Orbital Varieties and Unipotent Representations of Classical Semisimple Lie Groups

by

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Abstract

Let G be a complex semi-simple and classical Lie group. The notion of a Lagrangian covering can be used to extend the method of polarizing a nilpotent coadjoint orbit to obtain a unitary representation of G. W. Graham and D. Vogan propose such a construction, relying on the notions of orbital varieties and admissible orbit data.

The first part of the thesis seeks to understand the set of orbital varieties contained in a given nipotent orbit. Starting from N. Spaltenstein's parameterization of the irreducible components of the variety of flags fixed by a unipotent, we produce a parameterization of the orbital varieties lying in the corresponding fiber of the Steinberg map. The parameter set is the family of standard Young or domino tableau of a given shape. The key to the proof is understanding certain closed cycles as defined by D. Garfinkle. This parameterization is particularly useful; it provides a method of determining the τ-invariant of each orbital variety, as well as a way of relating an orbital variety in any classical group to one lying in type A.

The second part of the thesis addresses the representations $V(\mathcal{V}, \pi)$ constructed by Graham and Vogan. A natural question is how well the $V(\mathcal{V}, \pi)$ approximate the set of unipotent representations that ought to be attached to the nilpotent orbit $\mathcal{O}$. The answer is promising in the setting of spherical orbits. When it is possible to carry out the Graham-Vogan construction, the corresponding infinitesimal character lies in the set of characters suggested by W. M. McGovern. Furthermore, we show that it is possible to carry out the Graham-Vogan construction for a suffient number of orbital varieties to account for all the infinitesimal characters attached to $\mathcal{O}$ by McGovern.

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I would like to thank my family, for their love, understanding, and for teaching me what is important in life. Finally, I would like to thank my wife, Jennifer, for showing me how it's done. She has been a pillar of support, a terrific companion, and a partner in crime.

"The lyf so short, the craft so long to lerne."
- Masthead logo of *The Craftsman*, *ca.* 1910.

Contents

Chapter 1

Introduction

The orbit method seeks to classify irreducible unitary representations of a Lie group G by identifying them with the set of coadjoint orbits $\mathfrak{g}^*/G$. A classical theorem of Kostant and Kirillov for nilpotent Lie groups provides the motivation for this approach:

Theorem (Kirillov). *If G is a connected and simply connected nilpotent Lie group, then there is a bijective correspondence*

$$\mathfrak{g}^*/G \longrightarrow \widehat{G}_{unitary}$$

between the coadjoint orbits of G and the set of its irreducible unitary representations.

For semisimple Lie groups, however, this approach is far less understood. It is known which unitary representations correspond to *semisimple* coadjoint orbits. They are the representations obtained by parabolic or cohomological induction from a set of unitary characters of Levi subgroups of G. General orbits should correspond to representations that are parabolically and cohomogically induced from representations attached to nilpotent coadjoint orbits, which we denote $\widehat{G}_{fund}$. We call this set of representations *unipotent*. It is not known what this set of unipotent representations ought to be and in what manner such a correspondence should be carried out.

The paper [Graham-Vogan] constructs a collection $GV_{\mathcal{O}}$ of representation spaces for each nilpotent coadjoint orbit $\mathcal{O}$ of a complex reductive Lie group G. We denote each such space as $V(\mathcal{V}, \pi)$ and defer a more precise definition. Very little is known about which representations of G actually arise in this way, but conjecturally, they should be close to the set of unipotent representations corresponding to the nilpotent orbit $\mathcal{O}$, or $\widehat{G}^{\mathcal{O}}_{fund}$. We first address the data used to construct each $V(\mathcal{V}, \pi)$.

1.1 Orbital Varieties and Domino Tableaux

The method of polarization provides a motivated approach to the problem of quantizing nilpotent coadjoint orbits. Unfortunately, it requires a construction of certain Lagrangian foliations which often do not exist. The construction of Graham and Vogan is designed to mimic polarization, but replaces Lagrangian foliations by somewhat weaker structures called *Lagrangian coverings*. Work of V. Ginsburg imples that Lagrangian coverings always exist in the setting of coadjoint orbits of complex reductive Lie groups [Ginsburg].

The main ingredients in the construction of Lagrangian coverings of $\mathcal{O}$ are the *orbital varieties* $\mathcal{V} \subset \mathcal{O}$. For each choice of orbital variety $\mathcal{V}$ and a choice of an *admissible orbit datum* π, the Graham-Vogan construction provides a subspace $V(\mathcal{V}, \pi)$ of sections of a bundle $\mathcal{L}_{G/Q_{\mathcal{V}}}$ over $G/Q_{\mathcal{V}}$, where $Q_{\mathcal{V}} \subset G$ is the maximal subgroup of G stabilizing $\mathcal{V}$. The group $Q_{\mathcal{V}}$ contains a Borel subgroup of G and is hence parabolic.

To understand which representations arise as $V(\mathcal{V}, \pi)$, one would like to first parameterize orbital varieties in a way that also describes the parabolic $Q_{\mathcal{V}}$. We will use the term *standard tableau* to refer to a standard Young tableau when type $G =$ A, and to a standard domino tableau in the other classical types. A more complete description can be found in Section 3.1.3.

Theorem 3.3.2. *Among classical groups, the orbital varieties $\mathcal{V} \subset \mathcal{O}$ are parameterized by the set of standard tableaux of shape equal to the partition associated to $\mathcal{O}$. The only exception occurs in type D when $\mathcal{O}$ is very even. In this instance, the number of vertical dominos in the tableau is congruent to 0 or 2 mod 4 depending on the Roman numeral attached to the orbit $\mathcal{O}$.*

This mirrors the result obtained in [McGovern2] by examining certain equivalence classes on the Weyl group of G. The proof in this thesis relies on the work of N. Spaltenstein on the irreducible components of the unipotent variety $\mathcal{F}_u$ and the results of D. Garfinkle on domino tableaux. We can now write $\mathcal{V}_T$ for the orbital variety that corresponds to the standard tableau T. The description of the maximal stabilizing parabolic subgroup of $\mathcal{V}_T$ is now simple to state.

Theorem 3.5.1. *Consider an orbital variety $\mathcal{V} = \mathcal{V}_T$. The Lie algebra $\mathfrak{q}$ of $Q_{\mathcal{V}}$ contains the root space $\mathfrak{g}_{-\alpha_i}$ associated with the simple root $-\alpha_i$ iff*

(i) the entry $i+1$ appears strictly below i in T when type G is A, or

(ii) when type G is not A, either

- *$i = 1$ and the domino with label 1 is vertical, or*
- *$i \neq 1$ and the domino with label i lies strictly below $i-1$ in T.*

The standard tableaux parameterization has two more benefits. Using the work of [Carre-Leclerc], it is possible to define a projection map from orbital varieties of all classical types to ones of type A, both on the level of orbital varieties and their corresponding standard tableaux. This approach suggests a way of finding a *minimal representative* in each $\mathcal{V}_T$ in the sense of [Melnikov]. We carry this out in the setting of spherical orbits. See Definition 4.3.5.

The standard domino parameterization also admits a means of addressing some related calculations inductively. It sets up a framework for calculating data in the Graham-Vogan construction.

1.2 Infinitesimal Characters

Let $\mathcal{O}$ be a nilpotent coadjoint orbit for a semisimple Lie group G. [McGovern] determines a set $IC^1(\mathcal{O})$ of infinitesimal characters that ought to correspond to elements of $\widehat{G}^{\mathcal{O}}_{fund}$. This provides a convenient way of testing whether the Graham-Vogan construction provides good candidates for the representations in $\widehat{G}^{\mathcal{O}}_{fund}$; the set of infinitesimal characters of the representations attached to $\mathcal{O}$ should contain $IC^1(\mathcal{O})$. For spherical $\mathcal{O}$, this is exactly what happens:

Theorems 5.3.4 and 5.3.5. *Let $\mathcal{O}$ be a spherical nilpotent orbit of a complex classical semisimple Lie group G of rank n and suppose that it is possible to construct the space $V(\mathcal{V}, \pi)$. Let $\chi_{\mathcal{V}}$ be the infinitesimal character associated to $V(\mathcal{V}, \pi)$. Then,*

(i) If $\mathcal{O}$ is rigid, then $IC^1(\mathcal{O}) = \{\chi_{\mathcal{V}} \mid V(\mathcal{V}, \pi) \in GV_{\mathcal{O}}\}$,
(ii) If $\mathcal{O}$ is a model orbit and $n > 2$, then $IC^1(\mathcal{O}) \subset \{\chi_{\mathcal{V},\pi} \mid V(\mathcal{V}, \pi) \in GV_{\mathcal{O}}\}$.

The theorem implies that, at least for spherical nilpotent orbits, the Graham-Vogan spaces are strong candiates for unipotent representations. As for non-spherical orbits, it is apparent from Theorem 5.3.5 that the set $GV_{\mathcal{O}}$ is too large to be $\widehat{G}^{\mathcal{O}}_{fund}$. However, additional conditions on the closure $\overline{\mathcal{O}}$ not considered in [Graham-Vogan] should make it possible to restrict the set of possible infinitesimal characters.

Here is an outline of this thesis. Chapter 2 provides a short description of polarization and its refinement by Graham and Vogan. Chapter 3 addresses orbital varieties. After listing the possible approaches to the classification problem, we describe the work of N. Spaltenstein and M. A. van Leeuwen in this direction. A sequence of somewhat technical lemmas proves the parameterization of orbital varieties described above. We also address the τ-invariant of an orbital variety and describe a method of projecting all orbital varieties onto ones of type A. Chapter 4 begins by examining the Graham-Vogan construstion in a "model example." We restrict our attention to spherical orbits, construct a basepoint within each orbital variety, and exhibit an inductive construction that we will use to describe $V(\mathcal{V}, \pi)$. Finally, Chapter 5 addresses infinitesimal characters. We describe the work of McGovern describing a set $IC^1(\mathcal{O})$ of characters, provide a few examples of interesting behavior of $V(\mathcal{V}, \pi)$, and prove the theorems described above. We finish with some thoughts on further work.

Chapter 2

The Graham-Vogan Construction

The Graham-Vogan construction of representations associated to a coadjoint orbit $\mathcal{O}$ is an extension of the method of polarizing a coadjoint orbit. Polarization is a very effective tool for attaching representations to coadjoint orbits in the setting of connected nilpotent Lie groups, but unfortunately is not as useful among semisimple Lie groups. It relies on a construction of a Lagrangian foliation that may not always exist.

To amend this shortfall, [Graham-Vogan] replaces Lagrangian foliations used in polarization with the so-called Lagrangian coverings. By a theorem of V. Ginzburg, it is always possible to construct a Lagrangian covering of a coadjoint orbit $\mathcal{O}$. In fact, there is a unique one for each orbital variety contained in $\mathcal{O}$. The difficult task now becomes to mimic the construction of representations used in polarization in this more complicated setting.

For nilpotent coadjoint orbits, Graham and Vogan suggest a construction of a representation from each pair of the following objects:

· an *admissible orbit datum,* and

· an *orbital variety.*

The representation lies in the space of smooth vectors in a degenerate principal series representation induced from a representation of a parabolic subgroup of G.

2.1 Polarization

Given a coadjoint orbit $\mathcal{O}$, polarization attempts to find a smooth G-manifold M and a Hermitian line bundle $\mathcal{L}_M$ on M such that $\mathcal{O}$ is isomorphic as a symplectic G-space to a certain twisted cotangent bundle $T^*(M, \mathcal{L})$ such as that defined by [Kostant]. The unitary representation attached to $\mathcal{O}$ is then the space $\pi(M, \mathcal{L}_M)$ described in [Graham-Vogan].

This process provides a nice way of quantizing a coadjoint orbit, and hence we'd like to know under what circumstances such a manifold M and line bundle $\mathcal{L}_M$ exist. We note a few properties.

· $T^*(M, \mathcal{L}_M)$ is a symplectic manifold,

· $T^*_m(M, \mathcal{L}_M)$ are Lagrangian submanifolds,

· $\{T^*_m(M, \mathcal{L}_M)\}$ is a Lagrangian foliation of $T^*(M, \mathcal{L}_M)$.

If $\mathcal{O}$ is to be isomorphic as a symplectic G-space to $T^*(M, \mathcal{L}_M)$, it should be possible to find similar structures on $\mathcal{O}$. To this effect, we need to find a G-invariant Lagrangian foliation of $\mathcal{O}$. Because $\mathcal{O}$ is homogeneous, this can be reduced to finding a subgroup of G with certain properties. More precisely, if we fix a basepoint f, then $\mathcal{O} \cong G/G_f$, where G_f is the isotropy group. As $\mathcal{O}$ is homogeneous, so must be the space of leaves and the entire foliation is determined by the leaf Λ_f through f. The question of finding a Lagrangian foliation is so reduced to finding a subgroup H containing G_f such that:

- H is a closed Lie subgroup,
- $f|_{[\mathfrak{h},\mathfrak{h}]} = 0$,
- $\dim H/G_f = \frac{1}{2} \dim G/G_f$,

Given such a subgroup H, the manifold M is then isomorphic to the space of leaves G/H, and each leaf is isomorphic to H/G_f. Furthermore, the line bundle $\mathcal{L}_M$ is induced by a character

- $\tau \in \widehat{H}$ with $d\tau = 2\pi i f$.

Given a subgroup H and a character τ of H having these properties, the twisted cotangent bundle $T^*(M, \mathcal{L}_M)$ has the property that some open set is G-equivariantly symplectomorphic to a covering space of X.

The representation space $\pi(M, \mathcal{L}_M)$ attached to $\mathcal{O}$ in this situation is roughly speaking a set of sections of the line bundle $\mathcal{L}_M$. We can pull back $\mathcal{L}_M$ to a line bundle $\mathcal{L}_{\mathcal{O}}$ on $\mathcal{O}$. In this way, sections of $\mathcal{L}_M$ are identified with sections of $\mathcal{L}_{\mathcal{O}}$ that are constant along the leaves of the Lagrangian foliation, and we can realize $\pi(M, \mathcal{L}_M)$ among them.

$$\begin{array}{ccccc} G/G_f & \xrightarrow{\cong} & \mathcal{O} & \longleftarrow & \mathcal{L}_{\mathcal{O}} \\ & & \downarrow{\scriptstyle\rho} & & \downarrow \\ G/H & \xrightarrow{\cong} & M & \longleftarrow & \mathcal{L}_{\mathcal{M}} \end{array}$$

As hinted at before, when G is a nilpotent group, the family of subgroups of G is very rich, and it is always possible to find a group H that makes a given coadjoint orbit a twisted cotangent bundle. That is, polarization is enough to geometrically quantize the coadjoint orbits of nilpotent groups.

As G becomes more complicated, however, the pool of subgroups diminishes, and polarization becomes more difficult to carry out. Ozeki and Wakimoto [Ozeki-Wakimoto] show that among reductive Lie groups, subgroups satisfying just the isotropy and dimension requirements must be parabolic [Ozeki-Wakimoto]. Furthermore, they prove that polarization cannot be always carried out:

Corollary 2.1.1. *Suppose that G is a split simple group over $\mathbb{R}$ or $\mathbb{C}$ and is not of type A. If $\mathcal{O}$ is a coadjoint orbit of minimal non-zero dimension, then $\mathcal{O}$ is not locally isomorphic to a twisted contangent bundle for G.*

2.2 Lagrangian Coverings

Polarization, as described above, fails to suggest a representation that one could attach to every coadjoint orbit $\mathcal{O}$. To amend this situation, Graham and Vogan use an idea from

[Guillemin-Sternberg] and [Ginsburg] and replace the Lagrangian foliation above by a family of Lagrangian submanifolds that are allowed to overlap, called a *Lagrangian covering* of $\mathcal{O}$. Following a definition, we will describe a construction of Lagrangian coverings in the case of nilpotent coadjoint orbits. We then describe how [Graham-Vogan] mimics polarization in this setting to construct a space of representations attached to a nilpotent $\mathcal{O}$.

Definition 2.2.1. A *Lagrangian covering* of a symplectic manifold $\mathcal{O}$ is a pair (Z, M) of manifolds and smooth maps (τ, ρ)

$$\begin{array}{ccc} & & Z \\ & \stackrel{\tau}{\swarrow} & \downarrow \rho \\ \mathcal{O} & & M \end{array}$$

such that

- the diagram is a double fibration,
- each fiber of ρ is a Lagrangian submanifold of $\mathcal{O}$.

Theorem. *[Ginsburg] Let G be a complex reductive Lie group and $\mathcal{O}$ be a coadjoint orbit. Then there exists an equivariant Lagrangian covering of $\mathcal{O}$ with M a partial flag variety for G.*

We relate the construction in the case of a nilpotent coadjoint orbit. Fix a Borel subgroup B of G with unipotent radical N. Write

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{t} \oplus \mathfrak{n}$$

for the corresponding triangular decomposition. Let us restrict our attention to nilpotent coadjoint orbits $\mathcal{O}$, and consider the set $\mathcal{O} \cap \mathfrak{n}$. This is a locally closed subset of $\mathfrak{n}$ and can be expressed as a union of its irreducible components.

Definition 2.2.2. Consider a nilpotent coadjoint orbit $\mathcal{O}$. Denote the set of irreducible components of the variety $\mathcal{O} \cap \mathfrak{n}$ by $\mathrm{Irr}(\mathcal{O} \cap \mathfrak{n})$. Each element of $\mathrm{Irr}(\mathcal{O} \cap \mathfrak{n})$ is an *orbital variety* for $\mathcal{O}$.

Proposition 2.2.3. *Let $\mathcal{V}$ be an orbital variety for $\mathcal{O}$. Then*

- $|Irr(\mathcal{O} \cap \mathfrak{n})|$ *is finite,*
- $\dim \mathcal{V} = \frac{1}{2} \dim \mathcal{O}$,
- *Each $\mathcal{V}$ is a Lagrangian subvariety of $\mathcal{O}$.*

We are ready to construct a Lagrangian covering for the orbit $\mathcal{O}$. In fact, we will construct a distinct covering for each orbital variety contained in $\mathcal{O}$. Fix an orbital variety $\mathcal{V}$ and let $\mathcal{V}^0$ be its smooth part. Let Q be the subgroup of G that stabilizes $\mathcal{V}$, i.e.

$$Q = Q_{\mathcal{V}} = \{q \in G \mid q \cdot \mathcal{V} = \mathcal{V}\}.$$

This is a parabolic subgroup of G since $\mathcal{V}$ is B-stable. Furthermore, we can define the manifold M by

$$M = \{g \cdot \mathcal{V} \mid g \in G\} \cong G/Q.$$

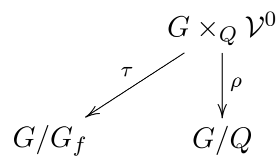
It is a partial flag variety for G.

Definition 2.2.4. For a subgroup $H \subset G$ and an H-space V, we define $G \times_H V$ to be the set of equivalence classes in $G \times V$ with $(gh, v) \sim (g, h \cdot v)$ for $g \in G, h \in H$, and $v \in V$.

The manifold Z in our Lagrangian covering of $\mathcal{O}$ associated to the orbital variety $\mathcal{V}$ is now defined as

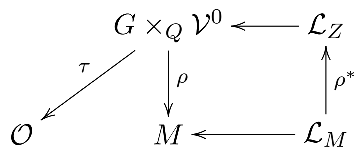
$$Z = G \times_Q \mathcal{V}^0.$$

We define $\rho : Z \longrightarrow M$ from the projection of G onto G/Q. The action of G on $\mathcal{O}$ gives natural map $G \times \mathcal{V} \to \mathcal{O}$. It descends to an algebraic map $\tau : Z \longrightarrow \mathcal{O}$. We now have a Lagrangian covering:



Because the diagram is a double fibration, we can identify fibers of ρ with subsets of $\mathcal{O}$. In fact, each fiber is Lagrangian in $\mathcal{O}$.

We would like to have a construction of representations reminiscent of polarization. To that effect, suppose that we have a G-equivariant line bundle $\mathcal{L}_M \to M$. We can again pull this bundle back along the fibration ρ, this time to obtain a bundle $\mathcal{L}_Z$.



Geometric quantization suggests that the representations attached to $\mathcal{O}$ should lie in the space of sections of $\mathcal{L}_M$, or in other words, in the space of sections of $\mathcal{L}_Z$ that are constant on the fibers of ρ. This is very similar to the situation arising in the polarization construction, as the fibers of ρ can again be identified with Lagrangian submanifolds of $\mathcal{O}$. This time, however, the full set of sections of $\mathcal{L}_M$ is too large to quantize $\mathcal{O}$. See [Graham-Vogan]. To this effect, it is necessary to pick out a subspace.

2.3 A Subspace of Sections of $\mathcal{L}_M$

Choosing an adequate subspace of the sections of $\mathcal{L}_M$ occupies most of [Graham-Vogan]. We relate only a general overview, and direct the reader to [Graham-Vogan] itself for the relevant details. The main idea is to prune the full space of sections of $\mathcal{L}_M$, leaving ones which also come from an *admissible orbit datum* of $\mathcal{O}$.

To do this, one must first attach a geometric structure to each orbit datum. This is achieved by mimicking the construction of a Hermitian bundle that often arises in descriptions of geometric quantization of *integral* orbit data. The main difficulty then lies in finding a way of embedding the information from this bundle into the space of sections of $\mathcal{L}_M$.

Definition 2.3.1. An *admissible orbit datum* at $f \in \mathfrak{g}^*$ is a genuine irreducible unitary representation π of the metaplectic cover $\widetilde{G}_f$ satisfying

$$\pi(\exp Y) = \chi(f(Y))$$

for a fixed non-trivial character χ of $\mathbb{R}$.

The process of pruning the set of sections of $\mathcal{L}_M$ follows the following outline:

(i) Attach some geometric structure to each admissible orbit datum. This will be the set of sections of an infinite-dimensional bundle $\mathcal{S}_\pi^{even,\infty}$ over $\mathcal{O}$.
(ii) Represent the sections of $\mathcal{S}_\pi^{even,\infty}$ within the space of sections of a finite-dimensional bundle $\mathcal{V}_\pi$.
(iii) Embed the space of sections of $\mathcal{V}_\pi$ among sections of $\mathcal{L}_M$. More precisely, embed them among the space of sections of $\mathcal{L}_Z$ that are constant on fibers of ρ.
(iv) The space of the representation we want is the family of sections of $\mathcal{L}_Z$, constant on fibers of ρ, that also come from the sections of $\mathcal{S}_\pi^{even,\infty}$ via this embedding.

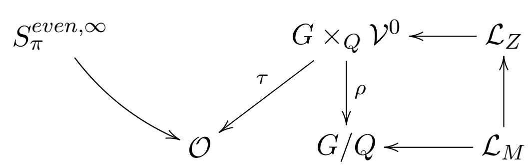
We begin by attaching geometric structure to an admissible orbit datum. Let us denote the metaplectic representation of $\widetilde{G}_f$ by τ_f and form the tensor product representation $\pi \otimes \tau_f$. While τ_f and π are genuine representations of $\widetilde{G}_f$, $\pi \otimes \tau_f$ in fact descends to a representation of G_f itself. This allows us to define a Hilbert bundle over the coadjoint orbit $\mathcal{O}$ by

$$\mathcal{S}_\pi = G \times_{G_f} (\pi \otimes \tau_f).$$

Following [Graham-Vogan], we call it the *bundle of twisted symplectic spinors* on $\mathcal{O}$. The metaplectic representation τ_f of $\widetilde{G}_f$ decomposes into two irreducible and inequivalent representations τ_f^{odd} and τ_f^{even}. Also write $\tau_f^{odd,\infty}$ and $\tau_f^{even,\infty}$ for the corresponding sets of smooth vectors. This decomposition passes to the bundle $\mathcal{S}_\pi$ and the geometric structure attached to the admissible orbit datum π is the subbundle of $\mathcal{S}_\pi$ defined by

$$\mathcal{S}_\pi^{even,\infty} = G \times_{G_f} (\pi \otimes \tau_f^{even,\infty}).$$

A similar equivariant Hermitian vector bundle appears in many descriptions of geometric quantization of *integral* orbit data. [Graham-Vogan] argues that the notion of admissible orbit data is more natural, and the corresponding bundle in this case is $\mathcal{S}_\pi^{even,\infty}$. The idea of [Graham-Vogan] is that the space of the representation attached to $\mathcal{O}$ should consist of the sections of the bundle $\mathcal{L}_M$ that somehow also come from sections of $\mathcal{S}_\pi^{even,\infty}$. Before this can be made more precise, a few problems must be overcome. First of all, the bundles $\mathcal{S}_\pi^{even,\infty}$ are infinite dimensional, while $\mathcal{L}_M$ may not be. Second, there needs to be a way of transferring sections of $\mathcal{S}_\pi^{even,\infty}$ to sections of $\mathcal{L}_M$.



By enlarging the base space, it is possible to construct a finite-dimensional bundle whose sections contain the sections of $\mathcal{S}_\pi^{even,\infty}$. The appropriate base space is the bundle of *infinitesimal Lagrangians* on the coadjoint orbit $\mathcal{O}$.

Definition 2.3.2. Suppose that X is a symplectic manifold. The *bundle of infinitesimal Lagrangians* on X is a fiber bundle $\mathcal{B}(X)$ over X. The fiber over each point $x \in X$ is the set of Lagrangian subspaces of the tangent space at x of X, denoted by $\mathcal{B}(T_xX)$.

Definition 2.3.3. Let $\mathcal{O}$ be a coadjoint orbit, and consider $\mathcal{V}$ a Lagrangian in the tangent space $\mathfrak{g}/\mathfrak{g}_f$. Write $\mathcal{L}(\mathcal{V})$ for the line defined in [Graham-Vogan] (7.4(c)) from the metaplectic representation τ_f. The admissible orbit datum π defines a G-equivariant vector bundle $\mathcal{V}_\pi$ on $\mathcal{B}(\mathcal{O})$ by letting the fiber at each $\mathcal{V}$ be $\mathcal{H}_\pi \otimes \mathcal{L}(\mathcal{V})$.

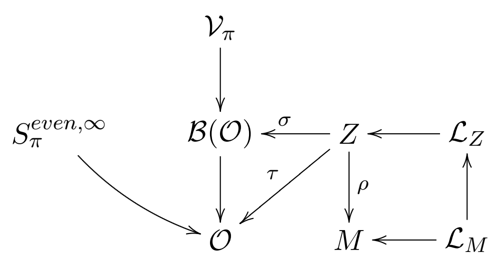
Theorem 2.3.4. *[Graham-Vogan] [Kostant] There exists a natural inclusion*

$$i : C^\infty(\mathcal{O}, \mathcal{S}_\pi^{even,\infty}) \hookrightarrow C^\infty(\mathcal{B}(\mathcal{O}), \mathcal{V}_\pi).$$

We would like to incorporate the bundle $\mathcal{V}_\pi$ over $\mathcal{B}(\mathcal{O})$ into our Lagrangian covering diagram. Define a map $\sigma : Z \to \mathcal{B}(\mathcal{O})$ as follows. Fix $z \in Z$. The definition of Lagrangian covering forces the fiber of ρ over $\rho(z) \in M$ to be a Lagrangian submanifold of $\mathcal{O}$ that contains $\tau(z)$. Hence its tangent space $T_{\tau(z)}(\rho^{-1}(\rho(z))$ is a Lagrangian subspace of $T_{\tau(z)}(\mathcal{O})$ and thus an element of $\mathcal{B}(\mathcal{O})$. Let

$$\sigma(z) = T_{\tau(z)}(\rho^{-1}(\rho(z)).$$

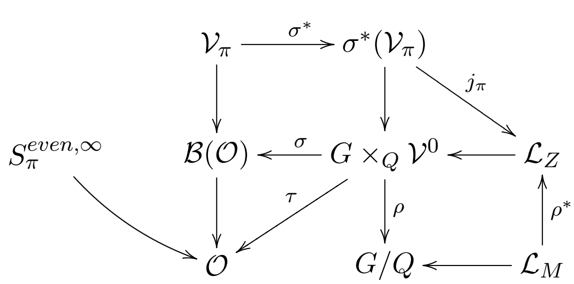
In this way, σ becomes a bundle map over $\mathcal{O}$.



To complete our task, note that we can pull back the bundle $\mathcal{V}_\pi$ along σ to a bundle $\sigma^*(\mathcal{V}_\pi)$ over Z. Smooth sections of $\mathcal{V}_\pi$ pull back to smooth sections of $\sigma^*(\mathcal{V}_\pi)$ and we have an injective map $\sigma^* \cdot i : C^\infty(\mathcal{O}, \mathcal{S}_\pi^{even,\infty}) \hookrightarrow C^\infty(Z, \sigma^*(\mathcal{V}_\pi))$. Provided that there is a G-equivariant vector bundle isomorphism $j_\pi : \sigma^*(\mathcal{V}_\pi) \to \rho^*(\mathcal{L}_M)$ we can define a smooth representation of G as:

$$V(\mathcal{V}, \pi) = \rho^*(C^\infty(M, \mathcal{L}_M)) \cap j_\pi(\sigma \cdot i(C^\infty(\mathcal{O}, \mathcal{S}_\pi^{even,\infty})))$$

If $\mathcal{L}_M$ is given by a representation γ of the parabolic subgroup Q, then $V(\mathcal{V}, \pi)$ lies in the space of smooth vectors of the degenerate principal series representation induced from γ. The entire construction may be summarized by the following diagram.



Chapter 3

Geometric Left Cells and the Unipotent Variety

The goal of this section is to obtain a useful parameterization of the orbital varieties contained in a given nilpotent coadjoint orbit. We begin by describing the motivation for our approach.

Let $\mathcal{N}$ denote the set of all complex nilpotent matrixes of size n. Jordan canonical form dictates that $\mathcal{N}$ decomposes into finitely many conjugacy classes, called nilpotent orbits, under the action of $SL(n,\mathbb{C})$. Each is indexed by a partition of n. The set of partitions of n, according to Frobenius and Young, also classifies the irreducible complex linear representations of S_n, the Weyl group of $SL(n,\mathbb{C})$.

