Parrondo's Games

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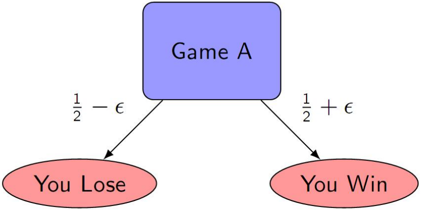
Game A

We'll toss an unfair coin.

- You win with probability $\frac{1}{2} + \epsilon$
- I win with probability $\frac{1}{2} - \epsilon$
- Winner gets \$1.



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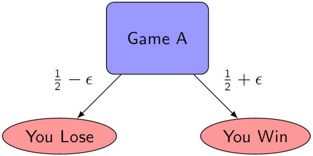
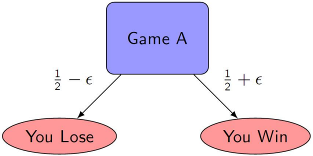


Let's fix $\epsilon = 0.005$ and agree that we'll play one hundred times.

QUESTION: Should you play?

Game A



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ANSWER: Most definitely!

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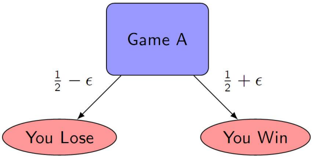


Let's fix $\epsilon = 0.005$ and agree that we'll play one hundred times.

QUESTION: Should you play?
ANSWER: Most definitely!

The expected return can be calculated by:

Prob. of losing $\times$ Amount of loss +
Prob. of winning $\times$ Amount of win

In this case:

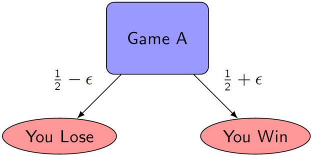
$$0.495 \cdot (-1) + 0.505 \cdot (1) = 0.01$$

Hence, on average, you stand to win

1 cent.

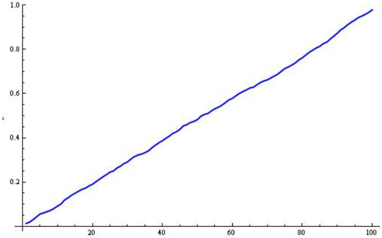
Over 100 repetitions, you stand to win 1 dollar.

Game A



Let's fix $\epsilon = 0.005$ and agree that we'll play one hundred times.

Just to make sure, I ran a simulation. The graph represents 50,000 runs.



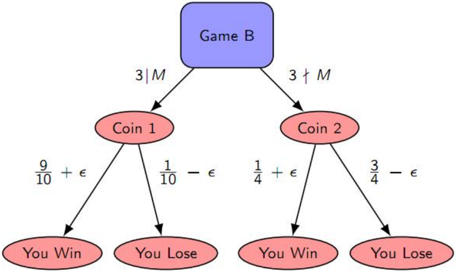
Things don't look too good for me.

Game B

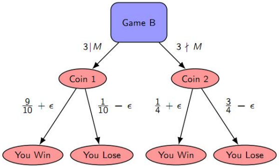
Let M = amount of \$ in your pocket. Assume this is an integer. We'll toss two coins:

- If M is divisible by 3, we will flip coin 1. Otherwise, we'll flip coin 2.

Coin 1: You win with probability $\frac{9}{10} + \epsilon$
Coin 2: You win with probability $\frac{1}{4} + \epsilon$.



Game B



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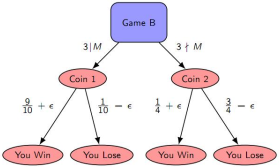
Game B



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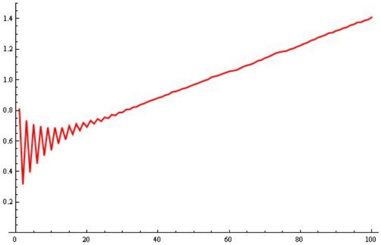

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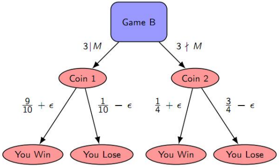
Analysis of Game B



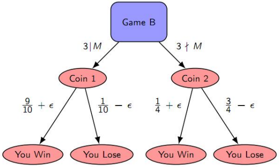
Expected Return:

Prob. of flipping coin 1 $\times$
Expected return from coin 1
$+$
Prob. of flipping coin 2 $\times$
Expected return from coin 2

We flip coin 1 when 3 divides M and flip coin 2 otherwise, so I expect we'll flip coin 1 once every three times we play this.

