Fast matrix multiplication: a brief adventure in neural networks and computational algebra

Thomas Pietraho

Fall, 2022

A Strange Theorem

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Let's look at 2×2 matrices:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{22}b_{12} + a_{22}b_{22} \end{bmatrix}$$

To compute this, $8 = 2^3$ scalar products must be found (and a few scalar sums). Thinking about this, we get

Theorem

Two $N \times N$ matrices can be multiplied using N^3 scalar multiplications.

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For computers

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If $N \approx 10^7$, then

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2. $N^3 \approx 10^{21}$

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For multiplication of $10^7 \times 10^7$ matrices, the "strange" theorem cuts the number of scalar multiplications by a factor of about $10^{1.35} \approx 22$.

Strassen's observation: 2×2 matrices

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

First form products:

$$\begin{aligned}
m_1 &= a_{11} b_{11} \\
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m_4 &= a_{12} b_{22} \\
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then combine them:

$$\begin{aligned} c_{11} &= m_1 + m_2 \\ c_{12} &= m_3 + m_4 \\ c_{21} &= m_5 + m_6 \\ c_{22} &= m_7 + m_8 \end{aligned}$$

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$$\begin{aligned}
c_{11} &= m_1 + m_4 - m_5 + m_7 \\
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2×2 *matrices can be multiplied using* 7 *only scalar multiplications!*

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Question: *Can one do better? 2×2 matrices using only 6 scalar multiplications? Or can one reduce the exponent $2.8074\ldots$ some other way?*

Other efforts and theoretical bounds

Definition

Let **MultRank**(N) be the minimum number of scalar multiplications necessary to multiply two $N \times N$ matrices. Let **MultExp**(N) be the corresponding exponent.

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Similar advances:

Theorem (Laderman, 1976)

$$\mathbf{MultRank}(3) \leq 23.$$

Theorem (Waksman, 1970)

$$\mathbf{MultRank}(2, 2, 3) \leq 11.$$

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Theorem (Strassen, 1969)

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Lower bounds:

Theorem (Winograd, 1971)

$$7 \leq \textbf{MultRank}(2)$$

Theorem (Bläser, 2003)

$19 \leq \textbf{MultRank}(3) \leq 23$

$10 \leq \textbf{MultRank}(2,2,3) \leq 11$

$14 \leq \textbf{MultRank}(2,3,3) \leq 15$

$33 \leq \textbf{MultRank}(4) \leq 49$

There is potential for significant improvement in existing algorithms when $N \geq 3$.

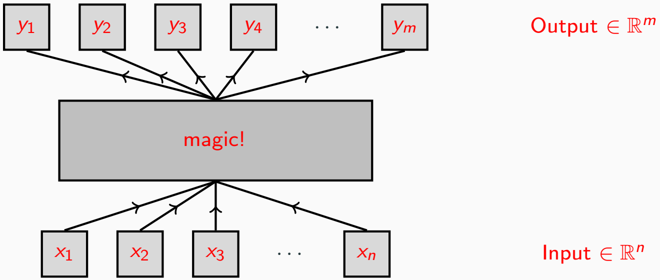
Question: *This is going to be a huge mess. How could one possibly improve any of these results without reams of computations?*

Obligatory "math on glass" image from *The Accountant*

Neural networks: a brief introduction

Neural Nets

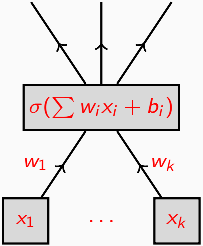
A neural net $\mathcal{N}$ is an object:



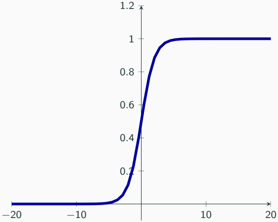
It is a fancy way to produce a function:

$$F_{\mathcal{N}} : \mathbb{R}^n \to \mathbb{R}^m.$$

Neural nets are made of “neurons”



Where σ is a function:



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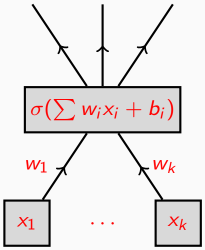