The theory of Springer explains this phenomenon [Springer]. Consider a unipotent element u of $G = SL(n,\mathbb{C})$ and let $\mathcal{F}$ denote the full flag variety. We write $\mathcal{F}_u$ for the variety of flags in $\mathcal{F}$ that are fixed under the action of u and call it the *unipotent variety*. Let $\mathcal{O}_u$ be the nilpotent orbit through the corresponding nilpotent element. For each $\mathcal{O}_u$, Springer constructs a linear action of S_n on the cohomology $H^*(\mathcal{F}_u)$. It commutes with the action of the isotropy group G_u and yields an irreducible representation π_u on $H^{\text{top}}(\mathcal{F}_u)^{G_u}$. The correspondence assigning π_u to each $\mathcal{O}_u$ is a bijection and gives a geometric explanation of the classical result.

Denote the Springer resolution by $\pi : T^*\mathcal{F} \longrightarrow \mathcal{N}$. A representation of S_n equivalent to π_u can also be constructed on the space $V_{\pi'_u} = H_*(\overline{\pi^{-1}\mathcal{O}_u})$ [Borho]. Both π_u and π'_u are equipped with canonical bases which respectively correspond to the irreducible components of $\mathcal{F}_u$ and the set of orbital varieties of $\mathcal{O}_u$. Both of these sets share an identical combinatorial description. Let λ_u denote the partition given by the Jordan canonical form of u.

Theorem. *[Spaltenstein] When $G = SL(n,\mathbb{C})$, the set of irreducible components of $\mathcal{F}_u$ corresponds to the set of standard Young tableaux with shape λ_u.*

Each orbital variety of a nilpotent orbit corresponds to a subset of the Weyl group called a *geometric left cell* [Joseph] . When $G = SL(n,\mathbb{C})$, the notion of a geometric left cell coincides with that of a Kazhdan-Lusztig left cell and the results of [Joseph] can be used to classify both. The key is the Robinson-Schensted algorithm which establishes a bijection between the permutations $w \in S_n$ and pairs of same-shape standard Young tableaux of size

n. We write

$$RS : S_n \longrightarrow \{(A, B) \in SYT(n) \times SYT(n) | \text{ shape } A = \text{shape } B\}$$

For an element $w \in S_n$, we denote the left tableau of its image by $RS_L(w)$ and the right tableau by $RS_R(w)$. Under this correspondence, two permutations in S_n belong to the same geometric left cell iff they share the same right Young tableau. Hence the set of all standard Young tableaux of size n parameterizes the set of geometric left cells. Joseph's identification now describes the orbital varieties contained in a given nilpotent orbit.

Theorem. *[Joseph] The set of orbital varieties contained in the nilpotent orbit $\mathcal{O}_u$ corresponds to the set of standard Young tableaux of shape λ_u.*

A similar description of orbital varieties for the other complex simple classical groups is complicated by two phenomena. First of all, the notions of Kazhdan-Lusztig left cells and geometric left cells no longer coincide. Second, the ordinary Robinson-Schensted algorithm needs to be adapted to the new Weyl group. The work of Garfinkle provides the initial steps [Garfinkle1]. Let W_m be the Weyl group of G. The generalized Robinson-Schensted algorithm now takes the form

$$RS : W_m \longrightarrow \{(A, B) \in SDT(m) \times SDT(m) | \text{ shape } A = \text{shape } B\}$$

where $SDT(m)$ consists of *domino tableaux* of size m. As in type A, two Weyl group elements belong to the same Kazhdan-Lusztig left cell if they share the same right domino tableau. However, the converse is no longer true. To remedy this, Garfinkle introduces an equivalence relation on domino tableaux by rearranging, or *moving through*, certain subsets of dominos called *open cycles*. Within each such equivalence class, there is a unique domino tableau of *special* shape [Lusztig]. This time, two Weyl group elements belong to the same Kazhdan-Lusztig left cell iff their right tableaux are equivalent to the same tableau of special shape. Because in type A all partitions are of special shape, this is a natural generalization of the original result.

Garfinkle's equivalence relation, however, is too strong to describe geometric left cells. By restricting the set of open cycles that one is allowed to move through, McGovern defines the appropriate equivalence relation [McGovern2]. In classical groups not of type A, not all partitions of m arise as Jordan block decompositions. Within each of McGovern's equivalence classes, however, there is a unique one which does.

Theorem. *[McGovern2] In types B and C, orbital varieties contained in the orbit $\mathcal{O}_u$ are parameterized by standard domino tableaux of shape λ_u. In type D, the result is the same unless λ_u has only even parts, when the number of vertical dominos should be congruent to 0 or $2 \bmod 4$ according as the Roman numeral attached to $\mathcal{O}_u$ is I or II.*

The natural question is whether a similar classification can also be obtained by studying the unipotent variety. In types B, C, and D, its irreducible components, $\text{Irr}(\mathcal{F}_u)$, are parameterized by *admissible* domino tableaux of shape λ_u and a choice of sign for certain disjoint subsets of dominos called *open* and *closed clusters* [Spaltenstein] [van Leeuwen]. The component group A_u of the centralizer G_u acts on $\text{Irr}(\mathcal{F}_u)$. Within the above parameterization, this action changes signs of the open clusters. As each orbital variety corresponds uniquely to an A_u-orbit on $\mathrm{Irr}(\mathcal{F}_u)$ [Borho-Brylinski], this parameterizes the orbital varieties in $\mathcal{O}_u$ by admissible domino tableaux of shape λ_u with a choice of sign for each of its closed clusters.

The key to reconciling the two parameterizations of orbital varieties lies in understanding the relationship between the cycles and clusters contained in the same tableau. In particular, we prove:

Lemma 3.3.1 *Each closed cluster $\mathcal{C}$ of a domino tableau contains the closed cycle $\mathcal{Y}_{\mathcal{C}}$ through its lowest-numbered domino.*

This observation allows us to define a map $\Phi : \Sigma DT_{cl}(\lambda) \longrightarrow SDT(\lambda)$ from admissible domino tableaux with signed closed clusters to the set of all standard domino tableaux of the same shape. The map Φ moves through the initial cycle of each closed cluster of positive sign while preserving the rest of the tableau.

Theorem 3.3.1 *The map Φ is a bijection between the set of domino tableaux and the set of admissible domino tableaux of the same shape with signed closed clusters.*

We conclude this chapter with two results. For an orbital variety $\mathcal{V}$, we describe the τ-invariant and the maximal stabilizing parabolic $Q_{\mathcal{V}}$. Then, we define a map from standard domino tableau to standard Young tableau which can be interpreted in terms of the corresponding orbital varieties.

3.1 Preliminary Definitions and Basic Facts

3.1.1 Unipotent Variety

For $\epsilon = \pm 1$, take $<,>_\epsilon$ be a non-degenerate bilinear form on $\mathbb{C}^m$ such that

$$< x, y >_\epsilon = \epsilon < y, x >_\epsilon \; \forall x, y \in \mathbb{C}^m.$$

The form $<,>_{-1}$ is symplectic and $m = 2n$ must be even. We call this a form of type C. The form $<,>_1$ is symmetric and m can be even or odd. When $m = 2n$ for some n, we say it is a form of type D and when $m = 2n + 1$, we say it is of type B. A *full isotropic flag* in $\mathbb{C}^m$ is a sequence

$$f_1 \subset f_2 \subset \cdots \subset f_n$$

of subspaces of $\mathbb{C}^m$ where each f_i is isotropic with respect to $<,>_\epsilon$, $\dim(f_i) = i$. Then f_n is a maximal isotropic subspace. Denote by $\mathcal{F}$ the set of all such flags. It has a natural structure of a projective algebraic variety which is irreducible in types B and C and has two connected components in type D.

Let G_ϵ be the isometry group of $<,>_\epsilon$ and take $\mathfrak{g}_\epsilon$ to be its Lie algebra. In types B and D, $G_1 = O(m)$ and $\mathfrak{g}_1 = \mathfrak{so}(m)$, while in type C, $G_{-1} = Sp(2n)$. Let us fix one of these types and simply refer to the isometry group as G. Let u be a unipotent element in G. Define *shape* u to be the partition whose parts are the sizes of the Jordan blocks of u arranged in decreasing order; it is well-defined and determined by the conjugacy class of u. The converse of this statement is also true: two unipotent elements of the same shape are conjugate in G. Hence to classify all unipotent conjugacy classes in G, one simply needs to determine which partitions appear as shapes of its unipotent elements.

Set $\mathcal{P}_\epsilon(m) = \{(\lambda_1, \ldots, \lambda_j)$ a partition of $m| \; \#\{k|\lambda_k = i\}$ is even $\forall i$ with $(-1)^i = \epsilon\}$. Then the classification takes the form:

Theorem. *[Gerstenhaber] Conjugacy classes of unipotent elements in G_ϵ are in one-to-one correspondence with the set of partitions of $\mathcal{P}_\epsilon(m)$.*

We write λ_u for the image of a unipotent element u under the above correspondence. The group G has an obvious action on the flag variety $\mathcal{F}$. Denote by $\mathcal{F}_u$ the fixed-point set of u on $\mathcal{F}$. In general, $\mathcal{F}_u$ is reducible, so let us define by $\mathrm{Irr}(\mathcal{F}_u)$ the set of its irreducible components. Let G_u be the centralizer of u in G and G_u^o the connected component of the identity. It acts on $\mathcal{F}_u$ and consequently on $\mathrm{Irr}(\mathcal{F}_u)$. Let $A_u = G_u/G_u^o$. We will write $A_u = A_{\lambda_u} = A_\lambda$ interchangeably when no confusion can arise. G_u^o acts trivially on $\mathrm{Irr}(\mathcal{F}_u)$ and hence A_u itself acts on $\mathrm{Irr}(\mathcal{F}_u)$. We will need to understand this action explicitly. For now, we describe A_u.

Fact. *For a partition λ, let B_λ be the set of its distinct parts λ_i satisfying $(-1)^{\lambda_i} = -\epsilon$. Then A_u is a 2-group with $|B_\lambda|$ components, i.e.*

$$A_u = \bigoplus_{|B_\lambda|} \mathbb{Z}_2.$$

3.1.2 Nilpotent Orbits and Orbital Varieties

The partition classification of 3.1.1 also can also be used to describe the nilpotent G_ϵ-orbits in $\mathfrak{g}_\epsilon$.

In cases B and C, nilpotent G_ϵ-orbits in $\mathfrak{g}_\epsilon$ are precisely the nilpotent orbits as defined by the group G_{ad}. However, in type D, G_ϵ is larger than G_{ad} whose action defines nilpotent orbits. To each *very even* partition λ_u, there correspond two nilpotent orbits whose elements have Jordan form λ_u. Nevertheless, there is only one such G_ϵ-orbit. Write $\mathcal{O}_u$ (with perhaps a Roman numeral) for a nilpotent orbit corresponding to λ_u and $\mathcal{O}_u{}'$ to the corresponding G_ϵ-orbit. We will also write $\mathcal{O}_u$ for the nilpotent orbit through u, and $\mathcal{O}_\lambda$ for the nilpotent orbit corresponding to the partition λ.

Let $\mathfrak{b} \subset \mathfrak{g}$ be a Borel subalgebra, $\mathfrak{h} \subset \mathfrak{b}$ a Cartan subalgebra, and $\mathfrak{n}$ the nilradical so that $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$. The irreducible components $\mathrm{Irr}(\mathcal{O}_{\lambda_u} \cap \mathfrak{n})$ are called *orbital varieties.* By [Joseph], they take the form $V(w) = \overline{B(\mathfrak{n} \cap w^{-1}\mathfrak{n})} \cap \mathcal{O}_{\lambda_u}$ for some $w \in W$, the Weyl group. The set of Weyl group elements which map to the same orbital variety under this correspondence is known as a *geometric left cell.* McGovern's parameterization of orbital varieties relies on this description. Our examination, however, will be based on a parameterization of $\mathrm{Irr}(\mathcal{F}_u)$ and the action of the component group A_u upon it. The key result is:

Fact. *[Borho-Brylinski] There is a bijective correspondence between A_u-orbits on Irr$(\mathcal{F}_u)$ and Irr$(\mathcal{O}'_u \cap \mathfrak{n})$.*

3.1.3 Domino Tableaux

Let T be an integer tableau, or a finite left-justified array of rows of squares, each labeled by an integer. To each tableau T one can assign a partition λ_T which we call the *shape* of T. We view T as a set of ordered pairs (k, S_{ij}) denoting that the square in row i and column j of T is labeled by the integer k. $D(k,T)$, a *domino with label k*, is a subset of T of the form $\{(k, S_{ij}), (k, S_{i+1,j})\}$ or $\{(k, S_{ij}), (k, S_{i,j+1})\}$. Call these, respectively, *vertical* and *horizontal dominos.* For convenience, we will refer to the set $\{(0, S_{11})\}$ as the 0-domino.

When not necessary, we will omit the labels of squares and write S_{ij} for (k, S_{ij}). In that case, we define *label* $S_{ij} = k$.

Definition 3.1.1. Let T be an integer tableau. If *shape* T is a partition of $2n+1$ of type B, T is a domino tableau of type B iff it is partitioned by the dominos $\{(0, S_{11})\}, D(1,T)$, $D(2,T), \ldots,$ and $D(n,T)$, in a way that the labels increase weakly along rows and columns of T.

If *shape* T is a partition of $2n$ and is of type C, (respectively D), T is a domino tableau of type C (respectively D) iff if it is partitioned by the dominos $D(1,T), D(2,T)$, $D(3,T), \ldots, D(n,T)$ such that the labels increase weakly along rows and columns of T.

For a partition λ, let $SDT(\lambda)$ denote the set of standard domino tableaux of shape λ. Implicit in this notation are the type of the partition λ and the type of domino tableaux. We will also need to consider domino tableaux all of whose subtableaux are of the same type.

Definition 3.1.2. For $T \in SDT(\lambda)$ let $T(i)$ denote the tableau formed be the dominos of T with labels less than or equal to i. Let X= B, C, or D, and take T to be a type X. The tableau T is *admissible* iff each $T(i)$ is also a domino tableau of type X for all i.

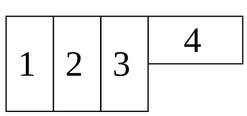
The dominos that appear within admissible tableaux fall into three categories.

Definition 3.1.3.
1. In types B and D (respectively C), a vertical domino is of type (I^+) if it lies in an odd- (respectively even-) numbered column.
2. A vertical domino not of type (I^+) is of type (I^-).
3. A horizontal domino is of type (N) if its left square lies in an even- (respectively odd-) numbered column.

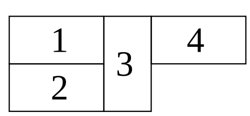
In fact, we can restate the above definition by noting that a domino tableau is admissible iff all of its dominos are of type $(I^+), (I^-)$, or (N), and perhaps the 0-domino. Finally, we would like to assign plus and minus signs to certain dominos.

Definition 3.1.4. A signed domino tableau is an admissible domino tableau with a sign label for each domino of type (I^+). Denote the set of all signed tableaux of shape λ by $\Sigma DT(\lambda)$. For a $T \in \Sigma DT(\lambda)$, let $|T|$ denote the underlying domino tableau.

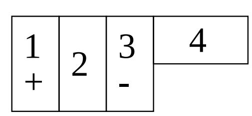
Example 3.1.5. Consider the partition [5,3] of type D and the three tableaux below. T is an admissible D-tableau; $D(1,T)$ and $D(3,T)$ are dominos of type (I^+), $D(2,T)$ is a domino of type (I^-), and $D(4,T)$ is of type (N).



(a) T



(b) T'



(c) T''

T' on the other hand is not admissible, as *shape* $T(1)$ is not a D-partition. Here, $D(1,T')$ and $D(2,T')$ are horizontal dominos not of type (N). Finally, T'' is a signed domino tableau; its dominos of type (I^+) are signed. Also, $|T''| = T$.

3.1.4 Cycles

Starting with a domino tableau T, Garfinkle defines a way to form a new domino tableau, not necessarily of the same type, that preserves the labels of certain alternate squares. There is a distinguished way to define this new domino tableau $MT(k,T)$ by requiring that a particular domino $D(k,T)$ be not be preserved by the process, but as much as possible of the rest of T and the labels of certain squares remain the same. The map is called *moving through* the domino $D(k,T)$.

We specify four choices of the squares of T whose labels should be fixed under moving through. For X = B or C, the square S_{ij} is said to be *X-fixed* iff $i+j$ is odd. For X=D or D', S_{ij} is *X-fixed* iff $i+j$ is even. Squares of T that are not X-fixed are called X-variable. When our choice of X is clear, we will refer simply to *fixed* and *variable* squares. Under the moving through map, the labels of fixed squares will be preserved while those of variable ones may change.

Recall Garfinkle's definition of a cycle [Garfinkle1]. We will think of cycles as both, subsets of dominos of T, as well as just the sets of their labels. We will call a cycle whose fixed squares are X-fixed an *X-cycle*. Because the definitions of B and C-fixed as well as D and D'-fixed are the same, the B and C-cycles as well as D and D'-cycles for a tableau T coincide.

For reference and to establish notation, we recall Garfinkle's definition of the image of a single domino under the moving through map.

Definition 3.1.6. Let $MT(D(k,T),T)$ be the image of the domino $D(k,T)$ under moving through. If $D(k,T) = \{(k, S_{i,j-1}), (k, S_{ij})\}$ or $\{(k, S_{ij}), (k, S_{i+1,j})\}$ and S_{ij} is fixed, let r be the label of $S_{i-1,j+1}$. Then

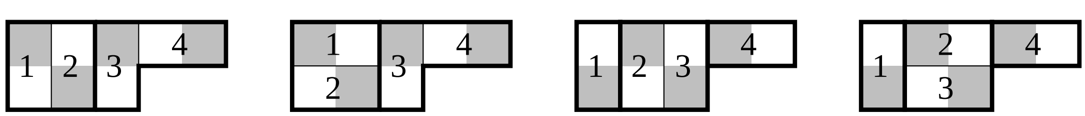
1. If $r > k$, let $MT(D(k,T),T) = \{(k, S_{ij}), (k, S_{i-1,j})\}$,
2. If $r < k$, let $MT(D(k,T),T) = \{(k, S_{ij}), (k, S_{i,j+1})\}$

On the other hand, If $D(k,T) = \{(k, S_{ij}), (k, S_{i,j+1})\}$ or $\{(k, S_{i-1,j}), (k, S_{ij})\}$ and S_{ij} is fixed, let r be the label of $S_{i+1,j-1}$. Then

1. If $r > k$, let $MT(D(k,T),T) = \{(k, S_{i,j-1}), (k, S_{ij})\}$,
2. If $r < k$, let $MT(D(k,T),T) = \{(k, S_{ij}), (k, S_{i+1,j})\}$

While $MT(D(k,T),T)$ denotes the image of the domino $D(k,T)$ under moving through, we let $MT(k,T)$ be the new tableau obtained by moving through the cycle containing $D(k,T)$. While this notation does not explicitly indicate the type of the cycle that is moved through, this will always be clear from the context.

Example 3.1.7. Consider again the D-tableau T from Example 3.1.5.



(a) T (b) $MT(2,T)$ (c) T (d) $MT(2,T)$

The shaded squares in (a) are D-fixed. With this choice, $MT(D(1,T),T) = \{S_{11}, S_{12}\}$ and the set $\{D(1,T), D(2,T)\}$ constitutes a D-cycle in T. The set $\{D(3,T), D(4,T)\}$ is T's other D-cycle. In the diagrams, the dark lines outline each cycle. Moving through $\{D(1,T), D(2,T)\}$ yields the inadmissible tableau in (b).

The shaded squares in diagram (b) are B-fixed. The B-cycles are the sets $\{D(1,T)\}$, $\{D(2,T), D(3,T)\}$, and $\{D(4,T)\}$. This time, moving through the cycle containing $D(2,T)$ gives the *admissible* tableau in (d).

3.2 Irreducible Components of $\mathcal{F}_u$

The irreducible components of the unipotent variety $\mathcal{F}_u$ were described by N. Spaltenstein in [Spaltenstein]. We present this parameterization as interpreted by M.A. van Leeuwen [van Leeuwen]. Its advantage lies in a particularly translucent realization of the action of A_u on these components that allows us to parameterize the orbital varieties $\mathrm{Irr}(\mathcal{O}'_u \cap \mathfrak{n})$.

Let us fix a unipotent u and a flag F of the appropriate type. Spaltenstein's parameterization of $\mathrm{Irr}(\mathcal{F}_u)$ begins by associating to F a signed domino tableau. Let $F \in \mathcal{F}_u$ be a flag of type X= B, C, or D, and recall the definition of the flags $F^{(i)}$. Let $\lambda^{(i)}$ be the shape of the Jordan form of the unipotent operator induced by u upon $F^{(i)}$. It turns out that for all i, the difference between $\lambda^{(i)}$ and $\lambda^{(i+1)}$ is precisely a domino [Spaltenstein]. By assigning this domino a label $i+1$, we obtain a domino tableau T of shape λ_u from any flag in $\mathcal{F}_u$. In fact, the construction implies that this domino tableau will be admissible.

Admissible tableaux, however, do not fully separate the components of $\mathcal{F}_u$. If two flags give rise to different domino tableaux in this way, they lie in different components of $\mathcal{F}_u$. However, the converse is not true. The inverse image $\mathcal{F}_{u,T}$ of a given admissible tableau T under this identification is in general not connected. Nevertheless, the irreducible components of $\mathcal{F}_{u,T}$ are precisely its connected components [van Leeuwen](3.2.3). Accounting for this disconnectedness yields a parameterization of $\mathrm{Irr}(\mathcal{F}_u)$. To this effect, we consider the set of signed admissible tableaux $\Sigma DT(\lambda_u)$. Itself, it is too large to parameterize $\mathrm{Irr}(\mathcal{F}_u)$, but with an appropriate equivalence relation, it will give us the parameterization. The manner in which this equivalence is defined separates [van Leeuwen] from [Spaltenstein]. We follow [van Leeuwen] and define the concept of *clusters*. Our definition differs from [van Leeuwen], but it is equivalent.

3.2.1 Clusters

Clusters partition the set of dominos of $T \in \Sigma DT(\lambda)$ into subsets. They are defined inductively and depend only on the underlying domino tableau $|T|$. Hence suppose we already know the clusters of $T(k-1)$ and would like to know how $D(k,T)$, the domino with label k inside T, fits into the clusters of $T(k)$. Here is a summary:

Definition 3.2.1. In type B and C, let $cl(0)$ be the cluster containing $T(1)$.

1. If $D(k,T) = \{S_{ij}, S_{i+1,j}\}$ and *type* $D = (I^-)$, then $D(k,T)$ joins the cluster of the domino containing $S_{i,j-1}$. If $j = 1$, then $D(k,T)$ joins $cl(0)$.
2. If $D(k,T) = \{S_{ij}, S_{i+1,j}\}$ and *type* $D = (I^+)$ then $D(k,T)$ forms a singleton cluster in $T(k)$, unless $i \geq 2$ and $S_{i-1,j+1}$ is not in T. In the latter case, $D(k,T)$ joins the cluster of the domino containing $S_{i-1,j}$.

3. Take $D(k,T) = \{S_{ij}, S_{i,j+1}\}$, so that *type* $D = (N)$. Let C_1 be the cluster of the domino containing $\{S_{i,j-1}\}$ but if $j = 1$, let $C_1 = cl(0)$. If $i \geq 2$ and $S_{i-1,j+2}$ is not in T, let C_2 be the cluster of the domino that containing $S_{i-1,j+1}$. If $C_1 = C_2$ or C_2 does not exist, the new cluster is $C_1 \cup D(k,T)$. If $C_1 \neq C_2$, the new cluster is $C_1 \cup C_2 \cup D(k,T)$.

4. The clusters of $T(k-1)$ left unaffected by the above simply become clusters of $T(k)$.

Definition 3.2.2. A cluster is *open* if it contains an (I^+) or (N) domino along its right edge and is not $cl(0)$. A cluster that is neither $cl(0)$ nor open is *closed*. Denote the set of open clusters of T by $OC(T)$ and the set of closed clusters as $CC(T)$.

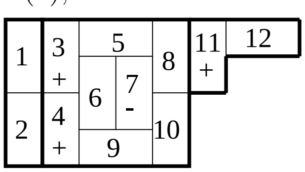
This definition differs from [van Leeuwen] as we do not call $cl(0)$ an open cluster. The open clusters of T correspond to the parts of λ contained in B_λ. As the latter set parameterizes the $\mathbb{Z}_2$ factors of A_λ, we will ultimately use open clusters to describe the action of A_λ on the irreducible components of $\mathcal{F}_u$. To be more precise, define a map

$$b_T : B_\lambda \longrightarrow OC(T) \cup cl(0).$$

For $r \in B_\lambda$, let $b_T(r)$ be the cluster that contains a domino ending a row of length r in T. This map is well-defined: any two dominos that end two rows of the same length belong to the same cluster; furthermore, such a cluster is always open or it is $cl(0)$. The map b_T is also onto $OC(T)$, but it is not one-to-one as T may have fewer open clusters than $|B_\lambda|$. For future reference, we will need this definition.

Definition 3.2.3. For a cluster $\mathcal{C}$, let $I_\mathcal{C}$ be the domino in $\mathcal{C}$ with the smallest label and take S_{ij} as its left and uppermost square. For X = B or C, we say $\mathcal{C}$ is an X-*cluster* iff $i + j$ is odd. For X = D or D', $\mathcal{C}$ is an X-*cluster* iff $i + j$ is even.

Example 3.2.4. Consider this admissible C-tableau for the partition $[8, 6, 5, 5]$.
Its clusters are the sets$\{1, 2\}$, $\{3, 4, 5, 6, 7, 8, 9, 10\}$, and $\{11, 12\}$. The set $\{1, 2\}$ is precisely $cl(0)$, the second set is a closed C-cluster, while the third is open and also of type C.



The clusters are outlined with darker lines. The C-cycles of S are $\{1, 2\}$, $\{3, 5, 8, 10, 9, 4\}$, $\{6, 7\}$, and $\{11, 12\}$, showing that the cluster and cycle structures for a given tableau may not coincide. Note, however, that each cluster contains the cycle of the same type through its smallest-numbered domino. This is true in general.

3.2.2 Equivalence Relations

Armed with the notion of open and closed clusters, we can now define two equivalence relations on $\Sigma DT(\lambda)$.

Definition 3.2.5. If $T, T' \in \Sigma DT(\lambda)$, let $T \sim_{op,cl} T'$ iff $|T| = |T'|$ and the products of signs in all corresponding open and closed clusters of T and T' agree. Denote the equivalence classes by $\Sigma DT_{op,cl}(\lambda)$ and write $[T]$ for the equivalence class of $T \in \Sigma DT(\lambda)$. We think of $\Sigma DT_{op,cl}(\lambda)$ as the set of admissible domino tableaux of shape λ with signed open and closed clusters.

Similarly, if $T, T' \in \Sigma DT(\lambda)$, let $T \sim_{cl} T'$ iff $|T| = |T'|$ and the products of signs in all corresponding closed clusters of T and T' agree. Denote the set of these equivalence classes by $\Sigma DT_{cl}(\lambda)$ and write $[[T]]$ for the equivalence class of T. Elements of $\Sigma DT_{cl}(\lambda)$ are represented by admissible domino tableaux of shape λ with signed closed clusters.

The set $\Sigma DT_{op,cl}(\lambda_u)$ will parameterize the irreducible components of $\mathcal{F}_u$. There is a considerable amount of freedom in how a flag of $\mathcal{F}_{u,T}$ can be assigned an equivalence class of signed admissible domino tableaux. A particular choice is presented in [van Leeuwen](3.4), and we call this map Γ_u.

To understand the action of A_u on the components of the unipotent variety in terms of this parameterization, we first describe an action of A_u on $\Sigma DT_{op,cl}(\lambda_u)$. Let

$$\xi_r : \Sigma DT_{op,cl}(\lambda) \longrightarrow \Sigma DT_{op,cl}(\lambda)$$

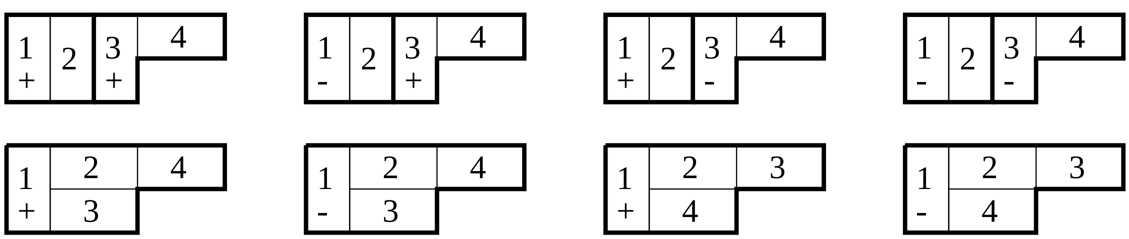
act trivially if $b_T(r) = cl(0)$ and by changing the sign of the open cluster $b_T(r)$ otherwise. For each $r \in B_\lambda$, let g_r denote the generator of the corresponding $\mathbb{Z}_2$ factor of A_u. We define the action of g_r on $\Sigma DT_{op,cl}(\lambda_u)$ by

$$g_r[T] = \xi_r[T].$$

This action accurately reflects the action of A_u on $\mathrm{Irr}(\mathcal{F}_u)$. We summarize this result below.

