On the sign representations for the complex reflection groups $G(r, p, n)$

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Abstract

We present a formula for the values of the sign representations of a complex reflection group $G(r, p, n)$ in terms of its image under a generalized Robinson-Schensted algorithm.

Keywords: Complex reflection groups, Robinson-Schensted map.

1 Introduction

The classical Robinson-Schensted algorithm establishes a bijection between permutations $w \in S_n$ and ordered pairs of same-shape standard Young tableaux of size n. This map has proven particularly well-suited to certain questions in the representation theory of both S_n and the semisimple Lie groups of type A. For instance, Kazhdan-Lusztig cells as well

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as the primitive spectra of semisimple Lie algebras can be readily described in terms of images of this correspondence.

Other sometimes more elementary representation-theoretic information requires more work to extract from standard Young tableaux. For instance, in independent work, A. Reifegerste [8] and J. Sjöstrand [10] developed a method for reading the value of the sign representation of a permutation $w \in S_n$ based on two tableaux statistics. Let $w \in S_n$ and write $RS(w) = (P, Q)$ for its image under the classical Robinson-Schensted map. If we write e for the number of squares in the even-indexed rows of P, let $sign(T)$ be the sign of a tableau T derived from its inversion number, and let sgn be the usual sign representation on S_n, then

$$sgn(w) = (-1)^e \cdot sign(P) \cdot sign(Q). \tag{1}$$

The focus of this note is to extend this result to the complex reflection groups $G(r, p, n)$. Its two main ingredients generalize readily to this setting. First, the Robinson-Schensted algorithm admits a straightforward extension mapping each $w \in G(r, p, n)$ to a same-shape pair of r-multitableaux, see R. Stanley [11, §6] and L. Iancu [4]. At the same time, the sign of a permutation in S_n extends to a family of r one-dimensional representations of $G(r, p, n)$. After defining new *spin* and *sign* statistics on r-multitableaux, we offer a short proof of the following:

Theorem. *Let $w \in G(r, p, n)$ and write $\mathbf{RS}(w) = (\mathbf{P}, \mathbf{Q})$ for its image under the generalized Robinson-Schensted map. Given a primitive r^{th} root of unity ζ and the associated family $\{sgn_i\}_{i=0}^{r-1}$ of representations of $G(r, p, n)$, we have*

$$sgn_i(w) = (-1)^{e(\mathbf{P})} \cdot (\zeta^i)^{spin(\mathbf{P})+spin(\mathbf{Q})} \cdot sign(\mathbf{P}) \cdot sign(\mathbf{Q}),$$

where $e(\mathbf{P})$ is the total sum of the lengths of the even-indexed rows of the component tableaux of $\mathbf{P}$.

A weaker version of this theorem has been used to verify a formula for the sign representation of the classical Weyl groups in type B for a family of domino tableaux Robinson-Schensted maps, see [7]. For classical Weyl groups, all of which appear among complex reflection groups, values of the sign representation can be used to compute the Möbius function of Bruhat order [12]. The above sign formulas show that the Möbius function is well-behaved with respect to the characterization of Kazhdan-Lusztig cells by equivalence classes of tableaux and multitableaux as in A. Joseph [5], S. Ariki [2], and C. Bonnafé and L. Iancu [3].

2 Preliminaries

After defining the family of complex reflection groups $G(r, p, n)$ and their one-dimensional representations, we define multipartitions, a generalization of the Robinson-Schensted algorithm, and tableaux statistics that we will use to describe these representations.

2.1 Sign representations

Consider positive integers r, p, and n with p dividing r and let $\zeta = \exp(2\pi\sqrt{-1})/r)$. We define the complex reflection groups $G(r,p,n)$ as subgroups of $GL_n(\mathbb{C})$ consisting of matrices such that

- the entries are either 0 or powers of ζ,
- there is exactly one nonzero entry in each row and column,
- the (r/p)-th power of the product of all nonzero entries is 1.