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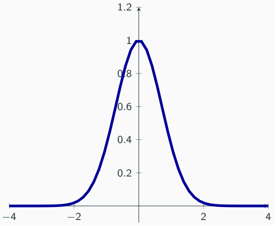

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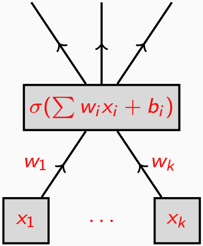

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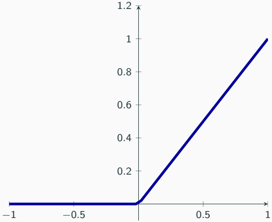

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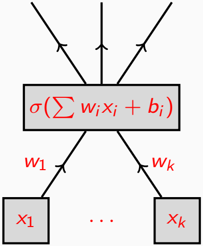

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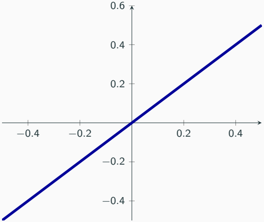
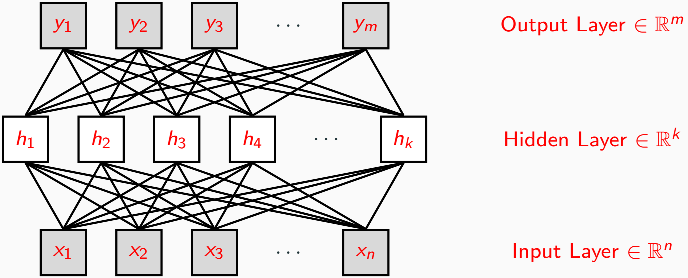
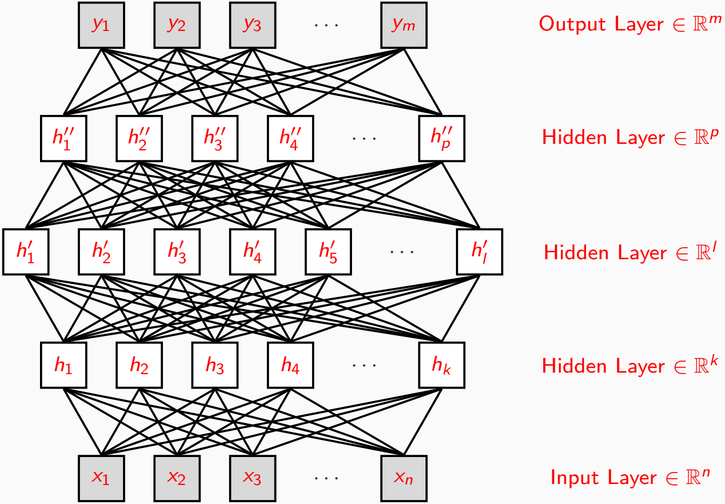


Neurons in layers make a neural network



Each edge may have a different w called its "weight". Each neuron may have a different b called its "bias."

Neurons in many layers make a "deep" neural net



The problem in deep learning:

Given, a perhaps not fully understood function F, find a neural network $\mathcal{N}$ that recovers F. That is:

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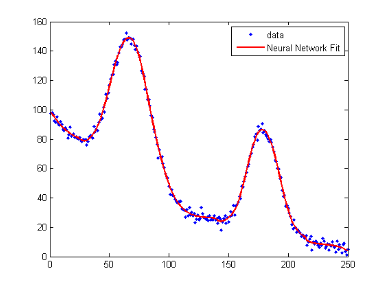

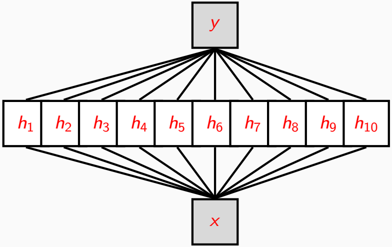

Image by R. Fithen

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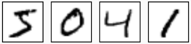

Here each handwritten digit is given by a 28×28 array of greyscale pixels. We'd like to understand

$$F : \mathbb{R}^{784} \to \mathbb{R}$$

or better still:

$$F : \mathbb{R}^{784} \to \mathbb{R}^{10}$$

This neural net is 85% accurate:

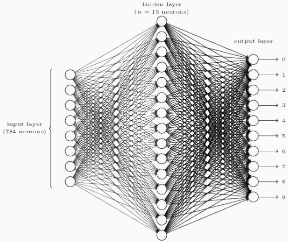

Image by A. Nielsen

ImageNet Challenge: *Given* 256×256 *RGB images classified into 1000 classes. Find a neural network* $\mathcal{N}$ *that describes the classification function:*

$$F : \mathbb{R}^{3\cdot 256^2} \to \mathbb{R}^{1000}.$$

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Google's `Inception` neural net $\mathcal{N}$ achieves 95% top-5 accuracy. The big picture of the neural net:

$\mathcal{N} =$

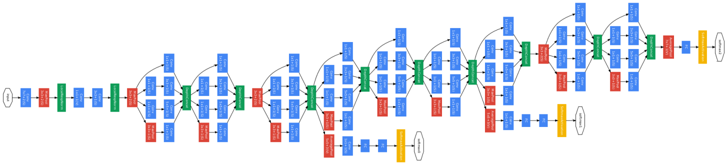


Image by Google

Fun Problem: *Predict species of bird based on photographic image.*

cardinal

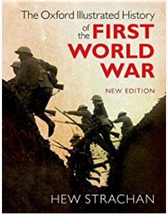
wood duck

anhinga

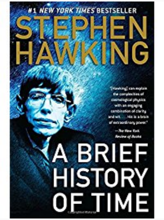
chickadee

Accuracy 87%. (P., 2017)

Fun Problem: *Predict book genre based on its cover.*

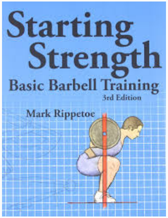


history



science

romance



sports

Accuracy 76%. (with Parikshit Sharma, '17, IndieBio)

The Whole Process

Goal: *Understand a, perhaps poorly defined, function F.*

1. Start with a set of data points:

$$(x_i, F(x_i))$$

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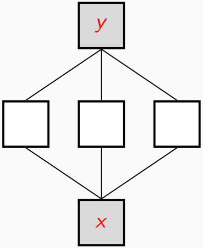

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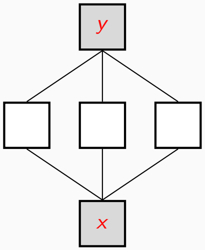

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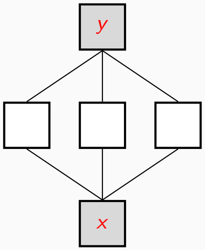
4. Tweak weights w and bias b decreasing

$$\text{Error} = \text{ave}|F(x_i) - F_{\mathcal{N}}(x_i)|$$



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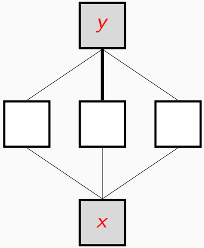

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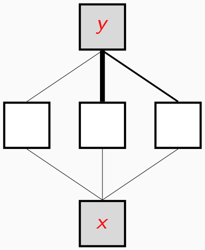

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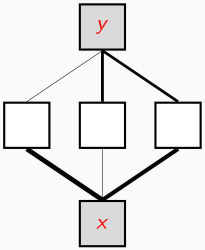

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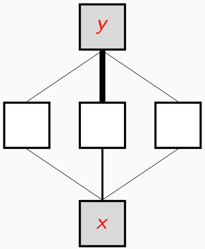

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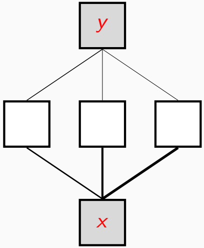

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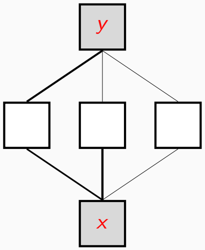

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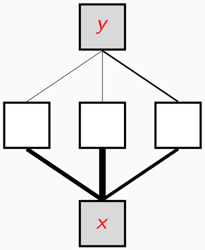

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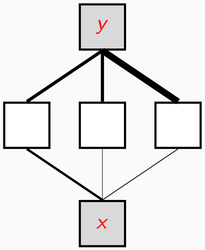

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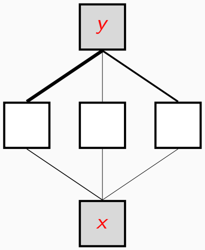

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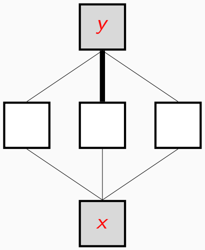
7. Buy fancy coffee maker for Math Dept.