Theorem. *The map Γ_u establishes an A_u-equivariant bijection between the components $Irr(\mathcal{F}_u)$ and $\Sigma DT_{op,cl}(\lambda_u)$, the set of admissible domino tableaux of shape λ_u with signed open and closed clusters.*

Example 3.2.6. Consider $\lambda_u = [5{,}3]$, a D-partition. The set $\mathrm{Irr}(\mathcal{F}_{[5,3]})$ is parameterized by $\Sigma DT_{op,cl}([5,3])$, or the following eight tableaux with signed open and closed clusters:

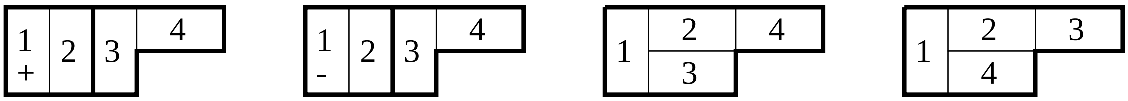


Tableaux (a)-(d) each have an open cluster $\{3,4\}$ and a closed cluster $\{1,2\}$, while (e)-(f) all have one open cluster $\{1,2,3,4\}$. $A_{[5,3]} = \mathbb{Z}_2 \times \mathbb{Z}_2$ and each of its factors acts in the same way by changing the sign of the sole open cluster in each of these tableaux.

According to Fact 3.1.2, the set of irreducible components of $\mathcal{O}'_u \cap \mathfrak{n}$, denoted $\mathrm{Irr}(\mathcal{O}'_u \cap \mathfrak{n})$, corresponds to the set of A_u-orbits of $\mathrm{Irr}(\mathcal{F}_u)$. The above characterization of $\mathrm{Irr}(\mathcal{F}_u)$ along with the description of the A_u action yield the following:

Corollary. *The orbital varieties $Irr(\mathcal{O}'_u \cap \mathfrak{n})$ are parametrized by $\Sigma DT_{cl}(\lambda)$, the set of admissible domino tableaux of shape λ with signed closed clusters.*

Example 3.2.7. With λ_u as in Example 3.2.6, $\mathrm{Irr}(\mathcal{O}_{[5,3]} \cap \mathfrak{n})$ is parameterized by $\Sigma DT_{cl}([5,3])$, or the following four domino tableaux with signed closed clusters. They are derived from the dominos of Example 3.2.6 by identifying the $A_{[5,3]}$-orbits.



3.3 The Components $\mathrm{Irr}(\mathcal{O}'_u \cap \mathfrak{n})$

We aim to reconcile Corollary 3.2.2 with McGovern's original parameterization of $\mathrm{Irr}(\mathcal{O}'_u \cap \mathfrak{n})$. To this effect, we define a bijection between $\Sigma DT_{cl}(\lambda)$ and $SDT(\lambda)$ by applying Garfinkle's moving through map to certain distinguished cycles.

3.3.1 A Bijection

Consider an X-cluster $\mathcal{C}$ and let I_C be the domino in $\mathcal{C}$ with the smallest label. Let $\mathcal{Y}_{\mathcal{C}}$ be the X-cycle through I_C. We call it the *distinguished* cycle of $\mathcal{C}$. We will use it to define a map between admissible domino tableaux with signed closed clusters and the set of domino tableaux of the same shape. Our construction relies on:

Lemma. *A closed cluster of an admissible domino tableau T contains its distinguished cycle, i.e. if $\mathcal{C} \in CC(T)$, then $\mathcal{Y}_{\mathcal{C}} \subset \mathcal{C}$.*

We defer the proof to its own section. Armed with this fact, we can now propose a map

$$\Phi : \Sigma DT_{cl}(\lambda) \longrightarrow SDT(\lambda)$$

by moving through the distinguished cycles of all closed clusters with positive sign. More explicitly, for a tableau $T \in \Sigma DT_{cl}(\lambda)$, let $CC^+(T)$ denote the set of closed clusters of T labeled by a $(+)$ and let $\sigma(T) = \{\mathcal{Y}_{\mathcal{C}} \mid \mathcal{C} \in CC^+(T)\}$ be the set of their distinguished cycles. Then we define

$$\Phi(T) = MT(|T|, \sigma(T)).$$

Theorem. $\Phi : \Sigma DT_{cl}(\lambda) \longrightarrow SDT(\lambda)$ *is a bijection.*

Proof. We check that this map is well-defined, that its image lies in $\Sigma DT(\lambda)$, and then construct its inverse. For this map to be well-defined, we need to know that the definition of Φ does not depend on which order we move through the cycles in $\sigma(T)$. It is enough to check that if $\mathcal{Y}_{\mathcal{C}}$ and $\mathcal{Y}'_{\mathcal{C}} \in \sigma(T)$, then $\mathcal{Y}'_{\mathcal{C}}$ is also lies in $\sigma(MT(|T|, \mathcal{Y}_{\mathcal{C}}))$. While this statement is not true for arbitrary cycles, in our setting, this is Lemma 3.4.7.

The image of Φ indeed lies in $\Sigma DT(\lambda)$. That $\Phi(T)$ is itself a domino tableau follows from the fact that moving through any cycle of T yields a domino tableau. That it is of the same shape as T follows as well because Φ moves through only closed cycles. Hence $\Phi(T) \in SDT(\lambda)$.

The definition of a cluster, and in particular 3.2.1(2), forces the initial domino $I_{\mathcal{C}}$ of every closed cluster to be of type (I^+). By the Definition 3.1.6, the image of $MT(I_{\mathcal{C}}, T)$

in $MT(\mathcal{Y}_{\mathcal{C}}, T)$ is inadmissible, i.e. it is a horizontal domino not of type (N). In general, all the inadmissible dominos in $\Phi(T)$ appear within the image of distinguished cycles under moving through. Furthermore, the lowest- numbered domino within each cycle is the image of the initial domino of some distinguished cycle. With this observation, we can construct the inverse of Φ. We define a map

$$\Psi : \Phi(\Sigma DT_{cl}(\lambda)) \longrightarrow \Sigma DT_{cl}(\lambda)$$

that satisfies $\Psi \circ \Phi =$ Identity. Let $\iota(\Phi(T))$ be the set of cycles in $\Phi(T)$ that contain inadmissible dominos. We define $\Psi(\Phi(T)) = MT(\Phi(T), \iota(\Phi(T)))$. By the above discussion, $\iota(\Phi(T))$ contains precisely the images of cycles in $\sigma(T)$. Hence

$$\Psi(\Phi(T) = MT(\Phi(T), \iota(\Phi(T))) = MT(MT(|T|, \sigma(T))) = T$$

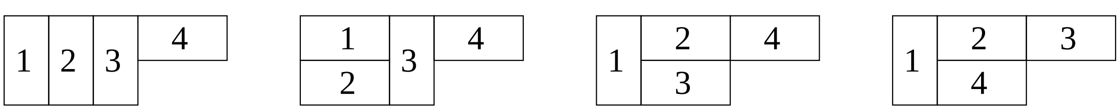
as desired. Thus Φ is a bijection onto its image in $SDT(\lambda)$. As we already know that the sets $\Sigma DT_{cl}(\lambda)$ and $SDT(\lambda)$ both parameterize the same set of orbital varieties, Φ must be a bijection between them. □

Corollary 3.3.1. *The orbital varieties Irr* $(\mathcal{O}'_u \cap \mathfrak{n})$ *are parameterized by the set* $SDT(\lambda_u)$.

We can translate this to:

Theorem 3.3.2. *In types B and C, orbital varieties contained in the orbit* $\mathcal{O}_u$ *are parameterized by standard domino tableaux of shape* λ_u*. In type D, the result is the same unless* λ_u *has only even parts, when the number of vertical dominos should be congruent to* 0 *or* $2 \, mod \, 4$ *according as the Roman numeral attached to* $\mathcal{O}_u$ *is* I *or* II.

Example 3.3.3. Recall our parameterization of $\mathrm{Irr}(\mathcal{O}'_{[5,3]} \cap \mathfrak{n})$ in Example 3.2.7 and first consider the tableau 3.2.7 (a). The set of its closed clusters labeled with (+) consists of $\{\{1, 2\}\}$ and hence $\mathcal{Y}_{\{1,2\}} = \{1, 2\}$. Its image under Φ is then the tableau of 3.1.5 (b). The tableaux 3.2.7 (b),(c), and (d), have no closed clusters labeled with (+) and their underlying domino tableaux are unaffected by Φ. Hence the image under Φ of the set $\Sigma DT_{cl}([5, 3])$ is the following set of domino tableaux:



or precisely the set $SDT([5, 3])$, as described by McGovern's original result.

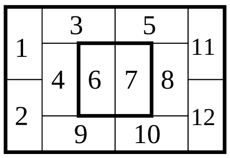
3.4 Proof of Lemma 3.3.1

We aim to show that a closed cluster $\mathcal{C}$ contains its distinguished cycle $\mathcal{Y}_{\mathcal{C}}$. The proof has two parts. First, we show that $\mathcal{Y}_{\mathcal{C}}$ is contained in a slightly larger set of clusters $\overline{\mathcal{C}}$, defined as the union of $\mathcal{C}$ with all of its *nested* clusters. Then, we show that $\mathcal{Y}_{\mathcal{C}}$ intersects each of these nested clusters trivially.

3.4.1 Closed and Nested Clusters

First, we define and describe the set $\overline{\mathcal{C}}$. Let $\mathcal{C}$ be a closed cluster of a tableau T and denote by $row_k\, T = \{S_{k,j} |\ j \geq 0\}$ the kth row of T. Define $col_k\, T$ similarly. If $row_k\, T \cap \mathcal{C} \neq \emptyset$, let $\inf_k\, \mathcal{C} = \inf\{j |\ S_{k,j} \in row_k\, T \cap \mathcal{C}\}$ and $\sup_k\, \mathcal{C} = \sup\{j \mid S_{k,j} \in row_k T \cap \mathcal{C}\}$.

Consider the following tableau of type D. It has two closed clusters given by the sets $\mathcal{C}_1 = \{1, 2, 3, 4, 5, 8, 9, 10, 11, 12\}$ and $\mathcal{C}_2 = \{6, 7\}$. $\mathcal{C}_1$ is a D-cluster while $\mathcal{C}_2$ is a B-cluster. $\mathcal{Y}_{\mathcal{C}_1}$



is then a D-cycle and equals $\{1, 3, 5, 11, 12, 10, 9, 2\}$. T has two other D-cycles, $\{4, 6\}$ and $\{7, 8\}$. Both intersect $\mathcal{C}_1$, but are not contained within it. The B-cluster $\mathcal{Y}_{\mathcal{C}_2}$ equals $\{6, 7\}$ and is contained in $\mathcal{C}_2$. This example shows that an X-cluster may not contain all the X-cycles through its dominos. However, it always contains its initial cycle. Also notice that $\mathcal{C}_1$ completely surrounds $\mathcal{C}_2$. We call such clusters *nested.*
Nested clusters complicate the description of closed clusters. To simplify our initial results, we would like to consider the set formed by a cluster together with all of its nested clusters. To be more precise:

Definition 3.4.1. Let $\mathcal{C}'$ be a cluster of T. $\mathcal{C}'$ is *nested* in $\mathcal{C}$ if all of the following are satisfied:

- $\inf\{k | row_k T \cap \mathcal{C}' \neq \emptyset\} > \inf\{k | row_k T \cap \mathcal{C} \neq \emptyset\}$
- $\sup\{k | row_k T \cap \mathcal{C}' \neq \emptyset\} < \sup\{k | row_k T \cap \mathcal{C} \neq \emptyset\}$
- $\inf\{k | col_k T \cap \mathcal{C}' \neq \emptyset\} > \inf\{k | col_k T \cap \mathcal{C} \neq \emptyset\}$
- $\sup\{k | col_k T \cap \mathcal{C}' \neq \emptyset\} < \sup\{k | col_k T \cap \mathcal{C} \neq \emptyset\}$

We define $\overline{\mathcal{C}} = \mathcal{C}\ \cup\ \{\mathcal{C}' \in CC(T) \mid \mathcal{C}' \text{ nested in } \mathcal{C}\}$ and, finally, let $periphery(\overline{\mathcal{C}}) = \{D(k, T) \in \overline{\mathcal{C}} \mid D(k, T) \text{ is adjacent to a square not in } \overline{\mathcal{C}}\}$. Note that $periphery(\overline{\mathcal{C}}) \subset \mathcal{C}$.

Example 3.4.2. In the above tableau, $\mathcal{C}_2$ is nested in $\mathcal{C}_1$. $\mathcal{C}_1 \cup \mathcal{C}_2 = \overline{\mathcal{C}} = T, periphery(\overline{\mathcal{C}}) = \mathcal{Y}_{\mathcal{C}_1} \subset \mathcal{C}_1$.

The next two propositions describe properties of dominos that occur along the left and right edges of $\overline{\mathcal{C}}$. They are essential to our goal of relating closed clusters to their distinguished cycles. Recall the our definition of the cycle $\mathcal{Y}_{\mathcal{C}}$ endows $\mathcal{C}$ as well as $\overline{\mathcal{C}}$ with a choice of fixed and variable squares by defining the left and uppermost square of $I_{\mathcal{C}}$ as fixed.

Proposition 3.4.3. *Take $\mathcal{C}$ a closed cluster of a domino tableau T and consider k such that $row_k T \cap \mathcal{C} \neq \emptyset$. Then:*

(i) $type\, D(label(T_{k,inf_k\, \mathcal{C}}), T) = type\, D(label(T_{k,inf_k\, \overline{\mathcal{C}}}), T) = (I^+)$

(ii) $type\, D(label(T_{k,sup_k\, \mathcal{C}}), T) = type\, D(label(T_{k,sup_k\, \overline{\mathcal{C}}}), T) = (I^-)$.

Proof. Part (i) is true for all non-0 clusters by 3.2.1(2) while part (ii) is the defining property of closed clusters.

∎

Proposition 3.4.4. *For a closed cycle $\mathcal{C}$, suppose that $I_{\mathcal{C}} = \{S_{ij}, S_{i+1,j}\}$ and that S_{ij} fixed. If $D = \{S_{pq}, S_{p+1,q}\}$ lies on periphery$(\overline{\mathcal{C}})$, then*

1. *S_{pq} is fixed if type $D = (I^+)$.*
2. *$S_{p+1,q}$ is fixed if type $D = (I^-)$*

Proof. (i) Assume the contrary, i.e. that there is a $D' \in periphery(\overline{\mathcal{C}})$ of type (I^+) whose uppermost square is not fixed. By examining such a domino that lies closest to $I_{\mathcal{C}}$, we find that $periphery(\overline{\mathcal{C}})$ must then contain two type (I^+) dominos $E = \{S_{kl}, S_{k+1,l}\}$ and $E' = \{S_{k+1,l}, S_{k+2,m}\}$ with the squares S_{kl} and $S_{k+2,m}$ fixed and $|m-l|$ minimal.

Assume $m < l$. The opposite case can be proved by a similar argument. Because E' is of type (I^+), there is an integer t such that $m < t < l$, $S_{k+1,t} \in periphery(\overline{\mathcal{C}})$, and t is maximal with these properties. Let F be the domino containing $S_{k+1,t}$. F has to be $\{S_{k+1,t}, S_{k+2,t}\}$ and of type (I^-). If its type was (I^-) or (N), 3.2.1(2) would force $S_{k+1,t+1}$ to be in $periphery(\overline{\mathcal{C}})$ as well. If F on the other hand was $\{S_{k+1,t}, S_{k,t}\}$, this would contradict the minimality of $|m-l|$. We now consider two cases.

(a) Assume $t = l-1$. Because E and F belong to $periphery(\overline{\mathcal{C}})$ and hence to $\mathcal{C}$, $\mathcal{C}$ must contain a domino of type (N) of the form $\{S_{u,l-1}, S_{u,l}\}$ with $u > k+2$ and u minimal with this property. The set of squares $\{S_{p,l-1} | k+2 < p < u\} \cup \{S_{pl} | k+1 < p < u\}$ must be tiled by dominos, which is impossible, as its cardinality is odd.

(b) Assume $t < l-1$. We will contradict the maximality of t. Because E and F both belong to $\mathcal{C}$, $\mathcal{C}$ must contain a sequence H_α of dominos of type (N) satisfying

$$H_\alpha = \{S_{k+1+f(\alpha),t+2\alpha}, S_{k+1+f(\alpha),t+2\alpha+1}\}$$

where $0 \leq \alpha \leq \frac{l-t+1}{2}$. We choose each H_α such that for all α, $f(\alpha)$ is minimal with this property. Because the sets $\{S_{k+p,l} | k+1 < p < k+1+f(\frac{l-t+1}{2})\}$ and $\{S_{k+p,t} | k+2 < p < k+1+f(0)\}$ have to be tiled by dominos of type (I^+) and (I^-) respectively, $f(0)$ has to be even and $f(\frac{l-t+1}{2})$ has to be odd. Hence there is a β such that $f(\beta)$ is even and $f(\beta+1)$ is odd.

Assume $f(\beta) < f(\beta+1)$, but the argument in the other case is symmetric. Let G be the domino containing the square $S_{k+1+f(\beta),t+2\beta+2}$. G must belong to $\mathcal{C}$, as H_β and G is either of type (I^-) or (N). The type of G cannot be (N), however, as this would contradict the condition on f. Hence G must be of type (I^-). If $G = \{S_{k+1+f(\beta),t+2\beta+2}, S_{k+f(\beta),t+2\beta+2}\}$. Then by successive applications of 3.2.1 (1) and (2), the set of dominos

$$\{\{S_{k+f(\beta)-\gamma\epsilon,t+2\beta+\epsilon}, S_{k+1+f(\beta)-\gamma-\epsilon,t+2\beta+\epsilon}\} |\ \epsilon = 1 \text{ or } 2 \text{ and } 0 \leq \gamma \leq f(\beta)-2\}$$

is contained in $\mathcal{C}$ as well. But this means that $t+2\beta+\epsilon$ for $\epsilon = 1$ or 2 satisfies the defining property of t, contradicting its maximality. This shows (i).

(ii) We would like to show that for every (I^-) domino in $periphery\overline{\mathcal{C}}$, the bottom square is fixed. It is enough to show that this is true for one such domino, as an argument similar to that in (i) can be repeated for the others. Let $l = \inf\{k | row_k T \cap \overline{\mathcal{C}} = \emptyset\}$. Then by 3.4.3 and the definition of fixed, we know that $S_{l,\inf_l \overline{\mathcal{C}}}$ is fixed. As $\{S_{l,\sup_l \overline{\mathcal{C}}}, S_{l+1,\sup_l \overline{\mathcal{C}}}\}$ is a domino of type (I^-) in $periphery(\overline{\mathcal{C}})$, we have found the desired domino.

□

3.4.2 Three More Lemmas

Lemma. *periphery* $(\overline{\mathcal{C}}) \subset \mathcal{Y}_{\mathcal{C}} \subset \overline{\mathcal{C}}$.

Proof. We prove the second inclusion. The first is apparent from the argument. We show that for each D in the $periphery(\overline{\mathcal{C}})$, as sets of squares, $MT(D,T) \subset \overline{\mathcal{C}}$. We differentiate cases accounting for different domino positions along $periphery(\overline{\mathcal{C}})$.

(i) Take $D = \{(k, S_{ij}), (k, S_{i+1,j})\}$ and suppose $typeD = (I^+)$. Because D lies on $periphery(\overline{\mathcal{C}})$, Proposition 3.4.4 implies that S_{ij} is fixed. Due to Definitions 3.2.1(2) and 3.2.2, $S_{i,j+1} \in \overline{\mathcal{C}}$.

  (a) Suppose $S_{i-1,j+1}$ in not in $\overline{\mathcal{C}}$. Then $r = label(S_{i-1,j+1}) < k$. Otherwise $S_{i-1,j}$ and S_{ij} would both belong to the same cluster by Definition 3.2.1(2); since $S_{i-1,j}$ and $S_{i-1,j+1}$ are in the same cluster by Definition 3.2.1(1) or (3), this contradicts our assumption. Now Definition 3.1.6(2) says

$$MT(D,T) = \{(k, S_{ij}), (k, S_{i,j+1})\},$$

  and since S_{ij} and $S_{i,j+1}$ both belong to $\overline{\mathcal{C}}$, so must $MT(D,T)$.

  (b) Suppose now that $S_{i-1,j+1} \in \overline{\mathcal{C}}$. Then $S_{i-1,j} \in \overline{\mathcal{C}}$ as well since by Definition 3.2.1(1) or (3), they both belong to the same cluster. Now Definition 3.1.6 implies $MT(D,T) \subset \{S_{ij}, S_{i-1,j}, S_{i,j+1}\}$. As all of these squares lie in $\overline{\mathcal{C}}$, we must also have $MT(D,T) \subset \overline{\mathcal{C}}$.

(ii) Suppose $D = \{(k, S_{ij}), (k, S_{i,j+1})\}$ and that $S_{i,j+1}$ is fixed. By Definitions 3.2.1(2) and 3.2.2 $S_{i,j+2} \in \overline{\mathcal{C}}$.

  (a) Suppose $S_{i-1,j+1}$ is not in $\overline{\mathcal{C}}$. Then $S_{i-1,j+2}$ is not in $\overline{\mathcal{C}}$, as by Definition 3.2.1(1) or (3) they both belong to the same cluster. By the definition of a cluster, $r = label(S_{i-1,j+2}) < k$ and by Definition 3.1.6(2) forces $MT(D,T) = \{S_{i,j+1}, S_{i,j+2}\}$. Since the squares $S_{i,j+1}$ as well as $S_{i,j+2}$ are both contained in $\overline{\mathcal{C}}$, so is $MT(D,T)$.

  (b) Suppose $S_{i-1,j+1} \in \overline{\mathcal{C}}$. Then because $MT(D,T) \subset \{S_{i,j+1}, S_{i,j+2}, S_{i-1,j+2}\}$, $MT(D,T) \subset \overline{\mathcal{C}}$.

(iii) Take $D = \{(k, S_{ij}), (k, S_{i,j+1})\}$ and suppose S_{ij} is fixed. Then $S_{i,j-1} \in \overline{\mathcal{C}}$ by Definition 3.2.1(3).

  (a) Suppose $S_{i+1,j-1}$ is not in $\overline{\mathcal{C}}$. Then $r = label(S_{i+1,j-1}) > k$ by Definition 3.2.1(2) or (3). Now Definition 3.1.6(1) implies that $MT(D,T) = \{S_{ij}, S_{i,j-1}\} \subset \overline{\mathcal{C}}$.

  (b) If $S_{i+1,j-1} \in \overline{\mathcal{C}}$, then $S_{i+1,j} \in \overline{\mathcal{C}}$ as well, since by Definition 3.2.1(1) or (3) they belong to the same cluster. But by Definition 3.1.6(1) or(2), $MT(D,T) \subset \{S_{ij}, S_{i+1,j}, S_{i,j-1}\} \subset \overline{\mathcal{C}}$.

(iv) Let $D = \{(k, S_{ij}), (k, S_{i+1,j})\}$ and take type D = (I^-). The square $S_{i+1,j}$ is then fixed and $S_{i+1,j-1} \in \overline{\mathcal{C}}$.

   (a) Assume that $S_{i+1,j-1} \in \overline{\mathcal{C}}$. Then $S_{i+2,j} \in \overline{\mathcal{C}}$ as well. Since $MT(D,T)$ equals $\{S_{i+1,j}, S_{i+1,j-1}\}$ or $\{S_{i+1,j}, S_{i+2,j}\}$, $MT(D,T) \in \overline{\mathcal{C}}$ as both possibilities are contained in $\overline{\mathcal{C}}$.

   (b) Assume $S_{i+1,j-1}$ is not in $\overline{\mathcal{C}}$. We have $r = \mathit{label}(S_{i+1,j-1}) > k$, for otherwise $D(r,T)$ and hence $S_{i+1,j-1} \in \overline{\mathcal{C}}$. But then $MT(D,T) = \{S_{i+1,j}, S_{i+1,j-1}\}$, so it is contained in $\overline{\mathcal{C}}$.

Because of Proposition 3.4.4, these are all the cases we need to consider. For instance, dominos of type D = (I^+) whose bottom squares are fixed cannot occur along $\mathit{periphery}(\overline{\mathcal{C}})$. □

What remains is to see that $\mathcal{Y}_\mathcal{C}$ is contained within the closed cluster $\mathcal{C}$ itself. It is enough to show that its intersection with any closed cluster nested in $\mathcal{C}$ is trivial. Our proof relies on the notion of X-boxing. For the following definition, recall Garfinkle's notion of ϕ_X-box [Garfinkle1](1.5.2)

Definition 3.4.5. For X = B, C, D, or D', we say a domino $D(k,T)$ is X-*boxed* iff it is contained in some ϕ_X-box.

The importance of this concept lies in its behavior with respect to cycles and moving through. The following proposition is a restatement of [Garfinkle1](1.5.9 and 1.5.22).

Proposition 3.4.6. *Suppose $D(k,T)$ and $D(k',T)$ both belong to the same X-cycle.*

(i) $D(k,T)$ is X-boxed iff $MT(D(k,T),T)$ is not X-boxed.

(ii) $D(k,T)$ and $D(k',T)$ are both simultaneously X-boxed or not X-boxed.

Armed with this notion, we can now address:

Lemma. *If $\mathcal{C}' \subset \overline{\mathcal{C}}$ is a closed cluster nested in $\mathcal{C}$, then $\mathcal{Y}_\mathcal{C} \cap \mathcal{C}' = \emptyset$.*

Proof. It is enough to show that $\mathit{periphery}\,\mathcal{C}' \cap \mathcal{Y}_\mathcal{C} = \emptyset$, as this forces $\mathcal{C}' \cap \mathcal{Y}_\mathcal{C} = \emptyset$. We divide the problem into a few cases.

(i) Suppose $\{\mathit{type}\ \mathcal{Y}_\mathcal{C}, \mathit{type}\ \mathcal{Y}_{\mathcal{C}'}\} = \{C, D'\}$. We investigate $\mathit{periphery}\,\mathcal{C}' \cap \mathcal{Y}_\mathcal{C}$. It cannot contain dominos of types (I^+) and (I^-); because the boxing property is constant on cycles according to Proposition 3.4.6(ii), such dominos would have to be simultaneously C and D-boxed, which is impossible. If $D(k,T) \in \mathit{periphery}\,\mathcal{C}' \cap \mathcal{Y}_{\mathcal{C}'}$ is of type (N), $D(k,T)$ and $MT(D(k,T),T)$ are both C and D'-boxed. This contradicts Proposition 3.4.6(i), forcing $\mathit{periphery}\,\mathcal{C}' \cap \mathcal{Y}_\mathcal{C} = \emptyset$. The proof is identical when $\{\mathit{type}\ \mathcal{Y}_\mathcal{C}, \mathit{type}\ \mathcal{Y}_{\mathcal{C}'}\} = \{B, D\}$.

(ii) Suppose $\{\mathit{type}\ \mathcal{Y}_\mathcal{C}, \mathit{type}\ \mathcal{Y}_{\mathcal{C}'}\} = \{C, D\}$. The proof is similar to the first case, except this time, dominos of type (N) cannot be simultaneously C and D-boxed. Again, the proof is identical when $\{\mathit{type}\ \mathcal{Y}_\mathcal{C}, \mathit{type}\ \mathcal{Y}_{\mathcal{C}'}\} = \{B, D'\}$.

(iii) Suppose $\{\mathit{type}\ \mathcal{Y}_\mathcal{C}, \mathit{type}\ \mathcal{Y}_{\mathcal{C}'}\} \subset \{B, C\}$ or $\{D, D'\}$. Then by by the definition of cycles, $\mathcal{Y}_\mathcal{C} \cap \mathcal{Y}_{\mathcal{C}'} = \emptyset$. We know $\mathit{periphery}(\mathcal{C}') \subset \mathcal{Y}_{\mathcal{C}'} \subset \overline{\mathcal{C}}'$ by Lemma 3.4.2, implying again that $\mathit{periphery}(\mathcal{C}') \cap \mathcal{Y}_\mathcal{C} = \emptyset$.

$\square$

Lemma 3.4.7. *Consider closed clusters $\mathcal{C}$ and $\mathcal{C}'$ and their initial cycles $\mathcal{Y}_{\mathcal{C}}$ and $\mathcal{Y}_{\mathcal{C}'}$. Then $\mathcal{Y}_{\mathcal{C}}$ is again a cycle in $MT(|T|, \mathcal{Y}_{\mathcal{C}'})$.*

Proof. If $\mathcal{C}$ and $\mathcal{C}'$ are clusters of the same type, then so are their initial cycles and the lemma is [Garfinkle 1.5.29]. Otherwise, without loss of generality, take $\mathcal{C}$ to be a C-cluster and $\mathcal{C}'$ to be a D-cluster. As the other cases are similar, we can also assume that $\mathcal{Y}_{\mathcal{C}}$ is C-boxed while $\mathcal{Y}_{\mathcal{C}'}$ is D-boxed.

Suppose that the dominos $D(r) \in \mathcal{Y}_{\mathcal{C}}$ and $D(s) \in \mathcal{Y}_{\mathcal{C}'}$ lie in relative positions described by the following diagram:

$$\begin{array}{|c|c|}\hline & s \\ \hline r & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

with the box labeled by r fixed. The same squares in $MT(|T|, \mathcal{Y}_{\mathcal{C}'})$ have the labels

$$\begin{array}{|c|c|}\hline & s' \\ \hline r & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

To prove the lemma, we need to show that $s < r$ implies $s' < r$ and $s > r$ implies $s' > r$. There are two possibilities for the domino $D(s)$. It is either horizontal or vertical and must occupy the following squares:

$$\text{Case (i)} \quad \begin{array}{|c|c|}\hline s & s \\ \hline r & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad \text{or} \qquad \text{Case (ii)} \quad \begin{array}{|c|c|}\hline & s \\ \hline & s \\ \hline r & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

- Case (i). In this case, $s < r$ always. Garfinkle's rules for moving through imply that $MT(|T|, D(r)) \cap \mathcal{C}' \neq \varnothing$. This is a contradiction since we know by hypothesis that $\mathcal{Y}_{\mathcal{C}} \neq \mathcal{Y}_{\mathcal{C}'}$. Hence this case does not occur.