$$\frac{1}{3}(0.095 \cdot (-1) + 0.905 \cdot (1)) +$$
$$\frac{2}{3}(0.745 \cdot (-1) + 0.255 \cdot (1))$$

Analysis of Game B



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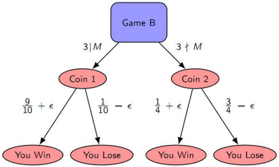
$$\frac{1}{3}(0.095 \cdot (-1) + 0.905 \cdot (1)) + \frac{2}{3}(0.745 \cdot (-1) + 0.255 \cdot (1))$$

$$\approx -0.056$$

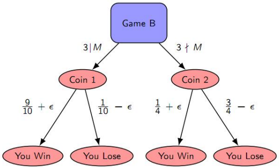
Hence, on average, you should lose 5.6 cents each time we play. That is, I should win 5.6 cents!

Analysis of Game B



My analysis does not agree with the computer!

What's wrong?!



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There are a few possibilities as to what went wrong:

- We were extremely unlucky and the computer simulation gave a statistically improbable result,

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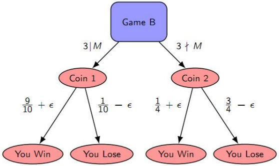

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- We've discovered a fundamental flaw in mathematics!

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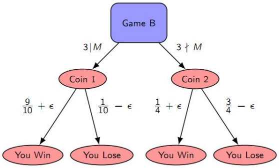

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- We were extremely unlucky and the computer simulation gave a statistically improbable result,
- We've discovered a fundamental flaw in mathematics!
- One of our assumptions was wrong.

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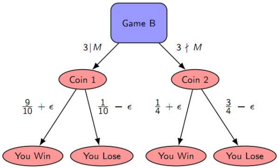

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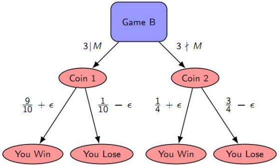
- We've discovered a fundamental flaw in mathematics!
- One of our assumptions was wrong.

I vote for the latter.

Analysis of Game B



We made one assumption analyzing Game B.

"I expect we'll flip coin 1 once every three times..."

This is in fact wrong. From the same computer simulation,

$M \equiv 0 \ (mod\ 3)$	38.4%	of the time
$M \equiv 1 \ (mod\ 3)$	15.4%	of the time
$M \equiv 2 \ (mod\ 3)$	46.2%	of the time

Hence we'll flip coin 1 38.4% of the time. The revised expected value:

$$0.384\big(0.095 \cdot (-1) + 0.905 \cdot (1)\big) + 0.616\big(0.745 \cdot (-1) + 0.255 \cdot (1)\big) \approx 0.009$$

On average, you should expect to win 0.9 cents each time we play.

Summary and Modest Proposal

If we play Game A for a while, you win. If we play Game B for a while, you win. I would like to propose:

Game C

- Randomly alternate playing Game A and Game B

Summary and Modest Proposal

If we play Game A for a while, you win. If we play Game B for a while, you win.
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Game C

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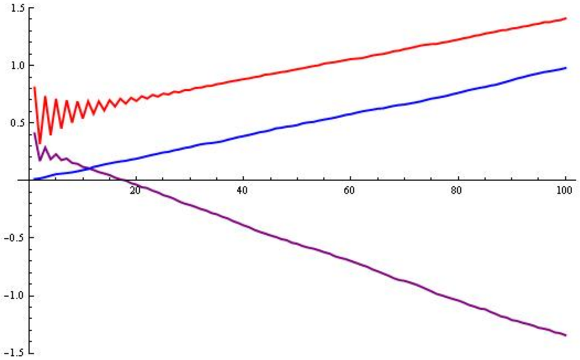
QUESTION: Any takers?

Computer simulation of Game C



Game C is actually a *winning* game for me!

Analysis of Game C

Recall:

- Randomly alternate playing Game A and Game B

Hence we are playing each game 50% of the time. If 3 divides M, the probability that you win is:

$$\begin{aligned} q_1 = & \quad 0.50 \cdot (\text{Prob. of winning game A}) \\ + & \quad 0.50 \cdot (\text{Prob. or winning game B}) \\ = & \quad 0.5 \cdot (0.505) + 0.5(0.905) = 0.71 \end{aligned}$$

If 3 does not divide M, this probability is

$$\begin{aligned} q_2 = & \quad 0.50 \cdot (\text{Prob. of winning game A}) \\ + & \quad 0.50 \cdot (\text{Prob. or winning game B}) \\ = & \quad 0.5 \cdot (0.505) + 0.5(0.255) = 0.38 \end{aligned}$$

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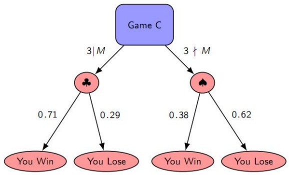
Let's draw a diagram:



But this is just Game B with a different set of probabilities! This means we can use the same method to find expected return:

Prob. that $3 \mid M$ × Expected return from ♣

$+$

Prob. that $3 \nmid M$ × Expected return from ♠

Just don't make false assumptions!