A machine learning approach to fast matrix multiplication

Back to matrix multiplication

Goal: Design a neural network that mimics 2×2 matrix multiplication:

$$F : \mathbb{R}^{2 \cdot 4} \to \mathbb{R}^4$$

$$F(A, B) = A \cdot B$$

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Step 1: Start with a set of data points:

$$(x_i, F(x_i))$$

This is easy. Generate lots of random 2×2 matrices $x_i = (A_i, B_i)$ as well as their products $F(x_i) = A_i \cdot B_i$.

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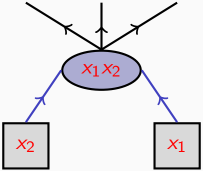
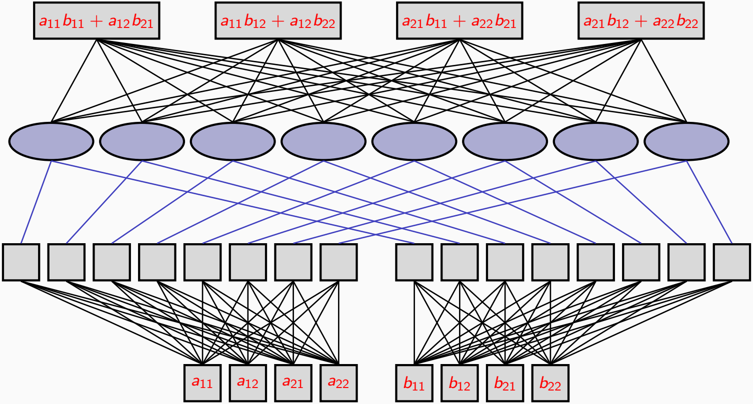
Step 2: Build a neural network $\mathcal{N}$

Need: A new type of neuron. One whose output is the product of its two inputs.



A new type of neural net!

Neural Net for matrix multiplication



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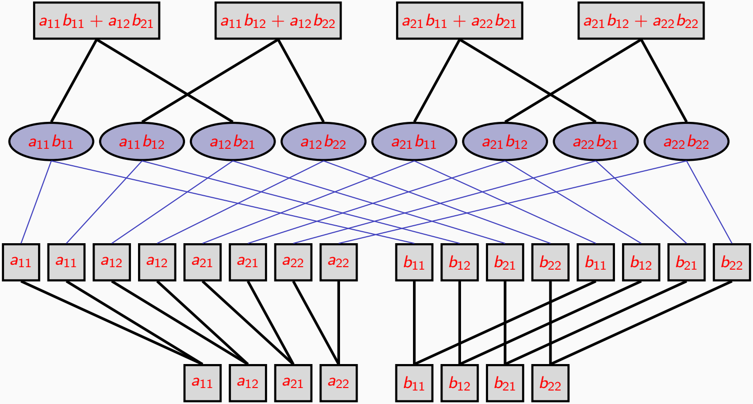

Step 3: *Compare with output of $\mathcal{N}$:*

$$(x_i, F_{\mathcal{N}}(x_i))$$

Step 4: *Tweak weights w and bias b for each edge so that*

$$\text{Error} = \text{ave}|F(x_i) - F_{\mathcal{N}}(x_i)|$$

decreases.