- Case (ii). First suppose $s > r$. Then the our squares within $MT(|T|, \mathcal{Y}_{\mathcal{C}'})$ must look like

  $$\begin{array}{|c|c|}\hline & s \\ \hline & s' \\ \hline r & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

  for some $s' \neq s$. Since the tableau $MT(\mathcal{Y}_{\mathcal{C}'}, T)$ is standard, this requires that $s' > s$ implying $s' > r$ which is what we desired. Now suppose $s < r$ and suppose the squares in our diagram look like

  $$\begin{array}{|c|c|}\hline & s \\ \hline t & s \\ \hline r & u \\ \hline\end{array}$$

  As in case (i), we find that $D(t) \notin \mathcal{C}'$. Since $D(t) \in \mathcal{C}$, type $D(s) = I^+$ implies type $D(t) = I^-$, type $D(r) = I^-$, and type $D(u) = I^+$. Otherwise, the rules defining clusters would force s to lie in the cluster $\mathcal{C}$. Now $D(u)$ lies in the initial cycle of a closed cluster of same type as $\mathcal{C}'$. Since it lies on the periphery and its type is

I^+, then its top square must be fixed. In particular, $D(u) \notin \mathcal{C}$. But $s < r$ implies $MT(D(r)) \cap D(u) \neq \varnothing$. This is a contradiction, implying that this case does not arise.

To finish the proof, we must examine the possibility that $D(s)$ and $D(r)$ lie in the relative positions described by

	r
s	

.

This case is completely analogous and we omit the proof. □

3.5 The τ-Invariant for Orbital Varieties

A natural question is whether the two methods of parameterizing orbital varieites by standard tableaux indeed yields the same parameterization. A partial answer is that the orbital varieties attached to the same standard tableau share the same generalized τ-invariant. This is a useful in our setting, as the construction of the Graham-Vogan space associated to an orbital variety $\mathcal{V}$ requires us to be able to explicity identify its τ-invariant. After a few definitions, we describe how to calculate the τ-invariant for an orbital variety $\mathcal{V}_T$ corresponding to a standard tableau T. We then show that this result is independent of the method of the parameterization.

Let Δ be the set of roots in $\mathfrak{g}$, Δ^+ the set of positive roots and Π the set of simple roots. Write $\mathfrak{g} = \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_{-\alpha} \oplus \mathfrak{t} \oplus \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha$ for the triangular decomposition of $\mathfrak{g}$ and let $\mathfrak{b} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha$. Write W for the Weyl group, and let P_α be the standard parabolic subgroup with Lie algebra $\mathfrak{p}_\alpha = \mathfrak{b} \oplus \mathfrak{g}_{-\alpha}$. Following [Joseph], for an element $w \in W$, an orbital variety $\mathcal{V}$, and a standard parabolic subgroup we define

$$\tau(w) = \Pi \cap w(-\Delta^+), \tag{3.1}$$

$$\tau(P) = \{\alpha \in \Pi \mid P_\alpha \subset P\}, \tag{3.2}$$

$$\tau(\mathcal{V}) = \{\alpha \in \Pi \mid P_\alpha(\mathcal{V}) = \mathcal{V}\}. \tag{3.3}$$

We would like to be able to read off $\tau(\mathcal{V})$ from the standard tableau parameterizing $\mathcal{V}$ as the maximal parabolic subgroup Q stabilizing $\mathcal{V}$ is precisely the standard parabolic subgroup satisfying $\tau(Q) = \tau(\mathcal{V})$. A result of Joseph suggests one approach to this problem.

Theorem. *[Joseph] Let $\mathcal{V}_w$ be the orbital variety of type A that corresponds to the Weyl group element $w \in S_n$. Then $\tau(w) = \tau(\mathcal{V}_w)$.*

The Robinson-Schensted algorithm now suggests that one should be able to describe the τ-invariant of $\mathcal{V}_w$ in terms of the left and right tableau $RS_L(w)$ and $RS_R(w)$. For a standard Young tableau T, write $r_T(i)$ for the number of the row that contains the square with label i.

Theorem. *[Joseph] Consider an element $w \in S_n$ and let $T = RS_L(w)$. The simple root $\alpha_i \in \Pi$ lies in $\tau(w)$ iff the square labelled i lies higher in T than the square with label $i+1$, or more precisely, iff*

$$r_T(i+1) > r_T(i).$$

This gives a quick way of describing the τ-invariant in type A. Garfinkle's generalization of the Robinson-Schensted algorithm to hyperoctahedral groups provides a means of using this approach in the setting of the other classical groups. For a standard domino tableau T, we say that a domino D *lies higher in* T than another domino D' iff there is an integer $j \geq 1$ such that D lies entirely within the union of rows $1, \ldots, j$ of T while D' lies entirely within the union of the rows $j+1, \ldots, l$ of T.

Theorem. *[Garfinkle2] Let w be an element of the Weyl group of a classical Lie group not of type A. Write $T = RS_L(w)$. The simple root $\alpha_i \in \Pi$ lies in $\tau(w)$ iff one of the following is satisfied:*

(i) $i = 1$ and the domino $D(1,T)$ is vertical,
(ii) $i > 1$ and $D(i-1,T)$ lies higher that $D(i)$ in T.

The above theorems find the τ invariant using the parameterization obtained from the Weyl group. To find the τ-invariant using the other approach, we rely on Spaltenstein's original work on the unipotent variety $\mathcal{F}_u$. For an irreducible component $\mathcal{C} \in \text{Irr } \mathcal{F}_u$, $\tau(\mathcal{C}) = \{\alpha \mid P_\alpha \mathcal{C} = \mathcal{C}\}$. Then

Theorem. *[Spaltenstein] Let $X = B$, C, or D. Consider $\mathcal{C} \in Irr\ \mathcal{F}_{u,|T|}$, that is, an irreducible component whose classifying tableau T in $SDT_{op,cl}$ has underlying domino tableau $|T|$. Then $\alpha_i \in \tau(\mathcal{C})$ iff one of the following is satisfied:*

(i) $i = 1$, $D(1,T)$ is vertical, and $X \neq D$,
(ii) $i > 1$ and $D(i-1,|T|)$ lies higher that $D(i,|T|)$ in $|T|$,
(iii) $i > 1$ and $\{D(i-1,T), D(i,T)\} \in CC^+(T)$,
(iv) If $X = D$, then $\alpha_1 \in \tau(\mathcal{C})$ iff $\{1,2\} \in CC^-(T)$ and $\alpha_2 \in \tau(\mathcal{C})$ iff $\{1,2\} \in CC^+(T)$.

This provides us with a means of finding the τ-invariant of an orbital variety independent of the above results on the Weyl group. Both approaches rely on a description of the relative positions of squares of dominoes within the standard tableau describing the orbital variety. It is not *a priori* apparent that the two parameterizations of orbital varieties by standard tableaux are the same, or have the same τ-invariant for that matter. However, deciphering Spaltenstein's description makes it possible to decide this question.

Theorem 3.5.1. *Consider an orbital variety $\mathcal{V}_T$ that corresponds to the standard domino tableau T under the correspondence of this chapter. The simple root α_i lies in $\tau(\mathcal{V}_T)$ iff one of the following conditions is satisfied:*

(i) $i = 1$ and the domino $D(1,T)$ is vertical,
(ii) $i > 1$ and $D(i-1,T)$ lies higher that $D(i)$ in T.

In particular, this means that $\tau(w) = \tau(\mathcal{V}_w) = \tau(\mathcal{V}_T) = \tau(Q_{\mathcal{V}_T})$.

Proof. Let π be the projection from $SDT_{op,cl}$ onto SDT_{cl} by identifying the A-group action. Also define a map $\tilde{\Phi} : SDT_{op,cl} \rightarrow SDT$ as the composition $\Phi \circ \pi$. We prove that if $T \in SDT_{op,cl}$ parameterizes $\mathcal{K}_T \in \text{Irr } \mathcal{F}_u$, then $\tau(\mathcal{K}_T) = \tau(\mathcal{V}_{\tilde{\Phi}(T)})$.

That $\alpha_1 \in \tau(\mathcal{K}_T)$ iff $\alpha_1 \in \tau(\mathcal{V}_{\tilde{\Phi}(T)})$ is clear in types B and C since $D(1,T)$ never lies within a closed cluster and hence remains unchanged under $\tilde{\Phi}$. For $i > 1$, suppose that

$D(i,T)$ or $D(i-1,T)$ lies in some $\mathcal{C} \in CC^+(T)$. If the number of dominos in $\mathcal{C}$ is greater than 2, then the fact that $\alpha_i \in \tau(\mathcal{K}_T)$ iff $\alpha_i \in \tau(\mathcal{V}_{\tilde{\Phi}(T)})$ follows from [Garfinkle III.1.4]. Now suppose that $\mathcal{C} = \{D(i), D(i-1)\}$. Then $\alpha_i \in \tau(\mathcal{K}_T)$. But $D(i-1)$ is higher than $D(i)$ in $MT(\mathcal{C}, T)$, implying by the definition of Φ that $\alpha_i \in \tau(\mathcal{V}_{\tilde{\Phi}(T)})$. The remaining possibility is that only one of $D(i)$ and $D(i-1)$ lies in $\mathcal{C}$. The fact that $\alpha_i \in \tau(\mathcal{K}_T)$ iff $\alpha_i \in (\mathcal{V}_{\tilde{\Phi}(T)})$ again follows by inspection. Finally, when $X = D$, the conditions for $\alpha_i, i \leq 2$ to lie in $\tau(\mathcal{K}_T)$ desribed by Spaltenstein translate exacly to our conditions for $\tau(\mathcal{V}_{\tilde{\Phi}(T)})$. □

These results; however, do not resolve the question of whether the two parameterizations of orbital varieties by standard tableaux are the same. Neverthelss, it is possible to prove a somewhat stronger statement than what we have. It turns out that the two parameterizations share the same *generalized* τ-invariant. This is readily verified by inspecting the actions of the $T_{\alpha\beta}$ operators in both settings. For the purposes of this work, however, only the results on the τ-invariant itself are necessary.

3.6 Projection of Orbital Varieties

A natural question to ask is how the orbital varieties in classical types not of type A relate to those of type A, and furthermore, can this relationship be easily interpreted through the corresponding standard tableaux? A result in this direction would be useful in subsequent work on Graham-Vogan spaces, as it may be used to break down the calculations required to understand these spaces.

Let $\mathfrak{g}$ be a classical complex Lie algebra of type $X_n = B_n$, C_n, or D_n and let $\mathfrak{n}$ be the unipotent part of $\mathfrak{b}$. There is a natural projection map π_A from $\mathfrak{n}$ to $\mathfrak{n}_A$, the corresponding unipotent part in type A_n. Let $\mathcal{O}$ be a nilpotent orbit of type X_n. It turns out that the image of an orbital variety for $\mathcal{O}$ under π_A is always an orbital variety for some nilpotent orbit $\mathcal{P}$ of type A. In fact, if $\mathcal{P}$ arises in this way, then *all* of its orbital varieties lie in the image of π_A for $\mathcal{O}$. We will use this idea to motivate our approach to describing the Graham-Vogan spaces for classical groups not of type A.

We interpret this approach in terms of the standard domino tableaux used to parameterize the relevant orbital varieties. In an effort to find certain Littlewood-Richardson coefficients, Carre and Leclerc define a bijection between the set of semistandard domino tableaux of a given shape and a set of pairs of Yamanouchi and semistandard Young tableaux. Because this map is very useful in answering the above questions, we state this more precisely.

Definition 3.6.1. A Yamanouchi domino tableau is a semistandard domino tableau whose column reading is a Yamanouchi word. We denote the set of Yamanouchi domino tableaux of shape λ and evaluation μ as $Yam_2(\lambda, \mu)$. A Yamanouchi word is a word $w = w_1 w_1 \dots w_l$ such that each right factor $w_i \dots w_l$ contains at least as many letters j than $j+1$. Finally, a column reading of a domino tableau T is the word obtained by reading the successive column of T from bottom to top and left to right, counting each horizontal domino only the first time that it arises.

Theorem 3.6.2 (Carre-Leclerc). *There is a bijection*

$$DT(\lambda, \mu) \xrightarrow[(\pi_1, \pi_2)]{} \amalg_{\nu \in L} Yam_2(\lambda, \nu) \times YT(\nu, \mu),$$

where the L contains all partitions ν such that the $Yam_2(\lambda, \nu)$ is not empty. Furthermore, its restriction to standard domino tableaux induces a bijection

$$SDT(\lambda) \xrightarrow[(\pi_1,\pi_2)]{} \amalg_\nu Yam_2(\lambda, \nu) \times SYT(\nu).$$

The bijection is an algorithm that takes a tableau T and modifies it succesively until its column reading becomes a Yamanouchi word. Simultaneously, it builds a standard Young tableau to record the sequence of moves.

Definition 3.6.3. Define a map

$$\pi_A : SDT(n) \longrightarrow SYT(n)$$

by $\pi_A(T) = \pi_2(T)$ where π_2 is the second component of the Carre-Leclerc map. We also denote by π_A the map induced on orbital varieties obtained by identifying T with $\mathcal{V}_T$.

Chapter 4

Restriction to Spherical Orbital Varieties

Armed with a description of the orbital varieties contained in a given nilpotent orbit as well as the corresponding τ-invariants, we can attempt to describe the Graham-Vogan representations attached to a nilpotent orbit in the setting of classical groups.

The Graham-Vogan construction takes on a particularly nice form for orbital varieties of *S-type*, to de defined in the following sections. We would like to restrict our attention to nilpotent orbits *all* of whose orbital varieties are S-type, and are therefore led to consider to the set of *spherical* nilpotent orbits.

After a brief description of S-type orbital varieties and how spherical orbits fit within the framework of all nilpotent coadjoint orbits, we introduce an inductive procedure that we will use in the next chapter while calculating the infinitesimal characters corresponding to $V(\mathcal{V}, \pi)$. It is suggested by a construction of a *minimal representative* of each orbital variety that we detail in this chapter.

We begin by illustrating our method with a "Model Example," which is sufficiently naïve to quickly describe our approach.

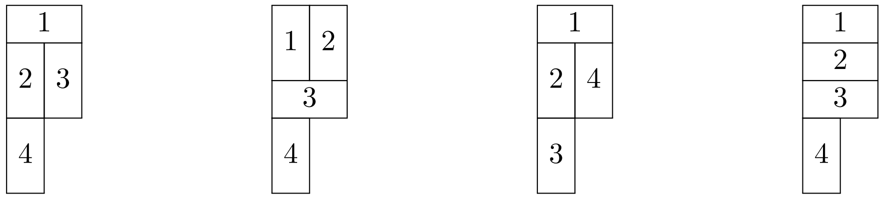
4.1 Model Example

We will calculate the infinitesimal character associated to $V(\mathcal{V}, \pi)$ constructed from a particular orbital variety in type C. Although the example we choose is a bit naïve from several perspectives, we use it to motivate our approach to the more general calculation. We will address its shortcomings within the following chapter.

Suppose $G = Sp(2 \cdot 4)$ and realize the Lie algebra $\mathfrak{g}$ as a set of 8×8 matrices of the form

$$\mathfrak{sp}(8) = \left\{ \mathfrak{m}(A, B, C) = \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \mid A, B, C \in M(4, \mathbb{C}) \text{ and } B, C \in SYM(4, \mathbb{C}) \right\}.$$

Let $\mathcal{O}$ be the nilpotent coadjoint orbit in $\mathfrak{g}^*$ corresponding to the partition $[2^3, 1^2]$. It has dimension 18. There are three orbital varieties contained in $\mathcal{O}$, corresponding to the domino tableaux



Let $\mathcal{V}$ be the orbital variety corresponding to the first domino tableau. Then $\dim \mathcal{V} = \frac{1}{2} \dim \mathcal{O} = 9$. As a representative, we take $f = \mathfrak{m}(A, B, 0)$ with

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

To calculate the infinitesimal character of the Graham-Vogan space for $\mathcal{V}$, we need to describe the parameters

(i) an admissible orbit datum $(\pi, \mathcal{H}_\pi)$,

(ii) $\mathcal{V}^o$, the smooth part of $\mathcal{V}$,

(iii) the stabilizing parabolic $Q_\mathcal{V} \subset G$,

(iv) a smooth representation (γ, W_γ) of $Q_\mathcal{V}$,

(v) and a G-equivariant isomorphism of vector bundles j_π.

where notation is as in Chapter 2.

Write G_f for the isotropy subgroup of f and $\mathfrak{g}_f$ for its Lie algebra. As G is complex, the metaplectic cover $\widetilde{G}_f$ is isomorphic to $G_f \times \mathbb{Z}/2\mathbb{Z}$. We choose one admissible orbit datum; it is trivial on G_f° and acts by the non-trivial character on $\mathbb{Z}/2\mathbb{Z}$.

The orbital variety $\mathcal{V}$ is smooth so that in the notation of the first chapter, $\mathcal{V}^o = \mathcal{V}$. From Theorem 3.5.1 , we find that the stabilizer of $\mathcal{V}$ is the standard parabolic subgroup $Q_\mathcal{V}$ with Levi factor isomorphic to $GL(2) \times GL(2)$. One can quickly check that, in this case, both $Q_\mathcal{V}$ and the standard Borel subgroup B act with dense orbit on $\mathcal{V}$.

This observation simplifies calculations, as it allows us to replace the Lagrangian covering $G \times_{Q_\mathcal{V}} \mathcal{V}$ by G/Q_f, where $Q_f = Q_\mathcal{V} \cap G_f$ and $\mathcal{V}$ contains $Q_\mathcal{V}/Q_f$ as a dense subset. We note that B/B_f is also dense in $\mathcal{V}$. The equivariant line bundle $\tau^* \mathcal{V}_\pi$ is induced by a character α of B_f. It is given by the square root of the absolute value of the real determinant of B_f acting on the tangent space $\mathfrak{b}/\mathfrak{b}_\mathfrak{f}$ of $\mathcal{V}$ at f. This is

$$\alpha \begin{pmatrix} A & * \\ 0 & A^{t^{-1}} \end{pmatrix} = |\, t_1^3 \, t_3^6 \,|^{-1} \text{ , where } A = \begin{pmatrix} t_1 & * & * & * \\ 0 & t_3 & * & * \\ 0 & 0 & t_3 & * \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Because we are looking for a map j_π, we would like to find a character γ of B whose restriction to B_f is α. Such a character is given by

$$\gamma \begin{pmatrix} A & * \\ 0 & A^{t^{-1}} \end{pmatrix} = |\, t_1\, t_2\, t_3\, t_4\,|^{-3} \text{ , where } A = \begin{pmatrix} t_1 & * & * & * \\ 0 & t_2 & * & * \\ 0 & 0 & t_3 & * \\ 0 & 0 & 0 & t_4 \end{pmatrix}.$$

The character γ extends uniquely from B to $Q_{\mathcal{V}}$. Let the half-density bundle on $G/Q_{\mathcal{V}}$ be given by the character $\rho_{Q_{\mathcal{V}}}$ and define another character γ' on $Q_{\mathcal{V}}$ to equal $\gamma \otimes \rho_{Q_{\mathcal{V}}}^{-1}$. Then

$$V(\mathcal{O}, \mathcal{V}, \pi, \gamma, j_{\gamma,\pi}) \subset Ind_Q^G(\gamma').$$

Hence the infinitesimal character that we associated to the representation space $V(\mathcal{V}, \pi)$ equals $-(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}) + \rho = (\frac{5}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2})$, where $\rho = (4, 3, 2, 1)$ and equality is up to Weyl group action. This is precisely the unique infinitesimal character attached to the orbit $\mathcal{O}_{[2^3,1^2]}$ by [McGovern]. Similar calculations for the other orbital varieties in this orbit yield the same infinitesimal character.

A significant simplification in this example came from the fact that the parabolic subgroup $Q_{\mathcal{V}}$ acted with dense orbit on $\mathcal{V}$. It made it easy to find the isomorphism j_π. Unfortunately, this is not always the case.

Example 4.1.1. [Melnikov] Let $G = SL_9$ and let

T=

1	2	3	6	9
4	5	8		
7				

Then $\mathcal{V}_T$, the orbital variety in $\mathcal{O}_{[5,3,1]}$ corresponding to T has dimension 31. However, $\dim Q_{\mathcal{V}} \cdot f \leq 30$ for all $f \in \mathcal{V}$.

This counterexample can be extended to produce others in larger groups of type A. Counterexamples in other classical types arise among orbital varieties $\mathcal{V}_T$ where $\mathcal{V}_{\pi_A(T)}$ produces a counterexample in type A. The reasons for this will become a bit more apparent in the next few sections.

Although we cannot always assume that $Q_{\mathcal{V}}$ acts with dense orbit on $\mathcal{V}$, fortunately, there are many instances where we can apply this simplifying assumption.

- There are important classes of nilpotent orbits *all* of whose orbital varieties *do* admit a dense orbit of their stabilizing parabolic.

- Furthermore, even among nilpotent orbits containing orbital varieties without this property, there are always some orbital varieties that admit the above simplification.

We start our investigation of the $V(\mathcal{V}, \pi)$ spaces with these. There are other complications that do not appear in this model example, but in order to spare the reader, we will address them when the need arises.

4.2 Spherical Orbital Varieties and Orbits of S-type

We would like to use the methods of our model example to calculate the infinitesimal character associated to $V(\mathcal{V}, \pi)$ for as many nilpotent orbits as feasible. The main assumption required is that the stabilizer of an orbital variety has a dense orbit in that variety. Such orbital varieties are called of *S-type*, as are the nilpotent orbits *all* of whose orbital varieties satisfy this condition. Among classical groups, there is a class of *small* nilpotent orbits that are of S-type. We first describe this set and then place it among other important nilpotent coadjoint orbits.

4.2.1 Spherical Orbits

Let G be a complex simple Lie group and B a Borel subgroup.

Definition 4.2.1. A nilpotent coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$ is spherical iff it contains an open B-orbit.

The work of Panyushev provides a concise description of spherical nilpotent orbits contained in classical groups.

Theorem 4.2.2. *[Panyushev] Spherical nilpotent orbits in complex classical Lie groups are precisely those that correspond to the following partitions:*

(i) $[2^b, 1^c]$ *in type* A,

(ii) $[3^a, 2^{2b}, 1^c]$ *with* $a \leq 1$ *in type* B,

(iii) $[2^b, 1^{2c}]$ *in type* C, *and*

(iv) $[3^a, 2^{2b}, 1^c]$ *with* $a \leq 1$ *in type* D.

These orbits arise in [McGovern3] for another reason. In the setting of complex simply connected semisimple Lie groups, there is an orbit whose G-module structure of the coordinate ring of regular functions on a nilpotent orbit $\mathcal{O}$, denoted $R(\mathcal{O})$, is more transparent than in general; all multiplicities in $R(\mathcal{O})$ are either 0 or 1. The last property characterizes spherical orbits. The largest such orbit is called the *model orbit.*

Theorem 4.2.3. *[McGovern3] In each of the classical types, the model orbit is the largest spherical nilpotent orbit and corresponds to the following partition. Let* $\epsilon = 0$ *or* 1.

(i) $[2^n, 1^\epsilon]$ *in type* $A_{2n+\epsilon-1}$,

(ii) $[3, 2^{4m-2\epsilon}, 1^{2\epsilon}]$ *in type* $B_{2(2m-\epsilon)+1}$,

(iii) $[2^n]$ *in type* C,

(iv) $[3, 2^{2m-2}, 1^{1+2\epsilon}]$ *in type* $D_{2(2m+\epsilon)}$.

Spherical orbits may be characterized alternately as those nilpotent coadjoint orbits contained in the closure of the model orbit. Yet another characterization of spherical orbits is that they are precisely the orbits that contain a representative which is a sum a root vectors corresponding to orthogonal simple roots [Panyushev2] [McGovern3].

4.2.2 Smith Orbits

Let $\mathcal{V}$ be an orbital variety and let $Q_{\mathcal{V}}$ be its stabilizer in G. To proceed as in our model example, we would like the action of $Q_{\mathcal{V}}$ on $\mathcal{V}$ to admit a dense orbit. As an example above showed, this is unfortunately not always true, but instead, leads us to a definition.

Definition 4.2.4. [Melnikov] An orbital variety $\mathcal{V} \subset \mathcal{O}$ is of S-type iff it admits a dense $Q_{\mathcal{V}}$ orbit. A nilpotent coadjoint orbit is a S-type iff all its orbital varieties are of S-type.

Fact 4.2.5. *If $\mathcal{O}$ is a spherical nilpotent orbit of a classical Lie group, then $\mathcal{O}$ is of S-type.*

Proof. This is a consequence of the dimension argument in Corollary 4.4.2. □

From now on, we will restrict our attention to the setting of spherical nilpotent orbits. For all spherical nilpotent orbits, we can use the methods of our model example to calculate the infinitesimal character associated to $V(\mathcal{V}, \pi)$. For completeness, we give a description of the S-type orbits in groups of type A.

Proposition 4.2.6. *[Melnikov] Suppose that $n \geq 13$, the partition λ has $\lambda_2 > 2$, and*

$$(5, 3, 1, \dots) \leq \lambda \leq (n-4, 4)$$

in the usual partial order on partitions. Then the orbit $\mathcal{O}_\lambda$ in type A_{n-1} is not of S-type.

In light of this, one should also expect orbits in other classical types not to be of S-type whenever a type A orbit in $\pi_A(\lambda)$ is not S-type. Nevertheless, in addition to spherical nilpotent orbits, Melnikov finds other classes of nilpotent orbits in type A that are S-type. Note, however, that her results are incomplete, as they fail to resolve the status of a number of nilpotent orbits in type A.

Proposition 4.2.7. *[Melnikov] A nilpotent orbit $\mathcal{O}_\lambda$ in type A is of S-type whenever λ satisfies one of the following:*

(i) $\lambda > (n-4, 4)$,
(ii) $\lambda = (\lambda_1, \lambda_2, 1, \dots, 1)$ *with* $\lambda_2 \leq 2$,
(iii) $\lambda = (2, \dots)$ *where* $\lambda_i \leq 2$ *for all* i.

The question arises as to why we do not address this larger class of S-type orbits instead of just the spherical orbits. We will address this issue later, but the short answer is that it may be necessary to again modify the Graham-Vogan construction for such results to be useful. We finish this section by listing how spherical orbits fit among two other important classes of nilpotent orbits.

4.2.3 Rigid and Special Orbits

There is an order reversing map d on the set of nilpotent orbits in $\mathfrak{g}$. The map d is involutive when it is restricted to its range.

Definition 4.2.8. An orbit in the range of d is called special.

A general characterization of special orbits appears in [Collingwood-McGovern](6.3.7). We interpret it among small orbits.

Fact 4.2.9. *The following is a complete list of special spherical orbits in classical Lie groups:*

(i) All orbits are special in type A,
(ii) the orbits corresponding to partitions of the form $[3, 2^{2b}, 1^c]$ and $[1^c]$ in type B,
(iii) the orbits corresponding to the partitions of the form $[2^{2b}, 1^{2c}]$ and $[2^b]$ in type C,
(iv) the orbits corresponding to the partitions of the form $[2^{2b}, 1^c]$ and $[3, 1^c]$ in type D.

Definition 4.2.10. A nilpotent orbit is *rigid* in $\mathfrak{g}$ if it is not induced from any proper parabolic subalgebra.

Proposition 4.2.11. *[Collingwood-McGovern] An nilpotent orbit corresponding to the partition $[p_1, p_2, \dots]$ is rigid iff*

- $0 \leq p_{i+1} \leq p_i \leq p_{i+1} + 1$ *for all i, and*
- $|\{j \mid p_j = i\}| \neq 2$ *if $\epsilon(-1)^i = -1$,*

where $\epsilon = \pm 1$ and is precisely defined as in Chapter 3.

Fact 4.2.12. *The following is a complete list of non-rigid spherical orbits in classical Lie groups:*

(i) All non-zero orbits are not rigid in type A,
(ii) the orbits corresponding to partitions of the form $[3, 1^{2c}]$ and $[2^{2b}, 1^2]$ in type B,
(iii) the orbits corresponding to the partitions of the form $[2^2, 1^{2c}]$ and $[2^{2c}]$ in type C, and
(iv) the orbits corresponding to the partitions of the form $[3, 1^c]$ and $[2^{2c}]$ in type D.

4.3 Basepoints in $\mathcal{V}_T$

From the previous sections, we know that each spherical orbital variety $\mathcal{V}$ contains a point whose orbit under the Borel subgroup is dense in $\mathcal{V}$. For our purposes, we would like a short expression for some such point to simplify the forthcoming calculations. For orbital varieties within classical nilpotent orbits, such an expression can be easily read off from the standard tableau corresponding to $\mathcal{V}$.

In type A, such a basepoint is essentially defined in [Melnikov]. We provide a slightly more general construction and extend the result to other classical types. The main tool for the latter is the surjection from domino tableaux onto standard Young tableaux defined in [Carre-Leclerc]. It induces a map on the level of orbital varieties that helps us define the basepoint in the "type A component" of each $\mathcal{V}$.