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As in the previous analysis, we need to glean this from our simulation:

$M \equiv 0 \ (mod\, 3)$	34.5%	of the time
$M \equiv 1 \ (mod\, 3)$	39.9%	of the time
$M \equiv 2 \ (mod\, 3)$	25.5%	of the time

We can plug those values into our expected return formula:

$$0.345\big(0.71 \cdot (1) + 0.3 \cdot (-1)\big) +$$
$$0.655\big(0.38 \cdot (1) + 0.62 \cdot (-1)\big)$$

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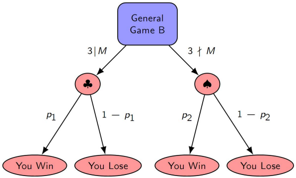
$$0.345\big(0.71 \cdot (1) + 0.3 \cdot (-1)\big) + \\ 0.655\big(0.38 \cdot (1) + 0.62 \cdot (-1)\big)$$

$$\approx -0.019$$

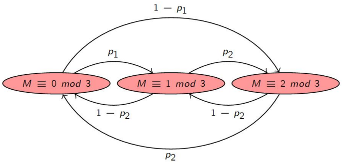
Hence, on average, you expect to lose 1.9 cents per repetition!

General Game B

Let's analyze the general version of Game B:

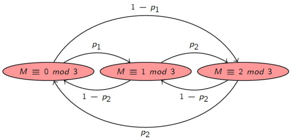


When we play this repeatedly, the game has three "states:"



We add arrows between states indicating the probability of changing from one state to the other. (This is a Markov chain of length 3).

General Game B



Let's assume that $M = 0$ when we start playing. Define

x_i = prob. that $M \equiv 0 \ (mod\ 3)$ after i-games
y_i = prob. that $M \equiv 1 \ (mod\ 3)$ after i-games
z_i = prob. that $M \equiv 2 \ (mod\ 3)$ after i-games

We would like to know what $x, y,$ and z are for large i.

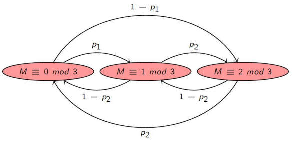
$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ p_1 \\ 1 - p_1 \end{pmatrix}$$

$$\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} (1 - p_2)p_1 \\ (1 - p_2)(1 - p_1) \\ p_2(1 - p_1) \end{pmatrix}$$

$$\begin{pmatrix} x_{i+1} \\ y_{i+1} \\ z_{i+1} \end{pmatrix} = \begin{pmatrix} (1 - p_2)y_i + p_2 z_i \\ p_1 x_i + (1 - p_2)z_i \\ (1 - p_1)x_i + p_2 y_i \end{pmatrix}$$

General Game B



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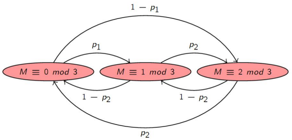

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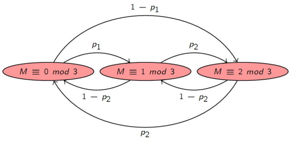
The final equation can also be expressed as:

$$\begin{pmatrix} 0 & 1-p_2 & p_2 \\ p_1 & 0 & 1-p_2 \\ 1-p_1 & p_2 & 0 \end{pmatrix} \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix} = \begin{pmatrix} x_{i+1} \\ y_{i+1} \\ z_{i+1} \end{pmatrix}$$

General Game B



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Suppose that there are numbers x, y, and z such that

$$\lim_{i \to \infty} x_i = x$$
$$\lim_{i \to \infty} y_i = y$$
$$\lim_{i \to \infty} z_i = z$$

Then

$$\begin{pmatrix} 0 & 1-p_2 & p_2 \\ p_1 & 0 & 1-p_2 \\ 1-p_1 & p_2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

In other words, (x, y, z) is an eigenvector of our matrix with eigenvalue 1.