Step 5: *Continue tweaking w and bias b until error is as small as possible*

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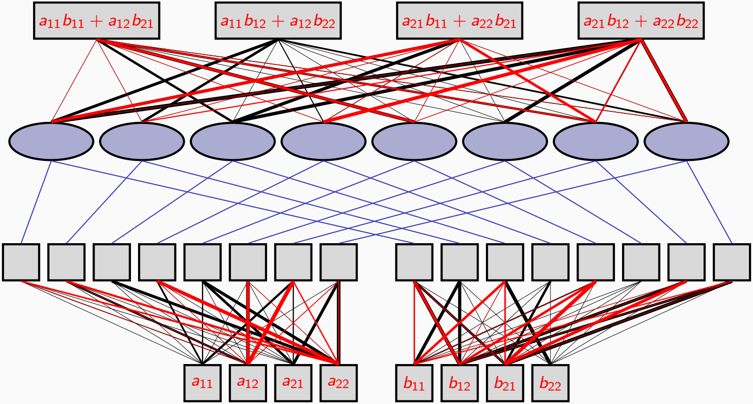
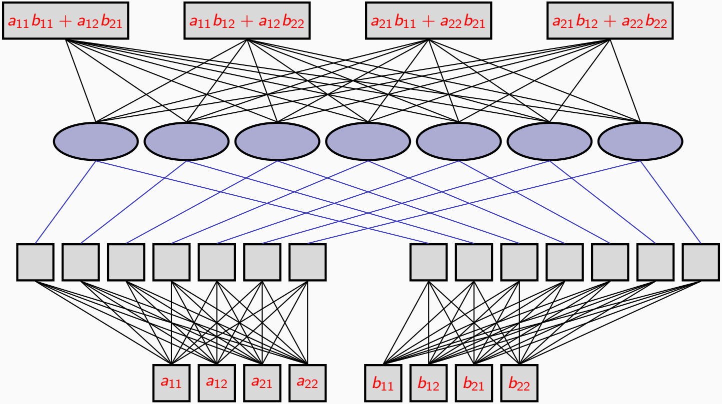
The Result:

Machine-trained neural net for matrix multiplication



Neural Net for Strassen's matrix multiplication



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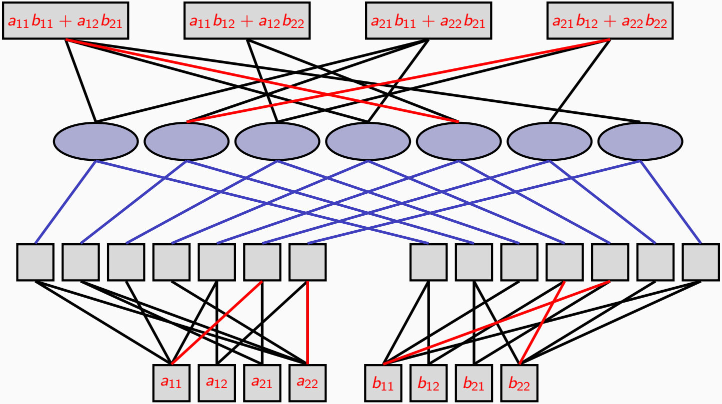
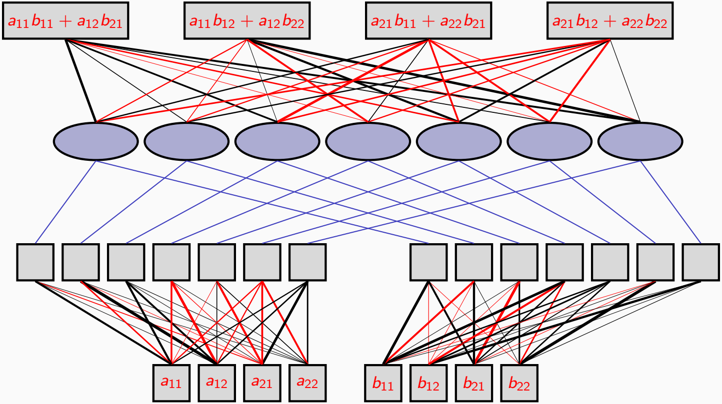


Machine-trained neural net for Strassen's matrix multiplication



In one day, our new fancied-up neural nets replicated:

Theorem (Strassen, 1969)

$\textbf{MultRank}(2) \leq 7$

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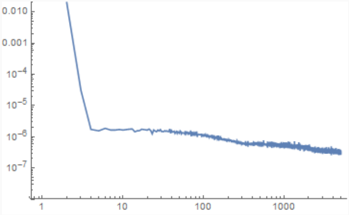