4.3.1 Notation

Let $\mathfrak{g}$ be a Lie algebra of classical type. For a fixed type X, we will write $\mathfrak{g}_n$ for the Lie algebra of rank n. Let Δ^+ be the set of its positive roots, and Π for its set of simple roots. To fix notation, we let Π equal

$$\begin{array}{ll} \{e_i - e_{i+1}\} & \text{(type } A) \\ \{e_i - e_{i+1}, e_n\} & \text{(type } B) \\ \{e_i - e_{i+1}, 2e_n\} & \text{(type } C) \\ \{e_i - e_{i+1}, e_{n-1} + e_n\} & \text{(type } D). \end{array}$$

Let $\mathfrak{g}_\alpha$ denote the root space corresponding to $\alpha \in \Delta$ and choose $E_\alpha \in \mathfrak{g}_\alpha$. Also choose T_i such that $\mathfrak{t} = \bigoplus \mathbb{C}T_i$. The triangular decomposition can then be written as

$$\mathfrak{g} = \sum_{-\alpha \in \Delta^+} \mathfrak{g}_\alpha \oplus \mathfrak{t} \oplus \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha.$$

Lie algebras of type A appear naturally inside Lie algebras of the other classical types, and inspired by the results on orbital varieties of the previous chapter, we would like be able to interpret results in type A within the context of the other classical types. To this effect, define two linear maps that relate the Lie algebras of type A to those of other classical types. Let $\mathfrak{g}$ be an algebra of type X_n and define the projection

$$\pi_A : \mathfrak{g} \to \mathfrak{gl}_n$$

by $\pi_A(E_{e_i - e_j}) = E_{e_i - e_j}$, $\pi_A(T_i) = T_i$, zero on the other root spaces, and extending linearly. Here we interpret $E_{e_i - e_j}$ and T_i differently on the domain and range of the map. Also define

$$\iota_A : \mathfrak{gl}_n \to \mathfrak{g}$$

by requiring that $\iota_A \circ \pi_A = I$ and Image $\iota_A = \bigoplus \mathfrak{g}_{e_i - e_j} \oplus \mathfrak{t}$.

To facilitate induction, we also define a linear map ι that embeds each Lie algebra of type X_{n-1} into the Lie algebra of type X_n. Somewhat informally

$$\iota : \mathfrak{g}_{n-1} \hookrightarrow \mathfrak{g}_n$$

is defined by adding one to each index that appears in the root space decomposition. For example, in type A, $\iota(\mathfrak{g}_{e_i - e_j}) = \mathfrak{g}_{e_{i+1} - e_{j+1}}$ and $\iota(\mathfrak{t}_i) = \mathfrak{t}_{i+1}$. Finally, for an integer i, define $\tilde{i} = n + 1 - i$, where n is the size of the underlying algebra. In particular, this means $\iota(\mathfrak{g}_{e_{\tilde{i}} - e_{\tilde{j}}}) = \mathfrak{g}_{e_{\tilde{i}} - e_{\tilde{j}}}$. While this notation is somewhat confusing, as it means different things for Lie algebras of different rank, it does simplify expressions in later calculations. We hope the reader forgives us, as the rank should always be clear from context.

4.3.2 Type A

Consider a spherical nilpotent orbit $\mathcal{O}$ and let $\mathcal{V}_T \subset \mathcal{O}$ be the orbital variety associated to the standard Young tableau $T \in YT(n)$. Let T^i denote the set of labels contained in the i-th column of T, so that in our case $T^i = \emptyset$ if $i > 2$. We will define a point f_T contained in $\mathcal{V}_T$ whose orbit under the Borel subgroup is dense in $\mathcal{V}_T$.

Fact 4.3.1. *Let $\phi : T^2 \longrightarrow T^1$ be an injection with the property that $\phi(k) < k$ for all $k \in T^2$. Such a map always exists, and furthermore, the point*

$$f_T = \sum_{k \in T^2} E_{e_{\tilde{k}} - e_{\widetilde{\phi(k)}}}$$

is contained in the variety $\mathcal{V}_T$.

Proof. The fact that a map ϕ always exists is clear by inspection. A spherical nilpotent orbit in type A is uniquely determined by the rank of its elements. For each f_T defined above, $f_T^2 = 0$, so it lies in *some* spherical orbital variety. That it lies precisely in $\mathcal{V}_T$ follows from induction and the above rank condition. $\square$

This definition includes Melnikov's construction as a special case. More precisely, it is always possible to choose ϕ in such a way so that $\phi(k) = k-1$ whenever $\alpha_{\widetilde{k-1}} \notin \tau(T)$. In this incarnation, f_T is a *minimal representative* of $\mathcal{V}_T$ in the sense described below. Let $f \in \mathfrak{n}$ and for its root space decomposition, let us write

$$f = \sum_{\epsilon \in \Delta^+} c_\epsilon(f) E_\epsilon.$$

Definition 4.3.2. An element $f \in \mathcal{V}$ is a representative of $\mathcal{V}$ if f does not belong to any other orbital varieties. A representative f of $\mathcal{V}$ is minimal if

1. each $c_\epsilon(f) \in \mathbb{Z}$,
2. for every $\alpha_i \notin \tau(\mathcal{V})$, $c_{\alpha_i}(f) \neq 0$,
3. If g is another representative of $\mathcal{V}$ satisfying the above, the the number of non-zero $c_\epsilon(g)$ will be greater than or equal to the number of non-zero $c_\epsilon(f)$.

We would like the basepoints we choose to be minimal representatives, as these conditions will simplify the ultimate infinitesimal character calculations. In type A, we have already seen that this is always possible and in further work we would like f_T to be close to satisfying this condition.

Example 4.3.3. Consider the orbital variety $\mathcal{V}_T$ associated with the standard Young tableau

$$\mathrm{T} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & \multicolumn{1}{c}{} \\ \cline{1-1} 5 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

The points

$$f_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } f_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

both lie in $\mathcal{V}_T$ and are both defined as f_T by Fact 4.3.1, each by different choice of ϕ. For the first, $\phi(3) = 2$ while for the second, $\phi(3) = 1$. However, only f_1 is a minimal representative of $\mathcal{V}_T$.

4.3.3 Other Classical Types

Let X = B, C, or D, and let $\mathcal{V}_T$ be the orbital variety in a spherical nilpotent orbit of type X associated with the standard tableau T. The basepoint that we define in $\mathcal{V}_T$ is the sum of two points,

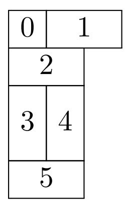
- $f_{\pi_A(T)}$, a version the basepoint of the type A orbital variety associated to $\mathcal{V}_T$ by the tableau projection map and
- M_T^X, a point defined from the set of horizontal dominos in T.

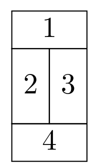
To define the second term, we first need to distinguish among the different types of horizontal dominos that appear in the domino tableaux of spherical orbital varieties. For each such tableau T, define the set of $N^T = \{k \in \mathit{Labels}(T) \,|\, D(k) \text{ is horizontal}\}$ to be the set of its horizontal dominos. Let S^T be the set of dominos in N^T that intersect the first column of T, or more formally, $S^T = \{k \in N^T \,|\, k \in T^1\}$. If M is a family of sets of integers, let M° denote the union of all integers contained in elements of M. We now inductively define a set ${N_1}^T$ of *pairs* of labels in T by $N_1^\varnothing = \varnothing$ and

$$N_1^T = \begin{cases} N_1^{T(n-1)} \cup \{\{k,n\}\} & \text{if } D(n) \in S^T \setminus (N_1^{T(n-1)})^\circ \\ & \text{ and if } X = C,\ k = n-1, \\ N_1^{T(n-1)} & \text{otherwise.} \end{cases}$$

Finally, let $N_2^T = S^T \setminus (N_1^T)^\circ$ and $N_3^T = N^T \setminus ((N_1^T)^\circ \cup N_2^T)$. Note that the set N_3^T is always empty in type C while N_2^T is always empty in types B and D.

Example 4.3.4. Suppose T and U are the following domino tableau:

$T =$ 

$U =$ 

For the tableau T of type B, $S^T = \{2,5\}$, $N_1^T = \{\{2,5\}\}$, $N_2^T = \varnothing$, and $N_3^T = \{1\}$. If we consider U as a domino of type C, then $S^U = \{1,4\}$, $N_1^U = \varnothing$, $N_2^U = \{1,4\}$, and $N_3^U = \varnothing$. If we consider U as a domino of type D, then $S^U = \{1,4\}$, $N_1^U = \{\{1,4\}\}$, $N_2^U = \varnothing$, and $N_3^U = \varnothing$.

We are now ready to define the point M_T^X. Let $M_T^X = \sum_{\{i,j\} \in N_1^T} E_{e_{\tilde{i}} + e_{\tilde{j}}} + \sum_{\alpha \in U_T^X} E_\alpha$, where for a standard tableau T in type X, U_T^X is the set of roots described by

$$U_T^X = \begin{cases} \{e_{\widetilde{k-1}} + e_{\tilde{k}} \,|\, k \in N_3^T\} & X = D \\ \{2e_{\tilde{k}} \,|\, N_2^T\} & X = C \\ \{e_{\tilde{k}} \,|\, k \in N_3^T\} \cup \{e_{\tilde{3}} \,|\, 2 \in T^3 \text{ and } 3 \in T^2\} & X = B \end{cases}$$

Definition 4.3.5. For X= B, C, or D, and a domino tableau T, let

$$f_T^X = f_{\pi_A(T)} + M_T^X$$

where $f_{\pi_A(T)}$ is interpreted as lying inside the Lie algebra of type X.

Lemma 4.3.6. *The point f_T is a minimal representative of $\mathcal{V}_T$.*

Proof. In type A, this is [Melnikov]. For other classical types, we define $T' = \Phi^{-1}(T) \in \Sigma DT_{cl}(\mathit{shape}\,T)$. We first show that $f_T \in \mathcal{V}_S$, where $S = \Phi(T^*)$ and $T^* \in \Sigma DT_{cl}(\mathit{shape}\,T)$ has the same underlying domino tableau as T'. We then show that T' and T^* must share the set of closed clusters with positive sign, which implies that $S = T$ by the definition of

Φ. This verifies that f_T is a representative of $\mathcal{V}_T$. Minimality of f_T may then be checked by inspection.

We would like to show that for all $k \leq n$, $f_{T(k)} \in \mathcal{O}_{\text{shape } T'(k)}$. By induction, it is enough to verify this for $k = n-1$. Note that for spherical orbits, the partition of the orbit containing a nilpotent element f is completely determined by rank f and rank f^2. The above statement can be now verified by inspecting the definition of f_T and comparing rank $f_{T(n-1)}$ and rank $f^2_{T(n-1)}$ with rank f_T and rank f^2_T. In this way, we have $f_T \in \mathcal{V}_S$, where $S = \Phi(T^*)$ and T^* is some tableau in $\Sigma DT_{cl}(shape\, T)$ sharing its underlying tableau with T'.

Now note that if $\mathcal{C}$ is a closed cluster of T' or T^*, then because the orbit $\mathcal{O}_{shape\, T}$ is spherical, the initial cycle $I_{\mathcal{C}}$ through $\mathcal{C}$ must have the form $I_{\mathcal{C}} = \{i, i+1, \ldots, j\}$. Theorem 3.5.1 implies that the simple root $\alpha_{\tilde{i}} \in \tau(T)$ iff there is a closed cluster $C \in \mathcal{C}^+$ with $I_{\mathcal{C}} = \{i, i+1, \ldots, j\}$ for some j. Further note that if $C \in \mathcal{C}^+$, then $E_{e_i+e_j}$ appears in the expansion of f_T with non-zero coefficient while $E_{e_i-e_j}$ has coefficient zero. Similarly, if $C \in \mathcal{C}^-$, then $E_{e_i-e_j}$ appears in the expansion of f_T with non-zero coefficient while $E_{e_i-e_j}$ has coefficient zero. But this forces $\Phi(T^*)$ to have the same τ-invariant as $\Phi(T')$, which implies that $\Phi(T^*) = \Phi(T')$. Hence f_T is a representative of $\mathcal{V}_T$. □

Lemma 4.3.7. *Consider an orbital variety $\mathcal{V}_T$ in a spherical nilpotent orbit of classical type that corresponds to the standard tableau T, and let $Q = Q_{\mathcal{V}_T}$ be the maximal parabolic stabilizing it. Then the orbits*

$$B \cdot f_T \text{ and } Q \cdot f_T$$

are dense in $\mathcal{V}_T$.

Proof. For the result in in type A, see [Melnikov](Proposition 4.13). In general, denseness follows by induction from Corollary 4.4.2 below. □

Example 4.3.8. Let $X = C$ and consider the orbital variety $\mathcal{V}_T$ associated with the domino tableau

T=
<table>
<tr><td colspan="2">1</td></tr>
<tr><td rowspan="2">2</td><td rowspan="2">3</td></tr>
<tr></tr>
<tr><td colspan="2">4</td></tr>
<tr><td colspan="2">5</td></tr>
</table>

. Then the Young tableau $\pi_A(T) =$
<table>
<tr><td>1</td><td>3</td></tr>
<tr><td>2</td></tr>
<tr><td>4</td></tr>
<tr><td>5</td></tr>
</table>

.

We have $N = \{1,4,5\}$, $N_1 = \{1\}$ and $N_2 = N \setminus N_2$. Finally, the basepoint $f_T = \begin{pmatrix} A & M \\ 0 & -A^t \end{pmatrix}$, where

$$A = f_{\pi_A(T)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

4.4 Induction

Our calculation of infinitesimal characters of Graham-Vogan representations attached to the orbital variety $\mathcal{V}_T$ will proceed by a type of induction on the standard tableau T. As in our model example, we would like to describe the action of Q_f on the space $\mathfrak{q}/\mathfrak{q}_\mathfrak{f}$. Because we consider only spherical orbits, it is equivalent to describe this action on the isomorphic space $\mathfrak{b}/\mathfrak{b}_\mathfrak{f}$. In this section, we describe this space inductively, verifying Lemma 4.3.7 in the process.

Fix a standard tableau T of a given classical type and write $\mathcal{V}_T$ for the orbital variety corresponding to it. Ideally, we would like to be able extract information about the orbital variety $\mathcal{V}_T$ from the orbital variety $\mathcal{V}_{T(n-1)}$ and in this manner set up a type of induction. Based on our construction of minimal representatives, there are two instances where an inductive step is not completely apparent. First, in type C, when both $D(n)$ and $D(n-1)$ are horizontal dominos, while $D(n-2)$ is not, it appears that one should let the induction take place from the orbital variety $\mathcal{V}_{T(n-2)}$ to $\mathcal{V}_{T(n)}$. A somewhat more serious problem is that in types B and D, the standard domino tableau $T(n-1)$ does not always correspond to an orbital variety of the same classical type as $\mathcal{V}_T$, so in order for induction to make sense, we have to be careful defining the appropriate orbital variety. To this effect, we define a standard tableau $T^\downarrow$ by

$$T^\downarrow = \begin{cases} MT(\mathcal{C}, T(n-1)) & X = B \text{ or } D, \text{ type } \mathcal{V}_{T(n-1)} \neq X \text{ and} \\ & \mathcal{C} \text{ the cycle in } T(n-1) \text{ through } n-1, \\ T(n-2) & X = C,\ D(n) \text{ and } D(n-1) \text{ are horizontal dominos,} \\ & \text{ while } D(n-2) \text{ is not,} \\ T(n-1) & \text{otherwise.} \end{cases}$$

With this definition, shape $T^\downarrow$ and shape T are partitions of the same classical type. Therefore, we are able to associate an orbital variety $\mathcal{V}_{T^\downarrow}$ of the same type as $\mathcal{V}_T$ to the standard tableau $T^\downarrow$. We will write $f^\downarrow$ for $f_{T^\downarrow}$, and $\mathfrak{b}^\downarrow, \mathfrak{q}^\downarrow$, and $\mathfrak{g}^\downarrow$ for the Lie algebras corresponding to $\mathcal{V}_{T^\downarrow}$.

As in our model example, we would like to describe the action of Q_f on the space $U = \mathfrak{b}/\mathfrak{b}_{f_T}$. If we think inductively, however, we can break this task down into a study of the quotients

$$U_n = (\mathfrak{b}/\mathfrak{b}_{f_T})/(\mathfrak{b}^\downarrow/\mathfrak{b}^\downarrow_{f^\downarrow}).$$

In fact,

$$\bigoplus_{k \leq n} U_k \simeq U.$$

It will be often convenient to divide our work into cases that arise from an inductive construction of of the representative f_T. The cases are distinguished by the possible forms of the difference $f_T - \iota(f_{T^\downarrow})$. We describe the possiblities along with what they imply on the level of tableaux.

(C1) When $f_T = \iota(f_{T^\downarrow})$, the domino $T \setminus T^\downarrow$ lies entirely in the first column of T.

(C2) When $f_T = \iota(f_{T^\downarrow}) + E_{e_1 - e_{\tilde{\phi}(n)}}$, the domino $T \setminus T^\downarrow$ lies entirely in the second column of T.

(N1) When $f_T = \iota(f_{T^\downarrow}) + E_{e_1+e_{\tilde{k}}}$ and $X = B$ or D, then this is the case when $T^\downarrow \neq T(n-1)$ and $\{k, k+1, \ldots, n-1\}$ is a cycle in $T(n-1)$. If $X = C$ and $\tilde{k} \neq 2$, then $k = n-1$ and $T^\downarrow = T(n-2)$.

(N2) When $f_T = \iota(f_{T^\downarrow}) + E_{2e_1}$, then $X = C$ and $T^\downarrow = T(n-1)$.

(N3) When $f_T = \iota(f_{T^\downarrow}) + E_{e_1-e_2} + E_{e_1}$, we have $X = B$ and $T \setminus T^\downarrow$ is a horizontal domino that intersects the third column of T. When $f_T = \iota(f_{T^\downarrow}) + E_{e_1-e_2} + E_{e_1+e_2}$, we have $X = D$ and again $T \setminus T^\downarrow$ is a horizontal domino that intersects the third column of T.

(*) When $f_T = \iota(f_{T^\downarrow}) + E_{e_1}$, we have $X = B$ and $T \setminus T^\downarrow = D(3) \in T^2$ while $D(2) \in N_3^T$.

We can now attack the description of the space U_n. Because we would like our description to reflect the original action of Q_{f_T}, we infact describe the quotient

$$U_n = (\mathfrak{b}/\mathfrak{b}_{f_T}) / \left(\iota(\mathfrak{b}^\downarrow)/\iota(\mathfrak{b}^\downarrow_{\iota(f^\downarrow)}) \right)$$

and embed it in $\mathfrak{b}$. There is a certain amount of choice possible in the parametrization of U_n; however, the determinant of the Q_{f_T} action is independent of these choices.

Lemma 4.4.1. *Consider a standard tableau T and recall our construction of f_T, a representative of the orbital variety $\mathcal{V}_T$. The group Q_{f_T} acts on the space $\mathfrak{b}/\mathfrak{b}_{f_T}$ and this action restricts to the quotient*

$$U_n = (\mathfrak{b}/\mathfrak{b}_{f_T}) / \left(\iota(\mathfrak{b}^\downarrow)/\iota(\mathfrak{b}^\downarrow_{\iota(f^\downarrow)}) \right).$$

In each of the above cases, we describe a space $U_n' \subset \mathfrak{b}$ with the property that the determinant of the natural action of Q_{f_T} equals the determinant of the Q_{f_T} action on the quotient U_n. Recall that for a standard tableau S, we denote the set of labels contained in column j of S by S^j. When the square or domino with label n lies entirely in the first column of T, that is, case (C1),

$$U_n = \begin{cases} \bigoplus_{T^2} \mathfrak{g}_{e_1 - e_{\tilde{i}}} & \textit{in type } A \\ \bigoplus_{\substack{(\pi_A T)^2 \\ N^T \cup *}} \mathfrak{g}_{e_1 - e_{\tilde{i}}} \oplus \bigoplus_{(\pi_A T)^2} \mathfrak{g}_{e_1 + e_{\widetilde{\phi(i)}}} & \textit{other types.} \end{cases} \tag{4.1}$$

When the square or domino with label n lies entirely in the second column of T, that is, case (C2), we define two spaces V and W by

$$V = \bigoplus_{\substack{j > \widetilde{\phi(n)} \\ j \notin \tilde{N} \cup \widetilde{(\pi_A T)}^2}} \mathfrak{g}_{e_{\tilde{\phi}(n)} - e_j} \oplus \bigoplus_{\substack{(\pi_A T(n-1))^2 \\ \phi(i) > \phi(n)}} \mathfrak{g}_{e_1 - e_{\tilde{i}}} \oplus \mathfrak{t}_1$$

$$W = \bigoplus_{\substack{j \neq \tilde{\phi}(n) \cup 1 \\ j \neq \tilde{\phi}(i) \\ i \in \widetilde{(\pi_A T^\downarrow)}^2}} \mathfrak{g}_{e_{\tilde{\phi}(n)} + e_j}.$$

We can then say

$$U_n = \begin{cases} V & \textit{in type } A \\ V \oplus W \oplus \mathfrak{g}_{e_1} \oplus_{N_3^T=\varnothing} \mathfrak{g}_{e_{\tilde{\phi}(n)}} & \textit{type } B \\ V \oplus W \oplus \mathfrak{g}_{2e_{\tilde{\phi}(n)}} \oplus \mathfrak{g}_{e_1+e_{\tilde{\phi}(n)}} & \textit{type } C \\ V \oplus W \oplus_{N_3^T} (\mathfrak{g}_{e_1-e_{\tilde{i}}} \oplus \mathfrak{g}_{e_1-e_{\widetilde{i-1}}}) & \textit{type } D. \end{cases} \tag{4.2}$$

Now suppose that we are in case (N1) and $\{\{k,n\}\} = N_1^T$. *Then in each of the classical types not equal to* A,

$$U_n = \bigoplus_{(\pi_A T^{\downarrow})^2} \mathfrak{g}_{e_1+e_{\tilde{\phi}(i)}} \oplus \bigoplus_{(N_1^{T^{\downarrow}})^{\circ}} \mathfrak{g}_{e_1-e_{\tilde{j}}} \oplus \bigoplus_{(\pi_A T^{\downarrow})^2} \mathfrak{g}_{e_1-e_{\tilde{i}}} \oplus \mathfrak{t}_1 \tag{4.3}$$

Case (N2) appears only in type C *and there*

$$\begin{aligned} U_n = & \bigoplus_{(\pi_A T^{\downarrow})^2} (\mathfrak{g}_{e_1-e_{\tilde{i}}} \oplus \mathfrak{g}_{e_2-e_{\tilde{i}}}) \oplus \bigoplus_{\substack{N^{T^{\downarrow}} \\ \textit{not } \mathfrak{g}_{e_2-e_1}}} (\mathfrak{g}_{e_1-e_{\tilde{i}}} \oplus \mathfrak{g}_{e_2-e_{\tilde{i}}}) \oplus \\ & \bigoplus_{(\pi_A T^{\downarrow})^2} (\mathfrak{g}_{e_1+\tilde{\phi}(i)} \oplus \mathfrak{g}_{e_2+\tilde{\phi}(i)}) \oplus \mathfrak{t}_1 \end{aligned} \tag{4.4}$$

Case (N3) appears only in types B *and* D.

$$U_n = \begin{cases} \bigoplus_{j>2} (\mathfrak{g}_{e_2-e_j} \oplus \mathfrak{g}_{e_2+e_j}) \oplus \mathfrak{g}_{e_2} \oplus \mathfrak{t}_1 \oplus \mathfrak{t}_2, & \textit{type } B \\ \bigoplus_{j>2} (\mathfrak{g}_{e_2-e_j} \oplus \mathfrak{g}_{e_2+e_j}) \oplus \mathfrak{t}_1 \oplus \mathfrak{t}_2 & \textit{type } D \end{cases} \tag{4.5}$$

Finally, while in the special case of (*),

$$U_3 = \mathfrak{g}_{e_1-e_2} \oplus \mathfrak{g}_{e_1+e_3} \oplus \mathfrak{g}_{e_2}. \tag{4.6}$$

Proof. First form a decomposition $\mathfrak{b} = \mathfrak{b}_1 \oplus \iota(\mathfrak{b}^{\downarrow})$ that is compatible with the root space decomposition. For $B \in \mathfrak{b}$, write $B = B_1 + B_2$ with $B_1 \in \mathfrak{b}_1$ and $B_2 \in \iota(\mathfrak{b}^{\downarrow})$. Note that $B \in \mathfrak{b}_{f_T}$ if and only if

$$[B, f_T] = 0. \tag{4.7}$$

To describe U_n, we assume that B_2 lies in $\iota(\mathfrak{b}_{f_{T^{\downarrow}}})$, i.e. that

$$[B_2, \iota(f_{T^{\downarrow}})] = 0. \tag{4.8}$$

We would like to know what additional conditions on B are necessary to make sure that it satisfies (4.7). If we write

$$B = \sum_{\alpha \in \Delta^+} c_{\alpha} E_{\alpha} + \sum_{i \leq n} c_i T_i,$$

then (4.7) imposes linear conditions on the coefficients c in the expansion of B. If we choose a representative α or i within each linear condition and denote the set of representatives by P, then

$$\mathfrak{b}/\mathfrak{b}_f \simeq \bigoplus_{\alpha\in P} \mathfrak{g}_\alpha \oplus \bigoplus_{i\in P} \mathfrak{t}_i.$$

The natural action of Q_f has the same determinant on both spaces. To describe U'_n, we only need to include representatives for linear conditions that do not already arise as conditions for (4.8). We carry out this plan in each of the cases by describing the set of representatives in each of the cases.

Case (C1). In this case, $f_T = \iota(f_{T^\downarrow})$. Condition (4.7) boils down to

$$[B_1, \iota(f_{T^\downarrow})] = 0. \tag{4.9}$$

Write $B_1 = \sum_S c_\alpha E_\alpha + c_1 T_1$. If we expand the left hand side of (4.9) in terms of root space coordinates, the resulting linear conditions imposed by (4.9) all take the form $c_\gamma = 0$ for γ in some set Ω. The quotient U'_n then takes the form $\bigoplus_\Omega \mathfrak{g}_\alpha$. Deciphering (4.9) explicitly leads to the description in the statement of the lemma.

Case (C2). In this case, $f_T = \iota(f_{T^\downarrow}) + E_{e_1 - e_{\tilde\phi(n)}}$. Equation (4.7) reduces to

$$[B_1, \iota(f_{T^\downarrow})] + [B_1, E_{e_1 - e_{\tilde\phi(n)}}] + [B_2, E_{e_1 - e_{\tilde\phi(n)}}] = 0. \tag{4.10}$$

We can again write $B_1 = \sum_S c_\alpha E_\alpha + c_1 T_1$ and expand (4.10) in terms of root space coordinates. This time, (4.10) imposes more complicated conditions on the coefficients. For each linear condition on the coefficient c obtained from (4.10), we select as representative the largest root γ such that c_γ appears within the linear equation. If , however, c_i also appears within a linear condition, we select the coefficient i instead. When we account for linear conditions that already appear in (4.8), we obtain the description of U'_n in the statement of the lemma.

Case (N1). In this case, $f_T = \iota(f_{T^\downarrow}) + E_{e_1 + e_{\tilde k}}$. Equation (4.7) reduces to

$$[B_1, \iota(f_{T^\downarrow})] + [B_1, E_{e_1 + e_{\tilde k}}] + [B_2, E_{e_1 + e_{\tilde k}}] = 0. \tag{4.11}$$

In types B and D, the method of case (C2) can be used verbatim, we only have to account for the different linear conditions imposed by (4.11). When $X = C$, we merely have to account for the different definition of $T^\downarrow$ in this case by letting $B_1 = \sum_S c_\alpha E_\alpha + c_1 T_1 + c_2 T_2$ for the appropriate set S.

Case (N2). In this case, $f_T = \iota(f_{T^\downarrow}) + E_{e_1 + e_2}$. Equation (4.7) reduces to

$$[B_1, \iota(f_{T^\downarrow})] + [B_1, E_{e_1 + e_2}] + [B_2, E_{e_1 + e_2}] = 0$$

and the method of case (C2) can again be used verbatim to describe U'_n.

Case (N3). In type B, $f_T = \iota(f_{T^\downarrow}) + E_{e_1 - e_2} + E_{e_1}$, while in type D, $f_T = \iota(f_{T^\downarrow}) + E_{e_1 - e_2} + E_{e_1 + e_2}$. In both cases, $f_{T^\downarrow} = 0$ and $\mathfrak{b}_{f_{T^\downarrow}} = \mathfrak{b}^\downarrow$. Hence equation (4.7) reduces to

$$[B, E_{e_1 - e_2}] + [B, E_{e_1 + e_2}] = 0 \tag{4.12}$$

in type B and

$$[B, E_{e_1-e_2}] + [B, E_{e_1+e_2}] = 0 \tag{4.13}$$

in type D.

Case (*). In this case, $f_T = \iota(f_{T\downarrow}) + E_{e_1}$.Equation (4.7) reduces to

$$[B_1, \iota(f_{T\downarrow})] + [B_1, E_{e_1}] + [B_2, E_{e_1}] = 0$$

and the method of case (C2) can again be used verbatim to describe U_n'. □

Corollary 4.4.2. *For a standard tableau T, $\dim U_n = \dim \mathcal{V}_T - \dim \mathcal{V}_{T\downarrow}$.*

Proof. We can compute $\dim \mathcal{V}_T - \dim \mathcal{V}_{T\downarrow}$ from the formula for the dimension of a nilpotent orbit. Let $[\lambda_1, \cdots \lambda_p]$ be the dual partition to shape T. In each of the cases, $\dim \mathcal{V}_T - \dim \mathcal{V}_{T\downarrow}$ equals

$$\frac{1}{2}(\dim \mathcal{O}_{shape\,T} - \dim \mathcal{O}_{shape\,T\downarrow}) = \begin{cases} \lambda_2 + \lambda_3 & \text{Case (C1)} \\ \lambda_1 & \text{Case (C2) and } X = A \\ \lambda_1 - 1 + \lambda_3 & \text{Case (C2) and } X = B \text{ or } D \\ \lambda_1 + 1 & \text{Case (C2) and } X = C \\ \lambda_1 & \text{Case (N1) and } X = B \text{ or } C \\ \lambda_1 - 1 & \text{Case (N1) and } X = D \\ 2\lambda_1 - 1 & \text{Case (N2)} \\ \lambda_1 & \text{Cases (N3) and (*)} \end{cases}$$

One can now check these are exactly the dimensions of the corresponding spaces U_n. We detail the calculation in the inductive case $(C2)$ when the group is of classical type C. The other cases are not dissimilar. Recall the two types of vertical dominos that arise within a domino tableau, and denote by I^- and I^+ the set of labels of the dominos of that type that are contained in the tableau T. Reading off the parameterization of U_n in this case implies that $\dim U_n$ equals

$$\begin{aligned} \#\left\{j < \phi(n) \,|\, j \notin N \cup (\pi_A T)^2\right\} &+ \#\left\{j \;\; j \neq \phi(i) \text{ for } i \in (\pi_A T)^2, j \neq n\right\} \\ &+ \#\left\{i \in (\pi_A T)^2 \,|\, \phi(i) > \phi(n)\right\} + 3 \\ &= \left\{j \in I^- \,|\, j \neq \phi(n), \text{ and if } j > \phi(n), \text{ then } j \in Im\,\phi\right\} \\ &+ \left\{j \in I^- \cup N \,|\, j \neq n\right\} \\ &= (\#I^- - 1) + (\#I^- + \#N - 1) + 3 \\ &= 2\#I^- + \#N + 1 = \lambda_1 + 1, \end{aligned}$$

as desired.