Eigenvectors

Does

$$\begin{pmatrix} 0 & 1-p_2 & p_2 \\ p_1 & 0 & 1-p_2 \\ 1-p_1 & p_2 & 0 \end{pmatrix}$$

have an eigenvector (x, y, z) with eigenvalue 1?

Note: Its columns all add up to one.

Eigenvectors

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Theorem. If the rows of a matrix M add up to one, then M has an an eigenvector with eigenvalue 1.

Proof: $(x, y, z) = (1, 1, 1)$ works!

Eigenvectors

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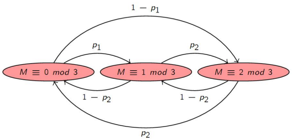
Theorem. The eigenvalues of M and M^t are the same.

Proof: $\det(M - \lambda I) = \det(M - \lambda I)^t = \det(M^t - \lambda I^t) = \det(M^t - \lambda I)$.

General Game B



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Original Game B

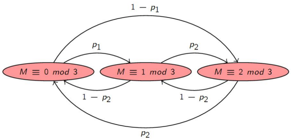
$$\begin{pmatrix} 0 & 1-p_2 & p_2 \\ p_1 & 0 & 1-p_2 \\ 1-p_1 & p_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & .745 & .255 \\ .905 & 0 & .745 \\ .095 & .255 & 0 \end{pmatrix}$$

The eigenvector in question is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} .3836.. \\ .1543.. \\ .4621.. \end{pmatrix}$$

or pretty much what we found from the simulation.

General Game B



$$\begin{pmatrix} x_{i+1} \\ y_{i+1} \\ z_{i+1} \end{pmatrix} = \begin{pmatrix} (1-p_2)y_i + p_2 z_i \\ p_1 x_i + (1-p_2) z_i \\ (1-p_1)x_i + p_2 y_i \end{pmatrix}$$

x_i = prob. that $M \equiv 0$ $(mod\ 3)$ after i-games
y_i = prob. that $M \equiv 1$ $(mod\ 3)$ after i-games
z_i = prob. that $M \equiv 2$ $(mod\ 3)$ after i-games

Original Game C

$$\begin{pmatrix} 0 & 1-p_2 & p_2 \\ p_1 & 0 & 1-p_2 \\ 1-p_1 & p_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & .62 & .38 \\ .71 & 0 & .62 \\ .29 & .38 & 0 \end{pmatrix}$$

The eigenvector in question is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} .3448.. \\ .3998.. \\ .2554.. \end{pmatrix}$$

or pretty much what we found from the simulation.

Markov Chains

This type of analysis works when the next state of the "game" depends only on current state of the game, and the outcome of some randomizer:

- Dice Games:
  - Monopoly (120 variables)
  - Snakes and Ladders
- Roulette
- Simple Weather Models
- Random Walk
- Google PageRank Algorithm

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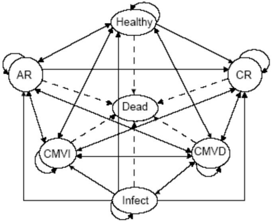
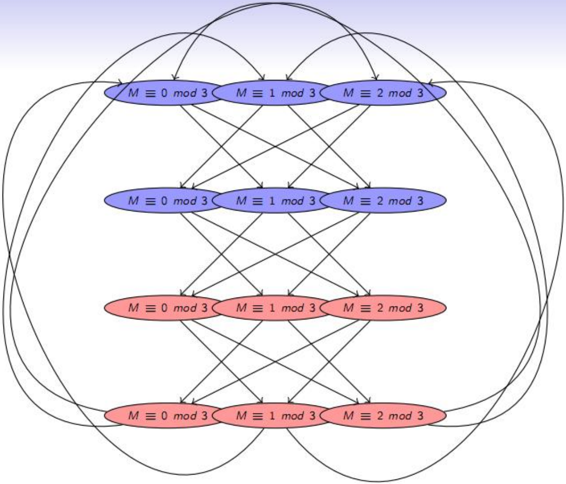


Not Markov Chains

Systems which have a memory of past moves do not lend themselves to this type of analysis. For instance, a player can gain an advantage by remembering which cards have already been shown and which cards are no longer in the deck. The transition probabilities depend not only on the current state of the game, but also on past events.

- Card Games:
  - Poker
  - Blackjack
- Real Weather

Markov Chain for Game C[2,2]



Parrondo's Games

Game C was discovered by J. M. R. Parrondo in 1996 to illustrate the concept of Brownian ratchets. It was presented in a talk called *How to Cheat a Bad Mathematician.*