Together with the thirty-four exceptional groups, the groups $G(r,p,n)$ account for all finite groups generated by complex reflections [9], and include among them all the classical Weyl groups. In our work the parameter r will generally be fixed allowing us to write simply W_n for the group $G(r,1,n)$. In order to establish succinct notation, we will write

$$[\zeta^{a_1}\sigma_1, \zeta^{a_2}\sigma_2, \dots, \zeta^{a_n}\sigma_n]$$

for the matrix whose nonzero entry in the ith column is ζ^{a_i} and appears in row σ_i. Utilizing this notation, define the set $S = \{s_0, \dots, s_{n-1}\}$ where

$$\begin{aligned} s_0 &= [\zeta \cdot 1, 2, 3, \dots, n], \text{ and} \\ s_i &= [1, 2, \dots, i-1, i+1, i, i+2, \dots, n]. \end{aligned}$$

Furthermore, let $S' = \{s_0^p, s_0 s_1 s_0, s_i \mid 1 \leqslant i \leqslant n-1\}$. The set S generates W_n with presentation given as

$$W_n = \langle s_i \mid s_0^r, s_m^2, (s_j s_k)^2, (s_0 s_1)^4, (s_l s_{l+1})^3,\ m \geqslant 1,\ |j-k| > 1,\ l \in [1, n-2] \rangle.$$

Subject to similar relations, S' generates a subgroup $G(r,p,n)$ of W_n of index p, see S. Ariki [1]. Let $\sigma = [\sigma_1, \dots \sigma_n] \in S_n$, and define $Inv(\sigma)$ to be the set of pairs (σ_i, σ_j) with $i < j$ and $\sigma_i > \sigma_j$.

There are exactly $2r$ one-dimensional representations of W_n; they divide naturally into two families.

Definition 2.1. For each integer i between 0 and $r-1$, we define representations ς_i and sgn_i of W_n by specifying their values on the generating set S. Let

$$\tau_i^\epsilon(s_j) = \begin{cases} \zeta^i & \text{if } j = 0, \text{ and} \\ (-1)^\epsilon & \text{if } j = 1, \dots, n-1 \end{cases}$$

and define $\varsigma_i = \tau_i^0$ and $sgn_i = \tau_i^1$. Each becomes a representation of the subgroup $G(r,p,n)$ by restriction.

2.2 Multitableaux

We write a partition λ of an integer m as a nonincreasing sequence of positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ and define its rank as $|\lambda| = m$. A *Young diagram* $[\lambda]$ of λ is a left-justified array of boxes containing λ_i boxes in its ith row. The shape of a Young diagram will refer to its underlying partition. With the integer r fixed, a *multipartition of rank* n is an r-tuple

$$\boldsymbol{\lambda} = (\lambda^0, \lambda^1, \ldots, \lambda^{r-1})$$

of partitions the sum of whose individual ranks equals n. The *Young diagram* $[\boldsymbol{\lambda}]$ of $\boldsymbol{\lambda}$ is the r-tuple $([\lambda^0], \ldots, [\lambda^{r-1}])$. We refer to $\boldsymbol{\lambda}$ as the *shape* of the diagram $[\boldsymbol{\lambda}]$ and define $|\boldsymbol{\lambda}| = n$. We will follow a convention of denoting objects derived from multipartitions in boldface while writing those derived from single partitions using a normal weight font.

A *standard Young tableaux of shape* $\boldsymbol{\lambda}$ is the Young diagram $[\boldsymbol{\lambda}]$ of rank n together with a labeling of each of its boxes with the elements of $\mathbb{N}_n := \{1, 2, \ldots, n\}$ in such a way that each number is used exactly once, and the labels of the boxes within each component Young diagram $[\lambda^i]$ increase along its rows and down its columns. Remembering that r is fixed, we will write $\mathbf{SYT}_n$ for the set of all standard Young tableaux of rank n whose shape is a multipartition with r components.

Definition 2.2. A tableau $\mathbf{T} = (T_0, T_1, \ldots, T_{r-1}) \in \mathbf{SYT}_n$ will be called *ascending* if the elements of the set of labels of the component tableau T_i are pairwise smaller than the elements of the set of labels of T_j for each pair of integers i, j such that $0 \leqslant i < j \leqslant r-1$.