□

4.5 The Trace of the Adjoint Action

Let $\mathfrak{t}_f$ be a maximal torus inside the Lie algebra $\mathfrak{q}_f$. It can be verified that the basepoint f_T was chosen so that $\mathfrak{q}_{f_T} \cap \mathfrak{t}$ is a maximal torus in $\mathfrak{q}_{f_T}$. The inductive procedure of the previous sections provides a quick description of the coordinates of $\mathfrak{t}_f$. The trace of the adjoint action of $\mathfrak{t}_f$ on $\mathfrak{q}/\mathfrak{q}_f$ can then be calculated as a sum of the traces of the actions of the quotient spaces U_i. In keeping with the inductive philosphy of this chapter, we compute this trace on the space U_n, separating each of the inductive cases.

Fact 4.5.1. *Let $f = f_T$ and write an element $a \in \mathfrak{t}$ as $a = \sum_{1\leq i\leq n} a_i \, \mathfrak{t}_i$. Then a lies in $\mathfrak{t}_f$ iff $\sum_{2\leq i\leq n} a_i \, \mathfrak{t}_i$ lies in the torus $\iota(\mathfrak{t})_{\iota(f^{\downarrow})}$ and additionally*

- $a_1 = a_{\tilde{\phi}(n)}$ *in case (C2),*
- $a_1 = -a_{\tilde{k}}$ *in case (N1), where $\{k, n\}$ is a pair in N_1^T,*
- $a_1 = 0$ *in cases (N2), (N3), as well as (*).*

Proof. This follows immediately from an inductive description of the basepoint f_T in each of the above cases. ☐

Proposition 4.5.2. *Let $[p_1, p_2, \ldots, p_l]$ be the partition that corresponds to the nilpotent orbit $\mathcal{O}$ through f_T. Write $[\lambda_1, \lambda_2, \ldots, \lambda_m]$ for its dual partition. Finally, write an arbitrary element of $\mathfrak{t}$ as $\sum_{i\leq n} a_i \, \mathfrak{t}_i$. The trace of the adjoint action of $\mathfrak{t}_f$ on the quotient U_n is listed below, sorted according to the inductive cases.*

(C1) In type A, the trace is $-\lambda_2 \, a_1 + \sum_{i\in T^2} a_{\tilde{\imath}}$, while it is $-(\lambda_2 + \lambda_3)a_1$ in the other classical types.

(C2) In type A, the trace is $-\lambda_1 \, a_1 + \sum_{i\in T^1} a_{\tilde{\imath}}$. In the other classical types with $N_3^T = \emptyset$ while in type D, it is

$$(-\lambda_1 - c)a_1,$$

where

- $c = 2$ *in type C,*
- $c = -2 + \lambda_3$ *in types B and D.*

(N1) Let $\{\{k, n\}\} = N_1^T$. In types B and D, the trace is

$$-(\lambda_1 - c)a_1$$

with c defined as in case (C2). In type C, the trace is 0.

(N2) This case occurs only in type C where the trace is 0.

(N3) This case occurs only in types B and D where the trace is $-\lambda_1 a_1$.

(*) Here , the trace is $-2a_1 - a_3$.

Proof. We use the description of the quotient U_n in Lemma 4.4.1 together with Fact 4.5.1. In type A, determining the trace is simply a matter of reading off the coordinates. We provide the calculations for the other classical types which are only a little more subtle. Write I^+ and I^- for the set of labels of the vertical dominos in T of the corresponding type, and write N for the set of horizontal dominos in T. We will use $|\cdot|$ to denote the order of each of these sets.

Case (C1). By reading off the coordinates, we find that the trace is

$$\sum_{\pi_A(T)^2} (-a_1 + a_{\tilde{\imath}}) + \sum_{\pi_A(T)^2} (-a_1 - a_{\tilde{\phi}(i)}) + \sum_{i \in N^T} (-a_1 - a_{\tilde{\imath}})\ (-a_1 + a_{\tilde{3}})$$

where the final parenthetical expression appears iff some sub-tableau of T lies in case (*). Applying (4.5.1) reduces the above to

$$-2\,|(\pi_A T)^2|\, a_1 - \ |N^T|\, a_1\ (-1) = -(\lambda_2 + \lambda_3) a_1.$$

Case (C2). We begin with type B. The trace is

$$\begin{aligned}
&\sum_{\substack{j > \tilde{\phi}(n) \\ j \notin \tilde{N} \cup \widetilde{(\pi_A T)^2}}} (-a_1 + a_j) + \sum_{\substack{(\pi_A T^{\downarrow})^2 \\ \phi(i) > \phi(n)}} (-a_1 + a_{\tilde{\imath}}) + \sum_{\substack{j \notin \tilde{\phi}(n) \cup 1 \\ j \neq \tilde{\phi}(i) \\ i \in (\pi_A T^{\downarrow})^2}} (-a_1 - a_j) \\
= &\sum_{\substack{i < \phi(n) \\ i \notin N \cup (\pi_A T)^2}} (-a_1 + a_{\tilde{\imath}}) + \sum_{\substack{i \in (\pi_A T^{\downarrow})^2 \\ \phi(i) > \phi(n) \\ i > \phi(n)}} (-a_1 + a_{\tilde{\imath}}) + \sum_{\substack{i \notin \phi(n) \cup n \\ i \neq Im(\phi) \\ i \in (\pi_A T^{\downarrow})^2}} (-a_1 - a_{\tilde{\imath}}) \\
= &\sum_{\substack{i < \phi(n) \\ i \notin N \cup (\pi_A T)^2}} (-a_1 + a_j) + \sum_{\substack{i > \phi(n) \\ i \notin N \cup (\pi_A T)^2}} (-a_1 + a_{\tilde{\imath}}) + \sum_{i \neq \phi(n)} (-a_1 - a_{\tilde{\imath}}) + ? + a_1 + (a_1) \\
= &- (2\,(|I^+| - 1) + |N| + 1\ (+1)) a_1
\end{aligned}$$

This is $(-\lambda_1 - \lambda_3 + 2)a_1$. In type C , the calculation is similar, with final line equal to

$$-(2\,(|I^-| - 1) + |N| + 2 + 1 + 1)a_1 = -(\lambda_1 + 2)a_1.$$

In type D, the final line is

$$-(2\,(|I^+| - 1) + |N|)a_1 - (a_1 - a_{\tilde{k}} + a_1 - a_{\tilde{k}}) = -(2\,(|I^+| - 1) + |N|)a_1 - (2a_1 - 2a_{\tilde{k}})$$

where the parenthetical expression appears iff $\{k\} = N_1^T$. This reduces to the statement in the Proposition.

Case (N1). The proof is similar to case (C2).

Case (N2). We find that the trace equals:

$$\sum_{i \in I^+} (a_1 - a_{\tilde{\imath}}) + \sum_{i \in N} (a_1 - a_{\tilde{\imath}}) + \sum_{i \in \phi(I^+)} (a_1 + a_{\tilde{\imath}}).$$

After applying Fact 4.5.1, this expression reduces to 0.

The remaining two cases are simple. □

For future use, let us define the vector $(c_n, c_{n-1}, \dots, c_1)$ by letting c_i equal the number of times the term a_i appears in the expression for the trace of the adjoint action on $\bigoplus_{i \leq n} U_i$ described by Proposition 4.5.2.

Chapter 5

Infinitesimal Characters

Armed with the constructions of the previous chapter, one is ready to examine the representations that arise from the Graham-Vogan construction. As in the model example, we restrict our examination to those representations that arise from spherical orbital varieties, as the corresponding Lagrangian coverings are then just quotients of the group G.

5.1 Characters, Weights, and Extensions

Let $\mathcal{O}$ be a spherical nilpotent orbit of a classical Lie group G, fix a Borel subgroup B, and consider the orbital variety $\mathcal{V} \subset \mathcal{O}$ that corresponds to the standard tableau T by Corollary 3.3.1. Write Q for its stabilizer in G; it is determined explicitly by Theorem 3.5.1. Recall the basepoint $f = f_T \in \mathcal{V}$ and its stablizer $Q_f \subset Q$. Lemma 4.3.7 implies that both, the parabolic Q, and in fact, the Borel B act on f with dense orbit in $\mathcal{V}$.

The Graham-Vogan construction examines the Lagrangian covering:

$$\begin{array}{ccc} & & G \times_Q \mathcal{V}^0 \\ & \swarrow^{\pi} & \downarrow{\scriptstyle \rho} \\ G/G_f & & G/Q \end{array}$$

Because Q/Q_f and B/B_f are dense in $\mathcal{V}$, Graham and Vogan's construction suggests looking at the character α of Q_f that is given by the square root of the absolute value of the real determinant of Q_f acting on the tangent space $\mathfrak{q}/\mathfrak{q}_f$ of $\mathcal{V}$ at f. Utilizing the notation of Chapter 1, one would like to know when a homomorphism j_π exists. This condition translates to the existence of an existence of a representation γ of Q that contains α on Q_f as a subrepresentation. That is,

$$\gamma|_{Q_f} \supset \alpha.$$

Ideally, one would like γ itself to be a character. In this case, j_π is an isomorphism, as required by [Graham-Vogan], rather than just an injection. We examine the weight w_α of the character α. First note that α is a real character. Recall the vector $(c_n, \dots, c_1)$ defined at the end of the previous chapter. If we split the weight of α into holomophic and anti-holomorphic parts, we obtain:

$$w_\alpha = (\tfrac{c_n}{2}, \tfrac{c_{n-1}}{2}, \dots, \tfrac{c_1}{2})(\tfrac{c_n}{2}, \tfrac{c_{n-1}}{2}, \dots, \tfrac{c_1}{2}).$$

Here, we interpret a weight of Q_f as an equivalence class of weights of Q and the above is just a representative of such an equivalence class.

We would like to answer the existence question for γ by examining its corresponding weights. To this effect, first suppose that γ is a real character. Write the Levi subalgebra $\mathfrak{l}$ as a sum of reductive parts as $\bigoplus_{i\leq s}\mathfrak{g}(l_i)$. A *real* character γ of L takes the form

$$\gamma(A) = \prod_{i\leq s} |\det A_i|^{\alpha_i} \tag{5.1}$$

where $A \in L$, $\alpha_i \in \mathbb{R}$ and A_i is the restriction of A to the ith reductive part of L. We can rewrite 5.1 as

$$\gamma(A) = \prod_{i\leq s} (\det A_i)^{\frac{\alpha_i}{2}} \overline{(\det A_i)}^{\frac{\alpha_i}{2}} \tag{5.2}$$

splitting it into holomorphic and anti-holomorphic parts. In this manner, we associate the weight

$$w_\gamma = (\tfrac{\alpha_n}{2}, \tfrac{\alpha_{n-1}}{2}, \dots, \tfrac{\alpha_1}{2})(\tfrac{\alpha_n}{2}, \tfrac{\alpha_{n-1}}{2}, \dots, \tfrac{\alpha_1}{2}).$$

We would like to know conditions under which w_γ lies in the same equivalence class of weights of Q as w_α. In the case of spherical orbits, Fact 4.5.1 imples that this occurs iff

- $\alpha_i + \alpha_j = c_i + c_j$ whenever $i = \phi(j)$,
- $\alpha_i - \alpha_j = c_i - c_j$ whenever $\{i, j\} \in N_1^T$, and
- $\alpha_i = c_i$ for all $i \notin N^T \cup T^2 \cup \phi(T^2)$.

If we write $w_\gamma = w_\alpha + \epsilon$ for some weight ϵ, then these conditions translate to

- $\epsilon_i + \epsilon_j = 0$ whenever $i = \phi(j)$,
- $\epsilon_i - \epsilon_j = 0$ whenever $\{i, j\} \in N_1^T$, and
- $\epsilon_i = 0$ for all $i \notin N^T \cup T^2 \cup \phi(T^2)$.

Denote the set of weights w_γ that satisfy the above conditions by $HW_r(w_\alpha)$. We would also like to know which weights in $HW_r(w_\alpha)$ correspond to a real character γ of Q. Write $HW_r^1(w_\alpha)$ for this set. First, let us define some notation. For a parabolic subgroup Q of G, we group the coordinates that correspond to the same reductive part of its Levi L by setting them off with an additional set of parentheses. If

$$\mathfrak{l} = \bigoplus_{i\leq s}\mathfrak{g}(l_i) \text{ and } \mathfrak{g}(l_j)\cap\mathfrak{t} = \bigoplus_{c_i\leq j\leq d_i} \mathbb{C}T_j,$$

then we will write a weight a as

$$a = ((a_n\ a_{n-1}\ \dots a_{d_1})(a_{c_2}\ \dots\ a_{d_2})\dots(a_{c_k}\dots a_{d_k})\dots(a_{d_l}\dots a_1)).$$

For instance, if $\mathfrak{l} = \mathfrak{gl}_3\oplus\mathfrak{gl}_2\oplus\mathfrak{gl}_1$, then we will write a weight as $a = ((a_6\ a_5\ a_4)\ (a_3\ a_2)\ (a_1))$. This notation provides us with a convenient way of testing which weights correspond to highest weights of one-dimensional representations of the parabolic Q.

Fact 5.1.1. *A weight $a \in HW_r(w)$ lies in $HW_r^1(w)$ iff all coefficients corresponding to a given reductive part of the Levi of Q are the same. That is, iff*

$$a_{c_k} = a_{c_k+1} = \ldots a_{d_k} \text{ for all } 1 \leq k \leq l.$$

Proof. Suppose that a satifies the above hypothesis. Then a character of Q with weight a is given by a product of exponents of absolute values of determinants of the reductive parts of L. The exponent of the determinant of the part corresponding to $\{c_k, c_k+1, \ldots, d_k\}$ is given by twice their common value, as per the descripition of real characters of 5.2. □

If γ is a real character of Q that restricts to α on Q_f, then $w_\gamma \in HW_r^1(w_\alpha)$. Conversely, given a weight w in $HW_r^1(w_\alpha)$, we can construct a real character γ_w according to the procedure described by 5.2.

Now suppose that γ is an arbitrary character of Q that restricts to the real character α on Q_f. Then γ takes the form

$$\gamma = \chi \cdot \gamma'$$

where γ' is a real character such that $\gamma'|_{Q_f} = \alpha$, and χ is a unitary character such that $\chi|_{Q_f} = 1$. In particular, this means that $\chi|_{T_f} = 1$. If we write $A \in T$ as $\sum a_i T_i$, then

$$\chi(A) = \prod_{i \leq n} \left(\tfrac{a_i}{|a_i|}\right)^{\beta_i} = \prod_{i \leq n} a_i^{\frac{\beta_i}{2}} (\overline{a_i})^{-\frac{\beta_i}{2}}.$$

In this manner, we define the weight

$$w_\chi = (\tfrac{\beta_n}{2}, \tfrac{\beta_{n-1}}{2}, \ldots, \tfrac{\beta_1}{2})(-\tfrac{\beta_n}{2}, -\tfrac{\beta_{n-1}}{2}, \ldots, -\tfrac{\beta_1}{2}).$$

The character χ restricts to the identity on Q_f iff w_χ lies in the equivalence class of 0 of weights of Q. This occurs iff

- $\beta_i + \beta_j = 0$ whenever $i = \phi(j)$,
- $\beta_i - \beta_j = 0$ whenever $\{i, j\} \in N_1^T$, and
- $\beta_i = 0$ for all $i \notin N^T \cup T^2 \cup \phi(T^2)$.

Furthermore, because χ should be a unitary character of L, its entries also need to satisfy the conditions of Fact 5.1.1. We can generalize the definitions of $HW_r(w_\alpha)$ and $HW_r^1(w_\alpha)$ in the obvious way.

For a weight w_γ, write w_γ^h and w_γ^a for its holomorphic and anti-holomorphic parts. Note that $w_\gamma \in HW(w_\alpha)$ iff $w_\gamma^h = (d_n, d_{n-1}, \ldots, d_1)$ satisfies:

- $d_i + d_j = c_i + c_j$ whenever $i = \phi(j)$,
- $d_i - d_j = c_i - c_j$ whenever $\{i, j\} \in N_1^T$, and
- $d_i = c_i$ for all $i \notin N^T \cup T^2 \cup \phi(T^2)$.

Furthermore, $w_\gamma \in HW^1(w_\alpha)$ iff $w_\gamma \in HW(w_\alpha)$ and the coefficients d_i of w_γ^h satisfy the conditions of Fact 5.1.1. This analysis simplifies notations by allowing us to refer to weights with only one part. With this characterization in mind, we redefine the sets HW and HW^1.

Definition 5.1.2. Let w be the weight of a one-dimensional representation of Q_f and define $HW(w)$ to be the set of weights of representations of Q that restrict to w on the torus $\mathfrak{t}_f$. Futhermore, let $HW^1(w)$ be the set of weights in $HW(w)$ that correspond to weights of characters of Q.

We would like to answer the following questions:

- As suggested by the Graham-Vogan construction, can α always be extended to a character γ of Q? What about a finite-dimensional representation of Q?

- What are the infinitesimal characters of the representations constructed in this way, and how do they fit into the set of infinitesimal characters that ought to be attached by the orbit method to the nilpotent orbit $\mathcal{O}$?

The arguments of this section reduce an answer to the first question to a description of the set $HW^1(w_\alpha)$. Proposition 4.5.2 together with the conditions of 5.1.1 calculate the weight w_α of the character α. A character γ that restricts to α on Q_f exists whenever $HW^1(w_\alpha)$ is non-empty.

To answer the second question, we turn to the work of W. M. McGovern, who describes a procedure for constructing a set of infintesimal characters that ought to be attached to an arbitrary nilpotent orbit among classical groups.

5.2 The Infinitesimal Characters $IC^1(\mathcal{O})$

A classification of unitary representations of complex reductive Lie groups can be obtained from a construction that begins with a set of the so-called *special unipotent representations* first suggested by Arthur (see [Barbasch]). This classification, however, is unsatisfactory from the point of view of the orbit method: special unipotent representations can only have as associated varieties the closures of *special* nilpotent orbits. To remedy this shortfall, [McGovern] suggests enlarging the set of special unipotent representations to a set of representations, called *q-unipotent*, with all possible associated varieties.

We recall McGovern's construction of the infinitesimal characters of q-unipotent representations for classical groups. However, not all q-unipotent infinitesimal characters obtained by his method can reasonably correspond to representations attached to nilpotent orbits. After describing this phenomenon more closely, we prune the set of q-unipotent infinitesimal characters to a set that should be attached to nilpotent orbits. We describe these infinitesimal characters explicitly for spherical nilpotent orbits. It should be noted that this can nevertheless serve only as an approximation to the set of infinitesimal characters that ought to be attached to a nilpotent orbit. For instance, there are unipotent representations that ought to be attached to certain nilpotent orbits that do not have half-integer coordinates (see for instance [McGovern4])(Section 5). Unfortunately, the methods of this section do not account for such infinitesimal characters.

5.2.1 Infinitesimal Characters of q-unipotent Representations

We reproduce the procedure for attaching infinitesimal characters to nilpotent orbits. Given a nilpotent orbit $\mathcal{O}$, we first decribe a way of producing an element $h_{\mathcal{O}}$ in a Cartan subalgebra of $\mathfrak{g}$.

Proposition 5.2.1. *For each nilpotent element $f \in \mathfrak{g}$, there is a homomorphism $\phi : \mathfrak{sl}_2 \longrightarrow \mathfrak{g}$ that maps the element $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ onto f. If the nilpotent orbit $\mathcal{O} = \mathcal{O}_f$ through f corresponds to the partition $[p_1, \ldots, p_l]$, then the matrix $h_{\mathcal{O}} = \phi \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ has the following eigenvalues:*

$$p_1 - 1, p_1 - 3, \ldots, -(p_1 - 1), p_2 - 1, \ldots, p_l - 1, \ldots, -(p_l - 1).$$

We can describe the element $h_{\mathcal{O}}$ more precisely in terms of its coordinates.

Proposition 5.2.2. *1. If $\mathfrak{g}$ is of type A, then the coordinates of $h_{\mathcal{O}}$, regarded as an element of a Cartan subalgebra of $\mathfrak{g}$, are its eigenvalues in non-increasing order.*

2. If $\mathfrak{g}$ is of type B, C, or D, embed it in some $\mathfrak{sl}(n)$ via the standard representation.

(a) Suppose the partition of $\mathcal{O}_f$ has the numeral I or none at all. Also suppose that 0 occurs as an eigenvalue of the matrix $h_{\mathcal{O}}$ with multiplicity k. Then the coordinates of $h_{\mathcal{O}}$ are its positive eigenvalues together with $[k/2]$ zeros, arranged in non-increasing order.

(b) If the numeral of $\mathcal{O}_f$ is II, then the coordinates of $h_{\mathcal{O}}$ are obtained in a similar manner, except that the final coordinate is replaced by its negative.

The element $h_{\mathcal{O}}$ lies in $\mathfrak{h}$ and not in its dual $\mathfrak{h}^*$. Therefore, it can be regarded as an infinitesimal character of the dual algebra ${}^L\mathfrak{g}$. We summarize Arthur's original construction of special unipotent representations. Consider a map

$$\phi : SL(2) \longrightarrow {}^L G.$$

Each such map gives a representation π_ϕ of G via a transfer of the parameters of the trivial representation of $PGL(2) = {}^L SL(2)$ to G. Representations obtained in this manner are called *spherical special unipotent*. For each π_ϕ, Arthur conjectured that there exist:

1. a finite set Π_ϕ of representations that contain π_ϕ providing an analogue of a Langlands L-packet, and

2. a linear combination of distribution characters of representations in Π_ϕ that is a stable distribution.

The representations contained in Π_ϕ are called *special unipotent*. Barbasch and Vogan gave a more precise definition of special unipotent representations.

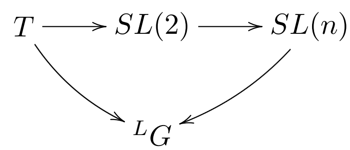
Definition 5.2.3. [Barbasch-Vogan] An irreducible representation of G is special unipotent if its annihilators equal $J_{max}(\lambda_{\mathcal{O}})$ for $\lambda_{\mathcal{O}} = \frac{1}{2} h_{\mathcal{O}}$.

A classification of unitary representations can be produced starting from this set of representations, see [Barbasch]. However, from the point of view of the orbit method, this approach is unsatisfactory. The associated variety of a special unipotent representation can only be the closure of a special nilpotent orbit. By enlarging the group ${}^L G$, McGovern suggests a way of enlarging the set of special unipotent representations to a set whose representations that have all possible associated varieties.

We follow [McGovern] in the definition of the enlargement of ${}^LG$. In each of the classical types except for type B, let n be the dimension d of the standard representation of ${}^LG$. In type B, however, let $n = d+1$. More precisely, the values of n are as in the following table:

Type	n
A_m	$m+1$
B_m	$2m+1$
C_m	$2m+1$
D_m	$2m$

The enlargement of ${}^LG$ that we seek is $SL(n)$. There is a natural injection ${}^LG \longrightarrow SL(n)$, of course with some choice in type B. Let T be the diagonal subgroup of $SL(2)$. We can consider all the possible maps $SL(2) \longrightarrow SL(n)$ and $T \longrightarrow {}^LG$ which make the following diagram commute:



Mimicking the Joseph construction, each such choice allows us to transfer the trivial representation from ${}^LT$ to a representation of G. The number n was chosen in such a way as to obtain as many representations as possible while making sure that the resulting representations of G depend only on a nilpotent orbit in $\mathfrak{sl}(n)$.

Definition 5.2.4. [McGovern] Let $\mathcal{U}$ be a nilpotent orbit in $\mathfrak{sl}(n)$, and recall the element $\lambda_{\mathcal{U}} = \frac{1}{2}h_{\mathcal{U}}$. Let $\lambda'_{\mathcal{U}}$ be any $SL(n)$-conjugate of $\lambda_{\mathcal{U}}$ lying inside a Cartan subalgebra of ${}^L\mathfrak{g}$. When regarded as an infinitesimal character of $\mathfrak{g}$, $\lambda'_{\mathcal{U}}$ is called *q-unipotent.*

We would like to know which nilpotent orbit in $\mathfrak{g}^*$ should be attached to each of the q-unipotent infinitesimal characters. The philosophy of the orbit method dictates that this is the open orbit $\mathcal{O}$ contained in the associated variety of $U(\mathfrak{g})/J_{max}(\lambda'_{\mathcal{U}})$.

Theorem 5.2.5. *[McGovern] Suppose that the orbit $\mathcal{U} \subset \mathfrak{sl}(n)^*$ corresponds to the partition p. The open orbit $\mathcal{O}$ in the associated variety of $U(\mathfrak{g})/J_{max}(\lambda'_{\mathcal{U}})$ has partition:*

1. *p^t in type A,*
2. *$(p^t)_B$ in type B,*
3. *$(l(p^t))_C$ in type C,*
4. *$(p^t)_D$ in type D, except when p is very even, in which case $\mathcal{O}$ depends on the choice of $\lambda_{\mathcal{U}}$ and can be either (p^t, I) or (p^t, II).*

Here the maps p_X are the X-collapses of the partition p and $l(p)$ is the partition obtained from p by subtracting 1 from its smallest term. For $\mathfrak{g}$ of a specified type, we can therefore define a map

$$M : \text{nilpotent orbits in } \mathfrak{sl}(n) \longrightarrow \text{nilpotent orbits in } \mathfrak{g}$$

by letting $M(\mathcal{U}) = \mathcal{O}$, the orbit in the associated variety of $U(\mathfrak{g})/J_{max}(\lambda'_{\mathcal{U}})$. With the help of the above theorem, we can also think of M as a map on partitions.

5.2.2 The Preimage $M^{-1}(\mathcal{O})$

According to the philosophy of the previous section, the q-unipotent infinitesimal characters that are attached to the nilpotent coadjoint orbit $\mathcal{O}$ of a classical group $\mathfrak{g}$ is the set

$$IC(\mathcal{O}) = \{\lambda'_{\mathcal{U}} \mid \mathcal{U} \in M^{-1}(\mathcal{O})\}.$$

This is at least a first approximation of the infinitesimal characters of the representations that should be attached to $\mathcal{O}$ by the orbit method. We describe this set explicitly for the spherical nilpotent orbits in classical Lie algebras.

Proposition 5.2.6. *Let $\mathcal{O} = \mathcal{O}_p$ be a spherical nilpotent orbit in a classical Lie algebra $\mathfrak{g}$. Then in each of the classical types, the set $M^{-1}(p)$ is as follows:*

Type A $\{p^t\}$,

Type B

When	$p = [2^{2k}, 1^{2n-4k+1}]$ $k \neq \frac{n}{2}$,	$\{[2n-2k+1, 2k], [2n-2k, 2k+1]\}$
	$p = [2^{2k}, 1^{2n-4k+1}]$ *n even,*	$\{[n+1, n]\}$
	$p = [3, 1^{2n-2}]$,	$\{[2n-1, 1^2], [2n-2, 2, 1], [2n-2, 1^3]\}$
	$p = [3, 2^{2k}, 1^{2n-4k-2}]$ $k \neq \frac{n-1}{2}, 0$,	$\{[2(n-k)-1-\epsilon, 2k+1+\epsilon, 1] \mid \epsilon = 0,\ 1\}$
	$p = [3, 2^{n-1}]$ *n odd;*	$\{[n^2, 1]\}$

Type C

When	$p = [1^{2n}]$,	$\{[2n+1]\}$
	$p = [2, 1^{2n-2}]$,	$\{[2n, 1]\}$
	$p = [2^k, 1^{2n-2k}]$, $k \neq 1$ *or* n,	$\{[2n-k+1, k], [2n-k+1, k-1, 1]\}$
	$p = [2^n]$;	$\{[n+1, n], [n^2, 1], [n+1, n-1, 1]\}$

Type D

When	$p = [2^{2k}, 1^{2n-4k}]$ $k \neq \frac{n}{2}$	$\{[2n-2k, 2k], [2n-2k-1, 2k+1]\}$
	$p = [2^n]$ *n even,*	$\{[n^2]\}$
	$p = [3, 1^{2n-3}]$,	$\{[2n-2, 1^2], [2n-3, 2, 1], [2n-3, 1^3]\}$
	$p = [3, 2^{2k}, 1^{2n-4k-3}]$ $k \neq \frac{n-2}{2}$,	$\{[2(n-k-1)-\epsilon, 2k+1+\epsilon, 1] \mid \epsilon = 0, 1\}$
	$p = [3, 2^{n-2}, 1]$ *n even,*	$\{[n, n-1, 1]\}$.