Figure 3: $N = 2$ **Rank** $= 7$

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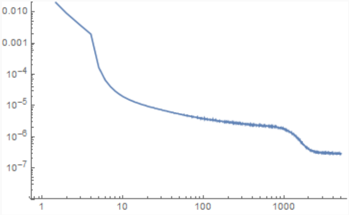

Figure 3: $N = 3$ **Rank** $= 23$

Failures

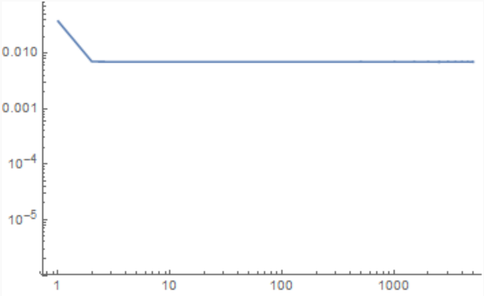

Figure 4: $N = 2$ **Rank** $= 6$

Failures

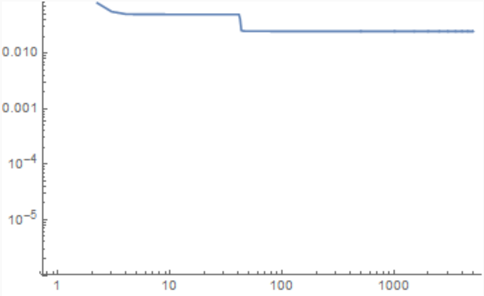

Figure 4: $N = 3$ **Rank** $= 22$

Failures

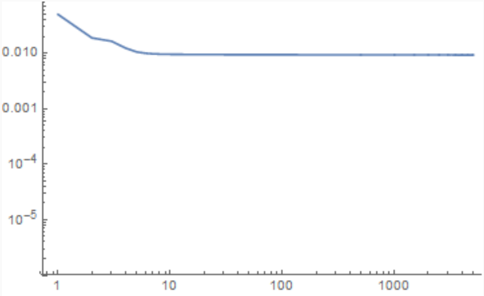

Figure 4: $N = 3$ **Rank** $= 21$

Failures

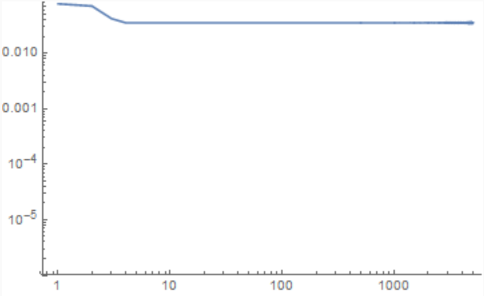

Figure 4: $N = 2, 2, 3$ **Rank** $= 10$

I was about to call an end to all of this, but then a recent preprint mentioned a forgotten result:

Theorem (Stothers, 2011)

$$\mathbf{MultRank}(4) \leq 48.$$

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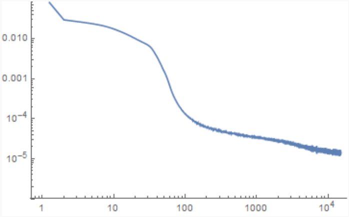

Figure 5: $N = 4$ **Rank** $= 48$

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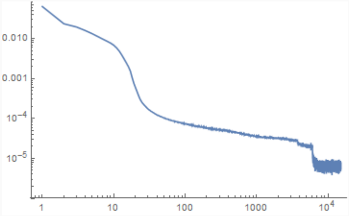

Figure 5: $N = 4$ **Rank** $= 47$

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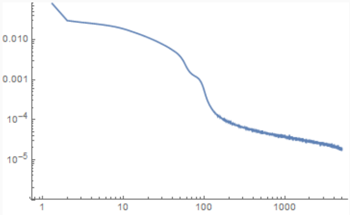

Figure 5: $N = 4$ **Rank** $= 46$

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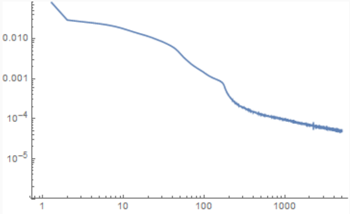

Figure 5: $N = 4$ **Rank** $= 45$

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OK, too much:

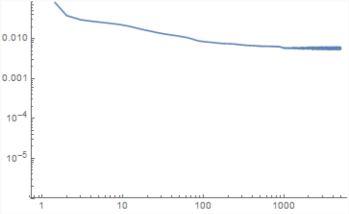

Figure 5: $N = 4$ **Rank** $= 33$

It looks like:

Conjecture: *One can multiply* 4×4 *matrices with fewer than 48 scalar multiplications. In fact, is seems that*

$$\mathbf{MultRank}(4) \leq 45.$$

Asymptotically, this would give $\mathbf{MultExp}(4) \leq \log_4 45 = 2.7459\ldots$.

