision theory.**

weatherunderground.com

Canonical problem: Probabilistic forecasting is in meteorology. What does an 86% chance of snow mean? Is this calibrated? If not, is there an appropriate bonus system to incentivize the meteorologist?

If you've studied measure theory, the Wikipedia article provides a good introduction.

T.Tao's approach

Criterion: *If a grading scheme is designed badly, a student may end up overstating or understating their confidence in an answer in order to optimise the (expected) grade: the optimal level of confidence q for a student to report on a question may differ from that student's subjective confidence p.*

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Expected score:

$$pS(q) + (1-p)S(1-q)$$

should me maximized when $p = q$. That is:

$$pS'(q) - (1-p)S'(1-q) = 0$$

when $p = q$. Solutions look like $S(p) = C \cdot \log p + D$.

Connections: Cauchy functional equation

Information theory
Extreme value theory
Backtest overfitting
Black swan events
Machine learning
Axiom of choice
Geometry
Pedagogy?????

Augustin Cauchy, from Science Photo Library