Proof. The proof is much simpler than the statement. It consists of understanding the above map and analyzing all the possibilities. The details are left to the interested reader. □

Unfortunately, even among this list, there already appear orbits $\mathcal{U}$ whose associated q-unipotent infinitesimal characters $\lambda'_{\mathcal{U}}$ cannot be attached to the nilpotent orbit $\mathcal{O} \subset \mathfrak{g}^*$ in any reasonable way. To say this more precisely, we need to define characteristic cycles. Let X be a finitely-generated Harish-Chandra $U(\mathfrak{g})$ bimodule. Take X_0 to be a finite-dimensional generating set for X that is stable under the adjoint action of $U(\mathfrak{g})$, and let $X_n = U_n(\mathfrak{g} \times \mathfrak{g})X_0$. Then X_n is a filtration of X.

The associated graded module grX is isomorphic to $S(\mathfrak{g} \times \mathfrak{g})$ as a $grU(\mathfrak{g} \times \mathfrak{g})$-module, and is annihilated by $S(\mathfrak{g}_\Delta)$. Hence grX can be regarded as a finitely-generated, one-sided module over $S(\mathfrak{g})$.

Let $\mathcal{V}_1, \ldots, \mathcal{V}_s$ be the irreducible components of the associated variety $\mathcal{V}(grX)$. Furthermore, let $P_1, \ldots P_s$ be the corresponding minimal primes over $AnngrX$. Then grX admits a finite filtration in which every graded su bquotient has the form $S(\mathfrak{g})/Q$ for $P_i \subset Q$ for some i. Let n_i be the number of graded subquotients isomorphic to $S(\mathfrak{g})/P_i$. This is independent of X_0, and we can define the *characteristic cycle* $Ch(X)$ to be $\sum n_i \mathcal{V}_i$. When X is irreducible, $s = 1$, and the sum contains only one term.

Definition 5.2.7. Let $\mathcal{U}$ be a nilpotent orbit in $\mathfrak{sl}(n)$ as before and recall the infinitesimal character $\lambda'_{\mathcal{U}}$. Write U for the spherical q-unipotent bimodule $U(\mathfrak{g})/J_{max}(\lambda'_{\mathcal{U}})$. We define the number $m_{\lambda'_{\mathcal{U}}}$ to be the multiplicity of $\mathcal{V}(U)$ in $Ch(U)$.

The orbit method dictates that in order for U to correspond to a cover of a nilpotent coadjoint orbit $\mathcal{O}$, $m_{\lambda'_{\mathcal{U}}}$ cannot exceed the order of the fundemental group of $\mathcal{O}$. That is, U should not be too large to meaningful ly correspond to $\mathcal{O}$. It turns out that for certain $\mathcal{U}$, this unfortunately does occur. Examples of this phenomenon arise already among spherical nilpotent orbits.

Example 5.2.8. Let $\mathcal{U}$ be the nilpotent orbit corresponding to the partition $[6,3]$ in $\mathfrak{sl}(9)$. Fix the type of the Lie algebra $\mathfrak{g}$ to be C. Then $M(\mathcal{U}) = \mathcal{O}_{[2^3,1^2]} \subset \mathfrak{sp}(8)^*$. Furthermore, $\lambda'_{\mathcal{U}} = (\frac{5}{2}, \frac{3}{2}, 1, \frac{1}{2})$. However, $m_{\lambda'_{\mathcal{U}}} = 4$ while $|\pi_1(\mathcal{O}_{[2^3,1^2]})| = 2$. According to the above philosophy, $m_{\lambda'_{\mathcal{U}}}$ should not be the infinitesimal character of a unipotent representation attached to $\mathcal{O}_{[2^3,1^2]}$. In fact, this is also true for any $\mathcal{U}$ with partition of the form $[2n-k+1, k]$. There are similar examples in the other classical groups not of type A. Based on this example, we know that in order to find the set of the infinitesimal characters of representations attached to the nilpotent orbit $\mathcal{O}$, we have to prune the set $IC(\mathcal{O})$.

5.2.3 Pruning of $IC(\mathcal{O})$

As in the above example, in some cases, $m_{\lambda'_{\mathcal{U}}} > |\pi_1(\mathcal{O})|$. We would like to exclude the infinitesimal characters that arise from these orbits, so our goal is to determine for which $\mathcal{U} \in M^{-1}(\mathcal{O})$ this occurs. First, we n eed to know the order of the fundamental group of each nilpotent orbit. Write a partition p as $[p_1, p_2, ..., p_l]$ and for each p define the numbers

$$a = \text{number of distinct odd } p_i,$$
$$b = \text{number of distinct even nonzero } p_i,$$
$$c = gcd(p_i).$$

Proposition 5.2.9. *Let $\mathcal{O} = \mathcal{O}_p$ be an orbit in a classical Lie algebra $\mathfrak{g}$. The order of the fundamental group is independent of the Roman numeral assigned to the orbit (if any) and* $|\pi_1(\mathcal{O}_p)| =$

1. *c in type A,*
2. *in type B,*
   - *2^a if p is rather odd,*
   - *2^{a-1} otherwise,*
3. *2^b in type C,*
4. *in type D,*
   - *$2 \cdot 2^{\max(0,a-1)}$ if p is rather odd,*
   - *$2^{\max(0,a-1)}$ otherwise.*

We follow [McGovern] in determining the multiplicity $m_{\lambda'_{\mathcal{U}}}$. The process is a bit complex and requires notation incompatible with some used here, so rather than referring the reader to [McGovern], we replicate the relevant parts here usi ng new notation.

Definition 5.2.10. Let $M(\mathcal{U}) = \mathcal{O}$, and suppose that $\mathcal{U}$ corresponds to the partition p. In each of the classical types $X = B, C$, and D, we define two numbers μ and ν.

When $X = D$, let $q = p_{odd} = (q_1^{\lambda_1}, \ldots, q_t^{\lambda_t})$ and break it up into chunks as follows. Starting from the left, each chunk takes on one of the forms: $(q_i^{\lambda_i}, q_{i+1}^{\lambda_{i+1}})$ with both λ_i and λ_{i+1} odd; $(q_i^{\lambda_i})$ with λ_i even; or $(q_i^{\lambda_i})$ with λ_i odd and λ_{i+1} even. Let ν be the number of chunks of the first two types. The number μ is defined the same way but with $q = [(p_{even})_D]_{odd}$.

When $X = B$, break up p_{odd} into chunks as in type D. Let ν_1 be the number of chunks of the first type. Let c be the leftmost chunk of the third type and let ν_2 be the number of chunks of the second type to the right of c, plus one. If no c exists, let $\nu_2 = 0$. Finally, let $\nu = \nu_1 + \nu_2$. The number μ is defined the same way but with $([r(p_{even})]_B)_{odd}$.

When $X = C$, define ν in the same way as in type B. To define μ, replicate its definition in type D but with the partition $[(p_{even})_D]_{odd}$.

Finally, in each of the cases let $\nu^* = \max(0, \nu - 1)$ and $\mu^* = \max(0, \mu - 1)$.

Definition 5.2.11. Write the infintesimal character $\lambda'_{\mathcal{U}}$ in coordinates as

$$((\tfrac{i}{2})^{r_i}, \ldots, (\tfrac{1}{2})^{r_1}, 0^{r_0}).$$

If $\lambda'_{\mathcal{U}}$ contains a coordinate of $-\frac{1}{2}$, simply write this as an additional $\frac{1}{2}$.

In type B, define the following numbers:

κ = number of even positive i with r_i odd and r_{i-1} even,

κ_1 = number of even positive i with r_i odd, r_{i-1} even, and either $r_{i-2} > r_i$ with $i > 2$, or $r_0 > \frac{1}{2}r_2$,

κ_2 = number of even positive i with r_i odd, r_{i-1} even positive, and the largest integer j with the following property is even: for even m, $i \leq m \leq j$, r_m is odd, while for odd m in the same range, r_m is positive even.

In type D, first let i_0 be the smallest odd integer i with r_i odd if one exists. Otherwise, let $i_0 = \infty$. Then define:

κ = number odd i with r_i odd and either r_{i-1} even or $i = i_0$,

κ_1 = number of odd $i > i_0$ with r_i odd, r_{i-1} even, and either $r_{i-2} > r_i$,

κ_2 = number of odd $i.i_0$ with r_i odd, r_{i-1} even positive, and the largest integer j with the following property is odd: for even m, $i \leq m \leq j$, r_m is positive even, while for odd m in the same range , r_m is odd.

In type C, the definition is a bit longer. Define a string of integers $i, \ldots j$ to be *relevant* if

$j > i \geq 0$,

for $i < m \leq j, r_m$ is odd,

either $i > 0$ and r_i is odd, or $i = 0$ and $r_i = \frac{1}{2}(r_{i+2} - 1)$,

the string is maximal subject to the above.

Now let

$E_S = \{$positive even integers i in S such that $r_i > 1$ and $i > 2$ or $r_{i-1} \neq 1\}$,

$F_S = \{$odd integers i in S with $r_i > 1\}$,

$\kappa'_S = \max(\#(E_S \cup F_S) - (length(S) - 2), 0)$.

We can now list the relevant strings as $S_1, \dots, S_r$ in such a way that the ones with $\kappa'_S = 2$ come first, followed by the ones with $\kappa'_S = 1$, and then the ones with $\kappa'_S = 0$. Enumerate the integers in $\cup_S E_S$ as $i_1, \dots i_s$ in such a way that the ones in S_1 come first, etc. Now let $\kappa(i_a) = 1$ iff $a \leq \nu^*$ and 0 otherwise. Also let $\kappa(j_b) = 1$ iff $b \leq \mu^*$ and 0 otherwise. Finally, for each relevant string S, we can define

$$\kappa_S = \sum_{i_a \in E_S} \kappa(i_a) + \sum_{j_b \in F_S} \kappa(j_b).$$

We are now ready to decribe the multiplicity $m_{\lambda'_{\mathcal{U}}}$. Let

$n_B = 2\kappa - \min(\nu^*, \kappa_1) - \min(\mu^*, \kappa_2)$,

$n_C = \sum_S \max(length(S) - 2 - \kappa_S, 0)$,

$n_D = 2\kappa - \min(\mu^*, \kappa_1) - \min(\nu^*, \kappa_2) + \kappa_3$.

Proposition 5.2.12 (McGovern). *Consdier the type A nilpotent orbit $\mathcal{U} = \mathcal{U}_q$. With notation as above, $m_{\lambda'_{\mathcal{U}}}$ equals*

1. 1 *in type A,*
2. 2^{n_B} *in type B,*
3. 2^{n_C} *in type C,*
4. $2^{max(n_D - 2, 0)}$ *in type D.*

Corollary 5.2.13. *Consider a spherical nilpotent orbit $\mathcal{O}$ and let $M(\mathcal{U}_p) = \mathcal{O}$. Then*

1. $n_B = 2\kappa$ *except when* $p = [2n-1, 1^2]$, *or* $[2(n-k)-1, 2k+1, 1]$, *in which case it equals* $2\kappa - 1$
2. $n_C = \kappa - 1$ *when q has the form* $[2n - k + 1, k]$,*and is* 0 *otherwise,*
3. $n_D = 2\kappa$.

Proof. In type B, both μ and ν are less than 2, except when $p = [2n-1, 1^2], [2(n-k)-1, 2k+1, 1], [2n-2, 2, 1]$, or $[2n-2k-2, 2k+2, 1]$. In the case of the former two, $\min(\nu^*, \kappa_1) = 1$, and in the case of all four, $\min(\mu^*, \kappa_2) = 0$. For spherical orbits of type D, both μ and ν are less than 2. Finally, in type C, relevant strings of length greater than 2 arise only when $p = [2n - k + 1, k]$. □

We are now ready to state a second approximation to the set of infintesimal characters that should appear as infinitesimal characters of representations attached to spherical nilpotent coadjoint orbits. As mentioned before, for a given nilpotent orbit $\mathcal{O}$ of a given classical type, this is the set of characters of the form $\lambda'_{\mathcal{U}}$ with $M(\mathcal{U}) = \mathcal{O}$ that also satisfy the condition

$$m_{\lambda'_{\mathcal{U}}} \leq |\pi_1(\mathcal{O})|.$$

We will denote this set by $IC^1(\mathcal{O})$, and compute it in the next proposition. We also adopt new notation for a q-unipotent infinitesimal character by associating it with the partition of the type A orbit that is used to compute it. For example, the orbit $\mathcal{U}_{[4^2,1]}$ lies in the preimage $M^{-1}(\mathcal{O}_{[2^4]}$ of the type C orbit with partition $[2^4]$. Then

$$\lambda'_{\mathcal{U}} = (\tfrac{3}{2}, \tfrac{3}{2}, \tfrac{1}{2}, \tfrac{1}{2})$$

which we write as

$$\lambda'_{\mathcal{U}} = [4^2, 1].$$

This expression is unique as long as the type of the orbit $\mathcal{O}$ is specified. In the case of very even orbit in type D, we take this to mean that the infinitesimal character with all terms nonnegative at tached to the orbit with numeral I and the infinitesimal character with one negative terms attached to the orbit with numeral II.

Proposition 5.2.14. *Let $\mathcal{O}_p$ be a spherical nilpotent orbit in a classical Lie algebra $\mathfrak{g}$ that corresponds to the partition p. The set $IC^1(\mathcal{O}_p)$ of infinitesimal characters attached to $\mathcal{O}_p$ by the above procedure is as follows:*

Type A	$\{p^t\}$	
Type B		
When	$p = [2^{2k}, 1^{2n-4k+1}]$	$\{[2n-2k, 2k+1]\}$,
	$p = [3, 1^{2n-2}], n \neq 2$	$\{[2n-2, 2, 1], [2n-2, 1^3]\}$
	$p = [3, 1^2]$,	$\{[2^2, 1], [2, 1^3], [3, 1^2]\}$
	$p = [3, 2^{2k}, 1^{2n-4k-2}]$ $k \neq \frac{n-1}{2}, 0$,	$\{[2n-2k-2, 2k+2, 1]\}$
	$p = [3, 2^{n-1}]$	$\{[n^2, 1]\}$
Type C		
When	$p = [1^{2n}]$,	$\{[2n+1]\}$
	$p = [2^k, 1^{2n-2k}] k \neq 2$	$\{[2n-k+1, k-1, 1]\}$
	$p = [2^2, 1^{2n-4}]$	$\{[2n-1, 1^2], [2n-1, 2]\}$
	$p = [2^n] n \neq 2$	$\{[n^2, 1], [n+1, n-1, 1]\}$;
	$p = [2^2]$	$\{[2^2, 1], [3, 1^2], [3, 2]\}$;
Type D		
When	$p = [2^{2k}, 1^{2n-4k}]$ $k \neq \frac{n}{2}$	$\{[2n-2k-1, 2k+1]\}$,
	$p = [2^n]$	$\{[n^2]\}$
	$p = [3, 1^{2n-3}]$,	$\{[2n-3, 2, 1], [2n-3, 1^3]\}$
	$p = [3, 2^{2k}, 1^{2n-4k-3}]$	$\{[2n-2k-3, 2k+2, 1]\}$

5.3 Infinitesimal Characters of $V(\mathcal{V}, \pi)$

Recall the character α, defined as the square root of the absolute value of the real determinant of the Q_f action on $\mathfrak{q}/\mathfrak{q}_f$ used to define $V(\mathcal{V}, \pi)$. Suppose that α extends to a character γ on Q. According to Section 5.1, such an extension exists whenever the set $HW^1(w_\alpha)$ is not empty. The first goal of this section is to decide whether and when this occurs. This is important as the construction of $V(\mathcal{V}, \pi)$ relies on the existence of a bundle isomorphism $j_{\gamma,\pi}$ defined in Chapter 2. In the setting of spherical nilpotent orbits, $j_{\gamma,\pi}$ exists precisely when there is a character γ of the parabolic Q stabilizing $\mathcal{V}$ that restricts to α on Q_f.

The second goal of the section is to decide how well the infinitesimal characters of $V(\mathcal{V}, \pi)$ fit within those that ought to be attached to the nilpotent orbit $\mathcal{O}$. Suppose that the half-density bundle on G/Q is given by the character $\rho_{G/Q}$, and define $\gamma' = \gamma \otimes \rho_{G/Q}^{-1}$. The space $V(\mathcal{V}, \pi)$ is then a subset of $Ind_Q^G(\gamma')$. If we write w_γ for the character of γ and ρ for the half-sum of the positive roots of G, then the associated infinitesimal character is $\chi_\gamma = w_\gamma + \rho$. One expects that χ_γ should be a character attached to $\mathcal{O}$ by the previous section, that is, it should lie in the set $IC^1(\mathcal{O})$.

We begin with a short litany of examples of what is *not* true. It turns out that the simplest solutions one would like to have for both of the above questions are not possible.

5.3.1 A Few Examples

First, we show that it is not always possible to find a character γ of Q that restricts to α on Q_f. This occurs already in type A for the minimal orbit in rank 5.

Example 5.3.1. Let $\mathfrak{g} = \mathfrak{gl}_5$ and consider the orbital variety $\mathcal{V}_T$ associated to the standard Young tableau

$$\mathrm{T} = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array}$$

The basepoint $f = f_T = E_{e_2-e_3}$ constructed in 4.3.5 has dense B orbit by Lemma 4.3.7. The τ-invariant of T and hence that of $\mathcal{V}_T$ can be gleaned from Theorem 3.5.1 and equals $\{e_1 - e_2, e_3 - e_4, e_4 - e_5\}$. If Q is the parabolic stabilizing $\mathcal{V}_T$ and L is its Levi subgroup, the τ-invariant forces $\mathfrak{l} = \mathfrak{gl}_2 \times \mathfrak{gl}_3$. We can now compute the weight of the square root of the absolute value of the determinant of the Q_f action on $\mathfrak{q}/\mathfrak{q}_f$. According to the inductive procedure of Proposition 4.5.2, the weight of α is $w_\alpha = -\mathfrak{t}_1 - \mathfrak{t}_3 + \mathfrak{t}_4 + \mathfrak{t}_5$ which we write as

$$w_\alpha = ((-1, 0), (-1, 1, 1))$$

by grouping terms that correspond to the same reductive part of the Levi. The set of weights that restrict to w_α consists of the one-parameter family

$$HW(w_\alpha) = \{w_\alpha + \epsilon = ((-1, \epsilon_1), (-1 - \epsilon_1, 1, 1))\}.$$

The weight $w_\alpha + \epsilon$ corresponds to a one-dimensional representation of Q iff the conditions described after Fact 5.1.1 are satisfied, that is, iff $-1 = \epsilon$ and $-1 - \epsilon = 1$. This is not possible, implying that $HW^1(w_\alpha) = \emptyset$ and that there does not exists a character γ of Q

that restricts to α on Q_f!

One can reasonably expect that this counterexample can be used to construct others in larger groups by considering orbital varieties associated to standard Young tableaux that contain the above T as a subtableau. In other words, the property that $HW^1(w_\alpha) = \emptyset$ should preserved by induction on tableau. Surprisingly, this too is false.

Example 5.3.2. Let $\mathfrak{g} = \mathfrak{gl}_6$ and consider

$$S = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 6 \\ \hline 3 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array}$$

Then S contains T as a subtableau. Following the procedure of the previous example, we find that $w_\alpha = ((-2), (0,0), (0,1,1))$ which extends to a two-parameter set of weights of the form

$$HW(w_\alpha) = \{w_\alpha + \epsilon = ((-2-\epsilon_1), (\epsilon_1, \epsilon_2), (-\epsilon_2, 1, 1))\}.$$

The weight $w_\alpha + \epsilon$ corresonds to a one-dimensional representation γ of Q whenever $\epsilon_1 = \epsilon_2 = -1$. The associated infinitesimal character is then

$$\begin{aligned} w_\gamma = w_\alpha + (1,-1,-1,1,0,0) &= (-1,-1,-1,1,1,1) + (\tfrac{5}{2}, \tfrac{3}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{3}{2}, -\tfrac{5}{2}) \\ &= (\tfrac{3}{2}, \tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{3}{2}). \end{aligned}$$

This is precisely the infinitesimal character attached to $\mathcal{O}_{[2^2,1^2]}$ by the procedure outlined in the previous section.

One also hopes that if there does exists a character γ that restricts to α, then the associated infinitesimal character is among the ones that ought to tbe associated to the original orbit $\mathcal{O}$, that is, $\chi_\gamma \in IC^1(\mathcal{O})$. Unfortunately, this also fails.

Example 5.3.3. Let $\mathfrak{g} = \mathfrak{gl}_6$ and consider the tableau

$$T = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & 6 \\ \hline\end{array}$$

The Levi of the parabolic stablizing $\mathcal{V}_T$ is $\mathfrak{l} = \mathfrak{gl}_2 \oplus \mathfrak{gl}_2 \oplus \mathfrak{gl}_2$. Proposition 4.5.2 implies that $w_\alpha = ((-\frac{3}{2}, -\frac{3}{2}), (\frac{1}{2}, -\frac{1}{2}), (\frac{3}{2}, \frac{3}{2}))$. The set of weights that restrict to α on $\mathfrak{t}_f$ is the three-parameter family

$$HW(w_\alpha) = \left\{ w(\epsilon_1, \epsilon_2, \epsilon_3) = ((-\tfrac{3+\epsilon_1}{2}, -\tfrac{3+\epsilon_2}{2}),\ (\tfrac{1+\epsilon_2}{2}, -\tfrac{1+\epsilon_3}{2}), (\tfrac{3+\epsilon_3}{2}, \tfrac{3+\epsilon_1}{2})) \right\}.$$

For $w(\epsilon_1, \epsilon_2, \epsilon_3)$ to lie in $HW^1(w_\alpha)$, we must have $\epsilon_1 = \epsilon_2 = \epsilon_3$ and $1 + \epsilon_2 = -1 - \epsilon_3$. This forces $\epsilon_1 = \epsilon_2 = \epsilon_3 = -1$. Hence

$$HW^1(w_\alpha) = \{w(-1,-1,-1) = ((-1,-1), (0,0), (1,1))\}$$

which corresponds to the character of the parabolic Q given by

$$\gamma\begin{pmatrix} A_1 & * & * \\ 0 & A_2 & * \\ 0 & 0 & A_3 \end{pmatrix} = \left(|A_1|^{-2}|A_3|^2\right)^{\frac{1}{2}}$$

The infinitesimal character of $Ind_Q^G(\gamma \otimes \rho_{G/Q}^{-1})$ is then

$$\begin{aligned}\chi_\gamma =&(-1,-1,0,0,-1,-1) + (\tfrac{5}{2},\tfrac{3}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{3}{2},-\tfrac{5}{2}) \\ =&(\tfrac{3}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{3}{2}).\end{aligned}$$

But χ_γ does not lie in $IC^1(\mathcal{O}_{[3,3]})$, which consists of the single infinitesimal character $(1\ 1\ 0\ 0\ -1\ -1)$. In fact, χ_γ lies in $IC^1(\mathcal{O}_{[4,2]})$!

5.3.2 Positive Results

The phenomenon of the final example above occurs only for certain model spherical orbits. For all other spherical orbits, the infinitesimal character χ_γ, if defined, is indeed attached to $\mathcal{O}$ by the method of the previous section.

Theorem 5.3.4. *Let $\mathcal{O}$ be a rigid, non-model spherical nilpotent orbit and consider an orbital variety $\mathcal{V}$ with stabilizer Q. Suppose that there exists a character γ of Q that restricts to the character α on Q_f defined as the absolute value on the real determinant of its action on $\mathfrak{q}/\mathfrak{q}_f$. Then the infinitesimal character χ_γ lies in $IC^1(\mathcal{O})$.*

We defer the proof to its own section. We have not yet answered the questions of when it is possible to extend the character α of Q_f to a character of Q, and whether the set of such extensions for a given orbit provides enough candidates whose associated infinitesimal characters exhaust $IC^1(\mathcal{O})$. Example 5.3.2 shows that it is certainly not always possible find an extension γ of α for every orbital variety $\mathcal{V} \subset \mathcal{O}$. However, there exists at least one orbital variety within each orbit whose associated α does admit such an extension. Futhermore, there exists a sufficient number of such orbital varieties in $\mathcal{O}$ to account for all infinitesimal characters in $IC^1(\mathcal{O})$.

Theorem 5.3.5. *Let $\mathcal{O}$ be a rigid spherical orbit or a model orbit with $n > 2$. For every $\chi \in IC^1(\mathcal{O})$, there exists an orbital variety $\mathcal{V} \subset \mathcal{O}$ satisfying*

- *$\alpha_{\mathcal{V}}$ extends to a character γ of Q, and*
- *$\chi_\gamma = \chi$.*

Proof. For every nilpotent orbit, we construct a set of standard tableaux. Given a nilpotent orbit $\mathcal{O}$, it is always possible to construct a unique domino tableau satisfying the following:

- There exists an integer k such that $\forall i \leq k$, $i \in T^1$ and $i \notin T^2$,
- k is maximal among all standard tableaux of shape equal to the partition corresponding to $\mathcal{O}$.

Write $T_{\mathcal{O}}$ for the above tableau. When $\mathcal{O}_T$ is a very even orbit in type D with Roman numeral II, define a tableau T_{II} satisfying:

· $\{n-1, n\} = N_1^T$,

· $\{1, 3, 5, \ldots\} \in T^1$, and

· $\{2, 4, 6, \ldots\} \in T^2$.

The Levi of the stabilizing subgroup of $\mathcal{V}_{T_{\mathcal{O}}}$ has exactly two reductive components. We first examine the case where the largest element of the partition p corresponding to $\mathcal{O}$ is 2 and the Roman numeral associated to $\mathcal{O}$, if any, is I. Let $[\lambda_1, \lambda_2]$ be the partition dual to p. The weight w_α has form:

$$w_\alpha = ((c_1, c_1, \ldots c_1), (c_2, c_2, \ldots c_2))$$

where

$$(c_1, c_2) = \begin{cases} (-\lambda_1, -\lambda_2) & \text{in type A,} \\ (-\lambda_1 + 2, 0) & \text{in types B and D,} \\ (-\lambda_1 - 2, 0) & \text{in type C.} \end{cases}$$

The elements of $HW(w_\alpha)$ have the general forms $w(\epsilon_1, \epsilon_2, \ldots, \epsilon_s) =$

$$\begin{array}{ll} ((c_1 - \epsilon_1, c_1 - \epsilon_2, \ldots c_1 - \epsilon_s), (c_2, \ldots, c_2 + \epsilon_2, c_2 + \epsilon_1)) & \text{if } \mathcal{O} \text{ is rigid,} \\ ((\epsilon_1, c_1 - \epsilon_2, \ldots c_1 - \epsilon_s), (c_2 + \epsilon_s, \ldots, c_2 + \epsilon_2, c_2 + \epsilon_1)) & p = [2^n],\ n \text{ is odd in type C,} \\ ((c_1 - \epsilon_1, c_1 - \epsilon_2, \ldots c_1 - \epsilon_s), (c_2 + \epsilon_s, \ldots, c_2 + \epsilon_2, c_2 + \epsilon_1)) & \text{otherwise.} \end{array}$$

In the first case, $w(\epsilon_1, \epsilon_2, \ldots, \epsilon_s) \in HW^1(w_\alpha)$ iff $\epsilon_i = 0$ for all i. In the third case, $w(\epsilon_1, \epsilon_2, \ldots, \epsilon_s) \in HW^1(w_\alpha)$ iff $\epsilon_i = \epsilon_j$ for all i and j. This produces a one-parameter family of weights that depends on the common value of the $\epsilon_i = \epsilon$. In the second case, $w(\epsilon_1, \epsilon_2, \ldots, \epsilon_s) \in HW^1(w_\alpha)$ iff $\epsilon_i = \epsilon_j$ for $i, j \geq 2$ and $\epsilon_1 = c_1 - \epsilon_2$. This again yields a one-parameter family of weights that depends on the common value of the $\epsilon_i = \epsilon$ with $i \geq 2$.

Whenever $\mathcal{O}$ is rigid, $|IC^1(\mathcal{O})| = 1$ and an easy comparison with Proposition 5.2.14 shows that $\{w(0, 0, \ldots 0) + \rho\} = IC^1(\mathcal{O})$. In the third case above in types B and D, $|IC^1(\mathcal{O})| = 1$ again and with $\epsilon = 0$, $\{w(0, 0, \ldots 0) + \rho\} = IC^1(\mathcal{O})$. In type C, when $n > 2$, $|IC^1(\mathcal{O})| = 2$. Note that $w(-1, -1, \ldots - 1) \neq w(0, 0, \ldots 0)$ and it is an easy check that $\{w(-1, -1, \ldots - 1) + \rho, w(0, 0, \ldots 0) + \rho\} = IC^1(\mathcal{O})$. The second case is similar. Now consider the case when $\mathcal{O}$ is very even in type D with numeral II, and consider the orbital variety $\mathcal{V}_{T_{II}}$. We find that

$$w_\alpha = \tfrac{1}{2}((-2n+2, -2n+2, -2n+4), (-2n+4, -2n+6), \ldots, (-4, -2), (-2, 0), (0)).$$

The elements in $HW(w_\alpha)$ have the form $w(\beta, \epsilon_1, \ldots, \epsilon_s) = \frac{1}{2}(-2n + 2 + \beta, -2n + 2 + \beta, -2n + 4 - \epsilon_1), (-2n + 4 + \epsilon_1, \ldots (-2 + \epsilon_{s-1}, -\epsilon_s), (\epsilon_s))$. The set of elements in $HW^1(w_\alpha)$ is a one-parameter family, consisting of $w(\epsilon_s) = \frac{1}{2}((\ldots(-4 + \epsilon_s, -4 + \epsilon_s), (-\epsilon_s, -\epsilon_s), (\epsilon_s))$. When n is odd, let $\epsilon_s = -1$ and when n is even, let $\epsilon_s = 3$. Inductively, it is now easy to show that $w(\epsilon_s) + \rho = (\frac{n-1}{2}, \frac{n-1}{2}, \frac{n-3}{2}, \frac{n-3}{2}, \ldots, 3, 3, 1, -1)$.