Example 2.3. Take $r = 3$. The following standard Young tableau $\mathbf{T}$ is of rank 11, has the shape $\boldsymbol{\lambda} = ((2,1), (1,1), (3,3))$, and is ascending:

$$\mathbf{T} = \left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\, , \begin{array}{|c|}\hline 4 \\ \hline 5 \\ \hline\end{array}\, , \begin{array}{|c|c|c|}\hline 6 & 7 & 10 \\ \hline 8 & 9 & 11 \\ \hline\end{array} \right)$$

Following [11, §6] and [4], we define a map from W_n to same-shape pairs of r-tuples of standard Young tableaux. Consider an element $w = [\zeta^{a_1}\sigma_1, \zeta^{a_2}\sigma_2, \ldots, \zeta^{a_n}\sigma_n] \in W_n$ and define the ordered sets $w^{(k)} = (\sigma_i \,|\, a_i = k)$ for $0 \leqslant k < r$. Let $Inv_P(w^{(k)}, w^{(l)})$ consist of $(i,j) \in Inv(\sigma)$ with $i \in w^{(l)}$ and $j \in w^{(k)}$, and let $Inv_Q(w^{(k)}, w^{(l)})$ consist of $(i,j) \in Inv(\sigma)$ with $i \in w^{(k)}$ and $j \in w^{(l)}$. Moreover, write $inv_P(w^{(k)}, w^{(l)})$ and $inv_Q(w^{(k)}, w^{(l)})$ for the respective cardinalities of these sets.

Let $RS(w^{(k)}) = (P_k, Q_k)$ be the image of the sequence $w^{(k)}$ under the usual Robinson-Schensted map, labeling squares of Q_k according to the relative positions of $i \in w^{(k)}$ within w, and define

$$\mathbf{P}(w) = (P_0, P_1, \ldots, P_{r-1}) \qquad \text{and} \qquad \mathbf{Q}(w) = (Q_0, Q_1, \ldots, Q_{r-1}).$$

The multitableaux Robinson-Schensted map is defined by $\mathbf{RS}(w) = (\mathbf{P}(w), \mathbf{Q}(w))$. It maps W_n onto the set of same-shape pairs of elements of $\mathbf{SYT}_n$ and is in fact a bijection.

2.3 Tableaux and multitableaux statistics

Our goal is to describe values of the sign representations on W_n under the above generalization of the Robinson-Schensted map. To do so, we rely on a few statistics that can be readily computed from multitableaux.

Definition 2.4. An *inversion* in a Young tableau T is a pair (i, j) with $j > i$ for which the box labeled by i is contained in a row strictly below the box labeled j. Let $Inv(T)$ be the set of inversions in T, and write $inv(T)$ for its cardinality. If $\mathbf{T} = (T_0, T_1, \dots T_{r-1})$ is a multitableau, we extend this notion and define:

$$Inv(\mathbf{T}) = \bigsqcup_k Inv(T_k) \sqcup \bigsqcup_{k<l} Inv(T_k, T_l)$$

where $Inv(T_k, T_l) = \{(j, i) \mid j > i, j \text{ is a label in } T_k, i \text{ is a label in } T_l\}$. We will be mainly interested in the parity of the size of this set and define

$$sign(\mathbf{T}) = (-1)^{inv(\mathbf{T})}.$$

Definition 2.5. For a Young tableau T, write $e(T)$ for the total number of boxes in its rows of even index. For a multitableau $\mathbf{T} = (T_0, T_1, \dots T_{r-1})$, we write $\mathrm{sh}(T_k)$ for the shape of the Young diagram underlying T_k and define the statistics e and $spin$ as follows:

$$e(\mathbf{T}) = \sum_{k=0}^{r-1} e(T_k) \quad \text{and} \quad spin(\mathbf{T}) = \frac{1}{2}\sum_{k=0}^{r-1} k \cdot |\,\mathrm{sh}(T_k)|.$$

The *spin* statistic provides a simple description of the image of the subgroup $G(r, p, n)$ under the r-multitableaux Robinson-Schensted map. The following is easy to verify:

Proposition 2.6. $(\mathbf{P}, \mathbf{Q}) \in \mathbf{RS}(G(r, p, n))$ *if and only if* $2\, spin(\mathbf{P}) \equiv 0 \pmod p$.