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Note this is only a conjecture. My neural networks were only **approximations**. What remains:

1. Find an **exact** version of this algorithm.
2. Find an equivalent **sparse** neural net. Preferably one whose non-zero weights equal ± 1.

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$$\lim_{N \to \infty} \mathbf{MultExp}(N) = 2$$

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Conjecture:

$$\lim_{N \to \infty} \textbf{MultExp}(N) = 2$$

Currently, it is known $\lim_{N \to \infty} \textbf{MultExp}(N) < 2.3728\ldots$ (Josh Alman and Virginia Williams, 2021).

Update

DeepMind

Discovering novel algorithms with AlphaTensor

This sheds light on a 50-year-old open question in mathematics about finding the fastest way to multiply two matrices.

3 weeks ago

New Scientist

DeepMind AI finds new way to multiply numbers and speed up ...

Matrix multiplication – where two grids of numbers are multiplied together ... But DeepMind's AI has now discovered a faster technique that...

3 weeks ago

Ars Technica

DeepMind breaks 50-year math record using AI; new record falls a week later

Last week, DeepMind announced it discovered a more efficient way to perform matrix multiplication, conquering a 50-year-old record.

2 weeks ago

Update

Nature announced in October, 2022 that:

Theorem (FBHHRBNRSSSHK)

MultRank$(4) \leq 47^*$ *and* **MultRank**$(5) \leq 96^*$

Update

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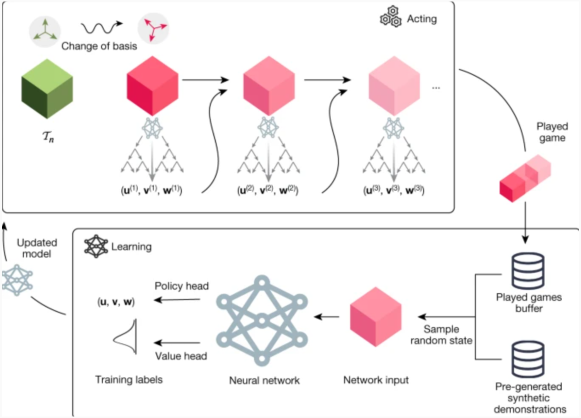
It made me feel better that to discover this results, this team used 64 state-of-the-art TPU cores, trained for 600,000 iterations: a non-academic battery of computational resources that cost somewhere between $10,000 and $100,000 to run.

* for 0,1-matrices.

Update

Figure 6: RL for AlphaTensor



A few days later...

THE FBHHRBNRSSSHK-ALGORITHM FOR MULTIPLICATION IN $\mathbb{Z}_2^{5\times 5}$ IS STILL NOT THE END OF THE STORY

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Abstract. In response to a recent *Nature* article which announced an algorithm for multiplying 5×5-matrices over $\mathbb{Z}_2$ with only 96 multiplications, two fewer than the previous record, we present an algorithm that does the job with only 95 multiplications.

1. Introduction

Ever since Strassen [8] discovered that 2×2-matrices can be multiplied with only 7 multiplications in the coefficient domain, there is a mystery around the complexity of matrix multiplication. For asymptotically large n, the best we know at the moment is a multiplication algorithm that requires $O(n^{2.3728596})$ operations [1], slightly improving upon the previous record $O(n^{2.3728639})$ [5]. For $n = 3$, it is known that 23 multiplications suffice in a non-commutative setting [4]. For $n = 4$, we can solve the problem with 49 multiplications by applying Strassen's algorithm recursively. In a recent article that received considerable media attention, Fawzi et al. [2] used a machine learning approach to find a multiplication scheme with 47 multiplications, applicable to coefficient domains of characteristic 2. Under the same