Now assume that p has largest part 3 and contains parts of size 2. Then according to Proposition 4.5.2,

$$w_\alpha = ((c_1, c_1, \ldots, c_1 - 1), (c_2, c_2, \ldots c_2))$$

where $(c_1, c_2) = (-\lambda_1 + 1, 0)$ in both tyles B and D. The elements in $HW(w_\alpha)$ have the general form

$$w(\epsilon_1, \dots \epsilon_{s+2}) = ((c_1 - \epsilon_1, \dots c_1 - \epsilon_s, c_1 - \epsilon_{s+1}), (c_2 + \epsilon_{s+2}, c_2 + \epsilon_s, \dots c_2 + \epsilon_1))$$

According to Fact 5.1.1, $w(\epsilon_1, \dots \epsilon_{s+2}) \in HW^1(w_\alpha)$ iff $\epsilon_i = 0$ for all $i \leq s$ and $s+2$, and $\epsilon_{s+1} = -1$. Furthermore, $\{w(0, 0, \dots, 0, -1, 0) + \rho)\} = IC^1(\mathcal{O})$.

Now if p has no parts of size two and $n > 2$, then $\mathcal{O}$ is neither rigid nor model, but the same result holds. We find that

$$w_\alpha = ((c_1)(c_2, \dots c_2))$$

and the general form of the elements in $HW(w_\alpha)$ is

$$w(\epsilon_1, \epsilon_2) = ((c_1 - \epsilon_1)(\epsilon_2, 0, 0 \dots 0))$$

where $c_1 = -\lambda_1$. Now $w(\epsilon_1, \epsilon_2)$ lies in $HW^1(\mathcal{O})$ iff $\epsilon_2 = 0$. It is an easy check that $\{w(1, 0) + \rho, w(0, 0) + \rho\} = IC^1(\mathcal{O})$. This finishes the proof of the theorem.

□

5.3.3 Proof of Theorem 5.3.4

We begin with an example.

Example 5.3.6. Let $\mathfrak{g} = \mathfrak{gl}_7$ and let $\mathcal{O}_{[4,3]}$ be the nilpotent orbit corresponding to the partition $[4, 3]$. Consider the standard Young tableau

T=

<table>
<tr><td>1</td><td>4</td></tr>
<tr><td>2</td><td>5</td></tr>
<tr><td>3</td><td>6</td></tr>
<tr><td>7</td><td></td></tr>
</table>

The orbital variety $\mathcal{V}_T$ has stabilizer Q with Levi L whose Lie algebra is $\mathfrak{l} = \mathfrak{gl}_4 \oplus \mathfrak{gl}_3$. We would like to know that if γ is a character of Q that restricts to α on Q_f, then $w_\gamma + \rho$ lies in $IC^1(\mathcal{O})$. By Proposition 4.5.2 and the analysis of Section 5.1,

$$w_\alpha = ((-\tfrac{3}{2}, -1, -1, -1), (\tfrac{3}{2}, \tfrac{3}{2}, \tfrac{3}{2})),$$

and

$$HW(w_\alpha) = \{w(\epsilon_1, \epsilon_2, \epsilon_3) = ((-\tfrac{3}{2}, -\tfrac{2+\epsilon_1}{2}, -\tfrac{2+\epsilon_2}{2}, -\tfrac{2+\epsilon_3}{2}), (\tfrac{3+\epsilon_3}{2}, \tfrac{3+\epsilon_2}{2}, \tfrac{3+\epsilon_1}{2})).$$

The conditions following Fact 5.1.1 now imply that $w(\epsilon_1, \epsilon_2, \epsilon_3) \in HW^1(w_\alpha)$ iff $\epsilon_i = 1$ for all i. Hence w_γ must equal $w(1, 1, 1)$ and

$$\begin{aligned} w_\gamma + \rho &= (-\tfrac{3}{2}, -\tfrac{3}{2}, -\tfrac{3}{2}, -\tfrac{3}{2}, 2, 2, 2) + (3, 2, 1, 0, -1, -2, -3) \\ &= (\tfrac{3}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{3}{2}, 1, 0, -1) \in IC^1(\mathcal{O}_{[4,3]}), \end{aligned}$$

as desired. Now note that

$$w_\alpha + \rho = w(0, 0, 0) + \rho = (\tfrac{3}{2}, 1, 0, -1, \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{3}{2}).$$

While w_α does not correspond to a character of Q, this is nevertheless a permutation of $w_\gamma + \rho$ and also lies in $IC^1(\mathcal{O}_{[4,3]})$. This observation suggests an approach to our problem. We will prove:

Lemma 5.3.7. *Suppose that we are in the setting of Theorem 5.3.4. Then there exists a weight w_β such that*

- $w_\beta + \rho \in IC^1(\mathcal{O})$, *and*
- $w_\beta + \rho$ *is in the same Weyl group orbit as* $w_\gamma + \rho$.

The lemma implies that $w_\gamma + \rho \in IC^1(\mathcal{O})$, proving Theorem 5.3.4.

We will first shed some light on the method of our proof. As in our examples,the form of a general element of $HW(w_\alpha)$ depends on a number of independent variables $\{\epsilon_i\}_{i\leq s}$ and can be written as $w(\epsilon_1, \epsilon_2, \dots \epsilon_s)$. The general form can also be expressed as $b = (b_n\ b_{n-1}\ \dots b_2\ b_1)$. Each entry b_i can be a constant or it may depend on a single independent variable, as prescribed by the conditions following Fact 5.1.1. At most two entries can depend on the same independent variable. We can divide the entries of b into disjoint maximal strings of entries of the form $b_l, b_{l-1}, \dots b_k$ which satisfy:

- b_l and b_k both depend on the same independent variable, and
- there is no pair (l', k') such that $b_{l'}$ and $b_{k'}$ both depend on the same independent variable and additionally $l' > l$ and $k' < k$.

For such a maximal string, call $I = (k, k+1, \dots, l)$ a *dependent interval* of b. It is an easy consequence of Fact 4.5.1 that if i lies in a dependent interval, the entry b_i is not constant. If $i \in I$ and b_i depends on the variable ϵ_{N_i}, we will say that ϵ_{N_i} *corresponds* to I. Note that each ϵ_i for $i \leq s$ corresponds to one and only one dependent interval I. Dependent intervals come in two flavors:

- $\forall i \leq \frac{l-k+1}{2}$, the entries b_{l-i} and b_{k-i} depend on ϵ_{N_i} for some integer $N_i \leq s$,
- there exists an non-negative integer $i \leq \frac{l-k+1}{2}$ such that b_{l-i} and b_{k_i} depend on different variables.

We will call dependent intervals of the first type *simple*. For each simple dependent interval $I = (k \dots l)$, we define a permutation σ_I as a product of transpositions by

$$\sigma_I = \prod_{i < \frac{l-k}{2}} (l - i \quad \tfrac{k+l}{2} + i).$$

The permutation σ_I simply interchanges the first $(l-k+1)/2$ entries of I with the second set of $(l-k+1)/2$ entries, preserving the relative order of elements in each set. By hypothesis, we know that there exists a character γ of Q that restricts to α on Q_f. Hence there exists a constant c_i for each variable ϵ_i such that $w(c_1, c_2, \dots) \in HW(w_\alpha)$ that equals w_γ. If there exists a $c_i \neq 0$ that corresponds to the dependent interval I, we say that I is *non-zero*.

Example. 5.3.6 (Again) The above example contains only one dependent interval. It equals $I = (1, \dots, 6)$ and corresponds to the entries

$$(b_6\ b_5\ b_4\ b_3\ b_2\ b_1) = (-\tfrac{2+\epsilon_1}{2}, -\tfrac{2+\epsilon_2}{2}, -\tfrac{2+\epsilon_3}{2}, \tfrac{3+\epsilon_3}{2}, \tfrac{3+\epsilon_2}{2}, \tfrac{3+\epsilon_1}{2}).$$

In fact, I is a simple dependent interval, and

$$\sigma_I = (3\ 6)\ (5\ 2)\ (4\ 1)$$

Now note that if we write $w_\alpha + \rho$ as $(c_7\ c_6 \dots c_1)$, then $w_\gamma + \rho = (c_{\sigma(7)}\ c_{\sigma(6)} \dots c_{\sigma(1)})$. Hence, at least in this case, we have produced a method of describing the permutation relating $w_\gamma + \rho$ and $w_\alpha + \rho$.

In general, after describing the weight w_β, we will show that if all non-zero dependent intervals in $HW(w_\alpha)$ are simple, then

Lemma 1. $w_\beta + \rho \in IC^1(\mathcal{O})$, and

Lemma 2. $w_\gamma + \rho = \sigma(w_\beta + \rho)$, where σ is the product of the σ_I taken over all non-zero simple dependent intervals I and acts by permuting the order of the entries of the weights.

Furthermore, we will show that

Lemma 3. a non-zero non-simple dependent interval cannot exist under the hypotheses of the Theorem.

We first describe w_β.

Definition 5.3.8. We define w_β inductively. Let $v_\delta = w_\delta - \iota(w_\delta^\downarrow)$, for $\delta = \alpha$ or β. Then let

$$v_\beta = v_\alpha + \begin{cases} -(n+2)T_1 & \text{Case (N1)}, X = C \\ -(n+1)T_1 & \text{Case (N2)}, X = C \\ T_1 & \text{Case (N3)}, X = B, D \\ T_3 & \text{Case (*)}, X = B. \end{cases}$$

Fact 4.5.1 implies that $w_\beta \in HW(w_\alpha)$. Because of the hypotheses of the Theorem, we know that there is a weight $w_\gamma \in HW^1(w_\alpha)$. If we write a general element of $HW(\alpha)$ as

$$w(\epsilon_1, \dots \epsilon_s) = (b_n, b_{n-1}, \dots, b_1),$$

then there exists constants $c_1, \dots c_s$ such that $w(c_1, \dots c_s) = w_\gamma$. Because $\mathcal{O}$ is rigid, there exists at least one entry b_p that is constant. Note that it does not belong to any dependent interval. We will prove:

Fact A. If b_p is adjacent to a non-zero non-simple dependent interval, then there are no constants $c_1, \dots c_s$ such that $w(c_1, \dots c_s) \in HW^1(w_\alpha)$.

Fact B. If $I_1 = (k_1, \dots l_1)$ is a non-zero non-simple dependent interval that is adjacent to a simple dependent interval $I_2 = (k_2, \dots l_2)$, then there are no constants $c_1, \dots c_s$ such that $w(c_1, \dots c_s) \in HW^1(w_\alpha)$.

If $w(\epsilon_1, \dots \epsilon_s)$ contains a non-zero non-simple dependent interval, it must contain at least one that is adjacent to either a simple dependent interval or a constant. Facts A and B then provide a contradiction, proving Lemma 3.

Fact C. If b_p is adjacent to a non-zero simple dependent interval $I = (k, \dots l)$, then

$$\sigma_I\left((w_\beta + \rho)_l \dots (w_\beta + \rho)_k\right) = \left((w_\gamma + \rho)_l \dots (w_\gamma + \rho)_k\right)$$

Fact D. If $I_1 = (k_1, \dots l_1)$ is a non-zero simple dependent interval that is adjacent to either a simple dependent interval or a zero non-simple dependent interval $I_2 = (k_2, \dots l_2)$, then

$$\sigma_I\left((w_\beta + \rho)_{l_1} \dots (w_\beta + \rho)_{k_1}\right) = \left((w_\gamma + \rho)_{l_1} \dots (w_\gamma + \rho)_{k_1}\right)$$

Lemma 3 shows that $w(\epsilon_1, \dots \epsilon_s)$ consists solely of simple dependent intervals and constants. For an integer i that either lies in a zero dependent interval or whose corresponding entry is a constant, we know that $(w_\gamma + \rho)_i = (w_\beta + \rho)_i$. If, however, i lies in a non-zero dependent interval, Facts C and D show that $(w_\gamma + \rho)_i = (w_\beta + \rho)_{\sigma(i)}$, which implies Lemma 2. It remains to prove Lemma 1 and the four Facts.

Proof of Lemma 1. We would like to show that $w_\beta \in IC^1(\mathcal{O})$. Let $S = \{IC^1(\mathcal{O}) - \iota(IC^1(\mathcal{O}^\downarrow))\}$ and define $w = w_\beta + \rho - \iota(w_\beta^\downarrow - \rho^\downarrow)$. It is easy to verify the lemma for small n. By induction, it is enough to show that $w \in S$. The proof in Type C includes all the essential elements of the general proof, and is particularly easy to state. We detail each inductive case.

(C1) Proposition 4.5.2 implies that $w = (\lambda_1, 0, \dots, 0)$. Recall the character notation of Section 5.2. Note that $k = \lambda_2$ and that the difference $w = [2n-k+1, k-1, 1] - [2n-2-k+1, k-1, 1]$ always lies in the one or two element set S.

(C2) This time, $w = (\lambda_2, 0, \dots, 0)$. Again using the notation of Section 5.2, we find that $w = [2n-k+1, k-1, 1] - [2n-k+1, k-3, 1]$ always lies in S.

(N1) Here, $w = (n-2, 0, \dots, 0)$. Using the notation of Section 5.2, $w = [n+1, n-1, 1] - [n+1, n-3, 1]$, which lies in S by 5.2.14.

(N2) Here, $w = (n-1, 0, \dots, 0)$. Using the notation of Section 5.2, $w = [n^2, 1] - [n, n-2, 1]$, which lies in S by 5.2.14.

This accounts for all the cases that arise in type C. For the other classical types, the proof requires the same inductive verification and follows inductively, except in one instance. When the partition corresponding to $\mathcal{O}$ has no parts of size 1, then $w_\beta \notin IC^1(\mathcal{O})$. This does not contradict the Lemma, as $\mathcal{O}$ is not rigid, but it does complicate the induction step. If $\mathcal{W}$ is an orbital variety such that $\mathcal{W}^\downarrow \subset \mathcal{O}$, then the associated w_β again lies in $IC^1(\mathcal{O})$. This proves Lemma 1.

Proof of Fact A. Write $I = [k, k+1, \dots l]$ for the non-zero non-simple dependent interval adjacent to b_p, and further assume that $p = k-1$. The proof for the other possibility is symmetric. Utilizing the notation suggested by Fact 5.1.2, the entries of I must have the form

$$b_l, b_{l-1}, \dots b_{m_1}), (b_{m_1-1}, \dots, b_{m_2}) \dots (b_{m_q-1}, \dots b_k.$$

We examine two possiblities. Either

· $l, k-1 \in T^2 \setminus T^1$ and $k \in T^1 \setminus T^2$, or

· $\{l, k\} \in N_1^T$.

The precise statement of the first case follows from the non-zero assumption on I; if a_{k-1} was constant and $k-1$ was contained in T^1, then I could not be non-zero. Hence consider the first possiblity. The entries of I must then have the form

$$a_l - \epsilon, a_{l-1} - \epsilon, \dots, a_{m_1} - \epsilon), (a_{m_1-1} + \epsilon, \dots a_{m_2} - \epsilon), \dots, (a_{m_q-1} + \epsilon, \dots a_k + \epsilon$$

for some ϵ since they must correspond to a weight in $HW^1(w_\alpha)$. Because all the entries grouped within parentheses must equal each other, according to Fact 5.1.1 this gives us the conditions

$$a_k + \epsilon = a_{k-1} \tag{5.3}$$

$$a_{m_i+1} + \epsilon = a_{m_{i+1}} - \epsilon \tag{5.4}$$

for all $i < q$, which translate to

$$\begin{aligned} \epsilon &= a_{k-1} - a_k \\ &= \tfrac{a_{m_{i+1}} - a_{m_i+1}}{2} \end{aligned} \tag{5.5}$$

for all $i < q$. We would like to show that these conditions are impossible to satisfy. Proposition 4.5.2 and Definition 5.3.8 give us a description of each of the a_i. We restrict the proof to case C, which contains all the elements of the general proof.

Let $[\lambda_1(i), \lambda_2(i)]$ be the partition dual to $\mathit{shape}\, T(i)$. Proposition 4.5.2 implies that

$$\begin{aligned} a_{k-1} &= -\lambda_1(k-1) + 2 \\ a_k &= -\lambda_2(k-1) + 2 \\ a_{m_2} &= -\lambda_1(m_2) + 2 \\ a_{m_1+1} &= -\lambda_2(m_1+1). \end{aligned}$$

Equations 5.5 translate to

$$\begin{aligned} \epsilon &= -\lambda_1(k-1) + \lambda_2(k-1) \\ &= \tfrac{-\lambda_1(m_2) + 2 + \lambda_2(m_1+1)}{2}. \end{aligned} \tag{5.6}$$

However, $\lambda_1(k-1) - \lambda_2(k-1) = \lambda_1(l) - \lambda_2(l)$ because I is a dependent interval. Furthermore, the form of the entries in I implies that $\lambda_2(l) > \lambda_2(m_1+1)$ and $\lambda_1(l) < \lambda_1(m_2)$. But this implies that it is impossible to satisfy equations 5.6. Hence we cannot find constants c_i so that $w(c_1, \dots c_s) \in HW^1(w_\alpha)$. The only difference in proof for the other classical types are the precise values for the a_i.

Now suppose that $\{k, l\} \in N_1^T$. The entries corresponding to the interval I must have the form

$$a_l + \beta, a_{l-1} - \epsilon, \dots, a_{m_1} - \epsilon), (a_{m_1-1} + \epsilon, \dots a_{m_2} - \epsilon), \dots, (a_{m_q-1} + \epsilon, \dots a_{k+1} + \epsilon, a_k + \beta.$$

Because I is non-simple, this means that the interval $\{k+1,\dots,l-1\}$ cannot be simple either. This time, we need to solve the equations

$$\begin{aligned}\epsilon &= a_k + \beta - a_{k+1} \\ &= a_{l-1} - a_l - \beta \\ &= \tfrac{a_{m_{i+1}} - a_{m_i+1}}{2}\end{aligned} \tag{5.7}$$

First, we find that $\beta = \frac{(a_{l-1}-a_l)+(a_{k+1}-a_k)}{2}$. This means that we still need to solve

$$\begin{aligned}\epsilon &= \tfrac{(a_{l-1}-a_l)-(a_{k+1}-a_k)}{2} \\ &= \tfrac{a_{m_{i+1}} - a_{m_i+1}}{2}\end{aligned} \tag{5.8}$$

By an analysis similar to the above, divided into each classical type, 5.8 again cannot be satisfied, and Fact A holds.

Proof of Fact B. If I_2 is a *zero* interval, then the proof is identical to the proof of Fact A, as the only property we needed was the expression for the term a_{k_1-1}, which is the same in the zero case. Now assume that I_1 is to the left of I_2 in the coordinate expression for w_γ of this section; the other possibility has a symmetric proof. There are again two cases in the proof. First assume that $\{k,n\} \notin N_1^T$. The two intervals must then have the form

$$a_{l_1} - \epsilon, a_{l_1-1} - \epsilon, \dots, a_{m_1} - \epsilon), (a_{m_1-1} + \epsilon, \dots a_{m_2} - \epsilon), \dots, (a_{m_q-1} + \epsilon, \dots a_{k_2} + \epsilon$$

and

$$a_{l_2} - \mu, a_{l_2-1} - \mu, \dots, a_{m'} - \mu), (a_{m'-1} + \mu, \dots, a_{k_2} + \mu$$

with the additional restriction that $a_{l_2} - \mu = a_{k_1} + \epsilon$. Write ρ in coordinates as $(\rho_n, \rho_{n-1}, \dots, \rho_1)$. The proof of Fact D and Fact C imply that either $\mu = 0$, or $\mu = a_{l_2} - a_{m'} + \rho_{l_2} - \rho_{m'}$. The first possiblity was considered above. As for the second, following the outline of the proof of Fact A, we would like to solve the equations

$$\begin{aligned}\epsilon &= a_{l_2} - \mu - a_k \\ &= \tfrac{a_{m_{i+1}} - a_{m_i+1}}{2}\end{aligned} \tag{5.9}$$

for all $i < q$. In each of the classical types, Proposition 4.5.2 gives us values for the a_i, and we can similarly give an explicit description of ρ. In a manner similar to the proof of Fact A, we can now show that a solution to 5.9 does not exists. A similar analysis works for the case when $\{k,l\} \in N_1^T$ and Fact B holds.

Proof of Fact C. Assume that $b_p = b_{k-1}$ as the proof for the other possiblity in symmetric. Utilizing the notation suggested by Fact 5.1.2, the entries of I must have the form

$$b_l, b_{l-1}, \dots b_m), (b_{m-1}, \dots b_k.$$

As in the proof of Fact A, there are two possiblities. Either $l, k-1 \in T^2 \backslash T^1$ and $k \in T^1 \backslash T^2$, or $\{l,k\} \in N_1^T$. The precise statement of the first case is due to the non-zero assumption on I. We examine the first case. The second in analogous. Write ρ in coordinates as

$(\rho_n, \ldots, \rho_1)$. The entries of w_γ have the form

$$a_l - \epsilon, a_{l-1} - \epsilon, \ldots, a_m - \epsilon), (a_{m-1} + \epsilon, \ldots, a_k + \epsilon$$

where entries grouped by parentheses must equal since $w_\gamma \in HW^1(w_\alpha)$. This condition further forces $a_{k-1} = a_k + \epsilon$, or in other words,

$$\epsilon = a_k - a_{k-1} \tag{5.10}$$

After examining the definition of the permutation σ_I, we need to verify that

$$a_{l+i} - \epsilon + \rho_{l+1} = a_{m+i} + \rho_{m+i} \tag{5.11}$$

for all $i < (l-k)/2$. This then implies Fact C. We proceed for type $G =$ A. First of all, $\rho_{l+i} = n + 1 - 2(l+i)$. Hence we would like to know whether the equality

$$a_{l_i} - \epsilon + n + 1 - 2(l+i) = a_{m+i} + n + 1 - 2(m+i)$$

holds. Proposition 4.5.2 implies that $a_{l+i} = a_l$ and $a_{m+i} = a_k$ for all of the above i. The above equation becomes $a_l - a_k + k - l + 1 = \epsilon$. This is possible iff this equation is compatible with 5.10. To verify this, we note that repeated application of Proposition 4.5.2 implies $a_{k-1} = -\lambda_1(k) + (\lambda_1 - \lambda_1(k)) = \lambda_1 - 2\lambda_1(k)$ which also equals $a_l + (k - l + 1)$. This implies that $a_l - a_k + (k - l + 1) = a_{k-1} - a_k$, as desired. Hence Fact C holds in type A. The proof for the groups of other types are analogous, only complicated by the appearance of horizontal dominos. However, dominos falling in cases (N2) or (N3) do not affect the dependent intervals because of Fact 4.5.1. Case (N1) is dealt with precisely as in the proof of Fact A.

Proof of Fact D. If I_2 is a *zero* dependent interval, then the proof is identical to the proof of Fact C. We would like to show that in fact, if I_1 is a non-zero simple dependent interval, then I_2 must be a zero dependent interval. We can assume that I_1 is to the left of I_2 in the coordinate notation we have grown accustomed to. As in Fact C, the interval I_1 has the form

$$a_{l_1} - \epsilon, a_{l_1-1} - \epsilon, \ldots, a_m - \epsilon), (a_{m-1} + \epsilon, \ldots, a_{k_1} + \epsilon$$

while the interval I_2 has the form

$$a_{l_2} - \mu, a_{l_2-1} - \mu, \ldots, a_{m'} - \mu), (a_{m'-1} + \mu, \ldots, a_{k_2} + \mu$$

with the additional constraint that $l_2 - 1 = k_2$. We would like to show that $\mu = 0$. Because $w_\gamma \in HW(w_\alpha)$, we know that $a_{l_2} - \mu = a_{k_l} + \epsilon$. But our proof of Fact C implies that infact, $a_{l_2} = a_{k_l} + \epsilon$, forcing μ to be zero. This implies Fact D.

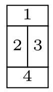
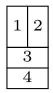
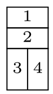
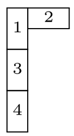
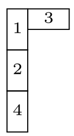
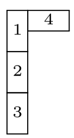
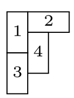
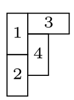
5.4 An Example

We find w_γ for all orbital varieties that arise in D_4.

Spherical Orbital Varieties in D_4

Partition of $\mathcal{O}$	$2 \cdot IC^1(\mathcal{O})$	ν_T	$2 \cdot w_\alpha$	$2 \cdot HW(w_\alpha)$	$2 \cdot HW^1(w_\alpha)$	$2 \cdot HW^1(w_\alpha) + 2\rho$
$[1^8]$	$(6,4,2,0)$	1 / 2 / 3 / 4	$((0,0,0,0))$	$((0,0,0,0))$	$((0,0,0,0))$	$(6,4,2,0)$
$[2^2,1^4]$	$(4,2,2,0)$	1 2 / 3 / 4	$((-2,-2,0),0)$	$((-2,-2,-\epsilon_1),\epsilon_1)$	$((-2,-2,-2),2)$	$(4,2,0,2)$
		1 3 / 2 / 4	$((-2,-2),(0,0))$	$((-2,-2-\epsilon_1),(\epsilon_1,0))$	$((-2,-2),(0,0))$	$(4,2,2,0)$
		1 4 / 2 / 3	$(-4,(0,0,0))$	$(-4-\epsilon_1,(\epsilon_1,0,0))$	$(-4,(0,0,0))$	$(2,4,2,0)$
		1 / 2 / 3 / 4	$((-2,-2,0,0))$	$((-2,-2,\epsilon_1,\epsilon_1))$	$((-2,-2,-2,-2))$	$(4,2,0,2)$
$[2^4]_I$	$(3,3,1,1)$	1 2 / 3 4	$(-2,(-2,0),0)$	$(-2-\epsilon_1,(-2+\epsilon_1,-\epsilon_2),\epsilon_2)$	$(-2-\epsilon_1,(-2+\epsilon_1,-2+\epsilon_1),2-\epsilon_1)$	$(4-\epsilon_1,2-\epsilon_1,\epsilon_1,2-\epsilon_1)$ $\epsilon_1=1$
		1 3 / 2 4	$((-2,-2),(0,0))$	$((-2-\epsilon_1,-2-\epsilon_2),(\epsilon_2,\epsilon_1))$	$((-2-\epsilon_1,-2-\epsilon_1),(\epsilon_1,\epsilon_1))$	$(4-\epsilon_1,2-\epsilon_1,2+\epsilon_1,\epsilon_1)$ $\epsilon_1=1$
		1 / 2 / 3 / 4	$((-2,-2,0,0))$	$((-2+\epsilon_1,-2+\epsilon_1,\epsilon_2,\epsilon_2))$	$((-2+\epsilon_1,-2+\epsilon_1,-2+\epsilon_1,-2+\epsilon_1,))$	$(4,2,0,2)$ $\epsilon_1=0$

Spherical Orbital Varieties in D_4 (cont.)

Partition of $\mathcal{O}$	$2 \cdot IC^1(\mathcal{O})$	ν_T	$2 \cdot w_\alpha$	$2 \cdot HW(w_\alpha)$	$2 \cdot HW^1(w_\alpha)$	$2 \cdot HW^1(w_\alpha) + 2\rho$
$[2^4]_{II}$	$(3,3,1,-1)$		$((-2,-2),(0,0))$	$((-2+\epsilon_1,-2-\epsilon_2),(\epsilon_2,\epsilon_1))$	$((-2-\epsilon_1,-2-\epsilon_1),(-\epsilon_1,\epsilon_1))$	$(4+\epsilon_1,2+\epsilon_1,2-\epsilon_1,\epsilon_1)$ $\epsilon_1 = -1$
			$((-2,-2,0),0)$	$((-2+\epsilon_1,-2+\epsilon_1,-\epsilon_2),\epsilon_2)$	$((-2+\epsilon_1,-2+\epsilon_1,-2+\epsilon_1),2-\epsilon_1)$	$(4+\epsilon_1,2+\epsilon_1,\epsilon_1,2-\epsilon_1)$ $\epsilon_1 = -1$
			$(-2,(-2,0,0))$	$(-2-\epsilon_1,(-2+\epsilon_1,\epsilon_2,\epsilon_2))$	$(-2-\epsilon_1,(-2+\epsilon_1,-2+\epsilon_1,-2+\epsilon_1))$	$(4-\epsilon_1,2+\epsilon_1,\epsilon_1,-2+\epsilon_1)$ $a = -1$
$[3,1^5]$	$(4,2,1,0)$ $(4,2,0,0)$		$((-2,-2,-2),0)$	$((-2,-2,\epsilon_1),\epsilon_2)$	$((-2,-2,-2),\epsilon_2)$	$(4,2,0,\epsilon_2)$ $\epsilon_2 = 0,\ 1$
			$((-2,-4),(0,0))$	$((-2,\epsilon_1),(\epsilon_2,0))$	$((-2,-2),(0,0))$	$(4,2,2,0)^*$
			$(-6,(0,0,0))$	$(-6+\epsilon_1,(\epsilon_2,0,0))$	$(-6+\epsilon_1,(0,0,0))$	$(\epsilon_1,4,2,0)$ $\epsilon_1 = 0,1$
$[3,2^2,1]$	$(3,2,1,0)$		$((-3,-4),(0,0))$	$((-3,\epsilon_1),(\epsilon_2,0))$	$((-3,-3),(0,0))$	$(3,1,2,0)$
			$(-3,(-2,-2),0)$	$(-3-\epsilon_1,(-2+\epsilon_1,\epsilon_1),\epsilon_2)$	$(-3-\epsilon_1,(-2+\epsilon_1,-2-\epsilon_1),\epsilon_2)$	$(3-\epsilon_1,2+\epsilon_1,\epsilon_1,\epsilon_2)$ $\epsilon_1 = 0,\ \epsilon_2 = 1$

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