2.4 A set of functions and an example

We define a family of functions on W_n. In the next section we will show that they coincide with the sign representations on W_n. Again, for $w \in W_n$, let $\mathbf{RS}(w) = (\mathbf{P}, \mathbf{Q})$. For $0 \leqslant i < r$, we will write

$$\pi_i(w) = (-1)^{e(\mathbf{P})} \cdot (\zeta^i)^{spin(\mathbf{P})+spin(\mathbf{Q})} \cdot sign(\mathbf{P}) \cdot sign(\mathbf{Q}).$$

Example 2.7. Consider $w = [\zeta^1 5, 1, \zeta^2 3, 6, \zeta^2 7, \zeta^1 4, 2, 8]$ in $G(4, 1, 8)$. Recalling the notation in Section 2.2, we have $w^{(0)} = (1, 6, 2, 8)$, $w^{(1)} = (5, 4)$, $w^{(2)} = (3, 7)$, and $w^{(3)} = \emptyset$. Furthermore,

$$RS(w^{(0)}) = \left(\begin{array}{|c|c|c|}\hline 1 & 2 & 8 \\ \hline 6 \\ \cline{1-1}\end{array}\, , \begin{array}{|c|c|c|}\hline 2 & 4 & 8 \\ \hline 7 \\ \cline{1-1}\end{array} \right) \qquad RS(w^{(1)}) = \left(\begin{array}{|c|}\hline 4 \\ \hline 5 \\ \hline\end{array}\, , \begin{array}{|c|}\hline 1 \\ \hline 6 \\ \hline\end{array} \right)$$

$$RS(w^{(2)}) = \left(\begin{array}{|c|c|}\hline 3 & 7 \\ \hline\end{array}\, , \begin{array}{|c|c|}\hline 3 & 5 \\ \hline\end{array} \right) \qquad RS(w^{(3)}) = (\,\emptyset, \emptyset\,).$$

From these we construct the Robinson-Schensted image of w:

$$\mathbf{RS}(w) = (\mathbf{P},\mathbf{Q}) = \left(\left(\begin{array}{|c|c|c|}\hline 1 & 2 & 8 \\ \hline 6 \\ \cline{1-1}\end{array}, \begin{array}{|c|}\hline 4 \\ \hline 5 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 3 & 7 \\ \hline\end{array}, \emptyset \right), \left(\begin{array}{|c|c|c|}\hline 2 & 4 & 8 \\ \hline 7 \\ \cline{1-1}\end{array}, \begin{array}{|c|}\hline 1 \\ \hline 6 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 3 & 5 \\ \hline\end{array}, \emptyset \right) \right).$$

We read off $inv(\mathbf{P}) = 10$, $inv(\mathbf{Q}) = 14$, $e(\mathbf{P}) = 2$, and $spin(\mathbf{P}) = spin(\mathbf{Q}) = 3$. Hence $\pi_i(w) = (\zeta^i)^2$ which coincides with $sgn_i(w)$.

3 Sign under the Robinson-Schensted map

With the appropriate definitions of the tableaux statistics in place, we can now verify the claimed formulas for the family of sign representations $\{sgn_i\}$. Recall our notation $w = [\zeta^{a_1}\sigma_1, \zeta^{a_2}\sigma_2, \ldots, \zeta^{a_n}\sigma_n] \in W_n$ where $\sigma = \sigma_1 \cdots \sigma_n \in S_n$. Directly from the definitions of Inv_P and Inv_Q, we obtain the following partition:

$$Inv(\sigma) = \coprod_{k=0}^{r-1} Inv(w^{(k)}) \coprod_{k<l} Inv_P(w^{(k)}, w^{(l)}) \coprod_{k<l} Inv_Q(w^{(k)}, w^{(l)}). \tag{2}$$

Applying this to sgn_i, we have:

$$sgn_i(w) = (\zeta^i)^{\sum_{k=1}^n a_k} \prod_{k=0}^{r-1} sgn(w^{(k)}) \prod_{k<l} (-1)^{inv_P(w^{(k)},w^{(l)}) + inv_Q(w^{(k)},w^{(l)})}. \tag{3}$$

Write $\sigma^{\text{rev}} := \sigma_n\sigma_{n-1}\cdots\sigma_1 \in S_n$ and for fixed integers k, l such that $0 \leqslant k < l \leqslant r-1$, let

$$\begin{aligned} \mathcal{I}_1 &= Inv_P(w^{(k)}, w^{(l)}) = \{(i,j) \in Inv(\sigma) \mid i \in w^{(l)} \text{ and } j \in w^{(k)}\} \\ \mathcal{I}_2 &= Inv_Q(w^{(k)}, w^{(l)}) = \{(i,j) \in Inv(\sigma) \mid i \in w^{(k)} \text{ and } j \in w^{(l)}\}, \text{ and} \\ \mathcal{I}_3 &= \{(i,j) \in Inv(\sigma^{\text{rev}}) \mid i \in w^{(k)} \text{ and } j \in w^{(l)}\}. \end{aligned}$$

Lemma 3.1. *For fixed integers k and l as above, we have*

$$Inv(P_k, P_l) = \mathcal{I}_2 \sqcup \mathcal{I}_3 \qquad \textit{and} \qquad |\, Inv(Q_k, Q_l)| = |\mathcal{I}_1 \sqcup \mathcal{I}_3|.$$

Proof. To prove the first claim, let $(\sigma_i, \sigma_j) \in Inv(P_k, P_l)$. Then $\sigma_i > \sigma_j$, $\sigma_i \in w^{(k)}$, and $\sigma_j \in w^{(l)}$. If $i < j$ then $(\sigma_i, \sigma_j) \in Inv(\sigma)$ and hence lies in $\mathcal{I}_2$. On the other hand if $i > j$, then $(\sigma_i, \sigma_j) \in Inv(\sigma^{\text{rev}})$ and hence lies in $\mathcal{I}_3$. To prove the second, let $(j, i) \in Inv(Q_k, Q_l)$. Then $j > i$, $\sigma_j \in w^{(k)}$, and $\sigma_i \in w^{(l)}$. If $\sigma_i > \sigma_j$, then $(\sigma_i, \sigma_j) \in Inv(\sigma)$ and hence lies in $\mathcal{I}_1$. On the other hand if $\sigma_i < \sigma_j$, then $(\sigma_j, \sigma_i) \in Inv(\sigma^{\text{rev}})$ and hence lies in $\mathcal{I}_3$. Thus each $(j, i) \in Inv(Q_k, Q_l)$ corresponds to either a $(\sigma_i, \sigma_j) \in \mathcal{I}_1$ or a $(\sigma_j, \sigma_i) \in \mathcal{I}_3$. □

Immediately, we obtain:

Corollary 3.2. *For any $k < l$,*

$$inv_P(w^{(k)}, w^{(l)}) + inv_Q(w^{(k)}, w^{(l)}) \equiv inv(P_k, P_l) + inv(Q_k, Q_l) \pmod 2.$$

We are now ready to prove that the functions π_i defined in Subsection 2.4 coincide with sgn_i, hence proving our main theorem.

Theorem 3.3. *Let $w \in W_n$ and write $\mathbf{RS}(w) = (\mathbf{P}, \mathbf{Q})$ for its image under the generalized Robinson-Schensted map. Given a primitive r^{th} root of unity ζ and the associated family $\{sgn_i\}_{i=0}^{r-1}$ of representations of W_n, we have*

$$sgn_i(w) = (-1)^{e(\mathbf{P})} \cdot (\zeta^i)^{spin(\mathbf{P})+spin(\mathbf{Q})} \cdot sign(\mathbf{P}) \cdot sign(\mathbf{Q}).$$

Proof. Observe that the functions π_i can be decomposed as follows

$$\begin{aligned}
\pi_i(w) &= (-1)^{e(\mathbf{P})} \cdot (\zeta^i)^{spin(\mathbf{P})+spin(\mathbf{Q})} \cdot sign(\mathbf{P}) \cdot sign(\mathbf{Q}) \\
&= (-1)^{\sum_{k=0}^{r-1} e(P_k)} \cdot (\zeta^i)^{\sum_{k=1}^{n} a_k} \cdot \prod_{k=0}^{r-1} sign(P_k) \cdot \prod_{k=0}^{r-1} sign(Q_k) \cdot \prod_{k<l} (-1)^{inv(P_k,P_l)+inv(Q_k,Q_l)} \\
&= (\zeta^i)^{\sum_{k=1}^{n} a_k} \cdot \prod_{k=0}^{r-1} \left((-1)^{e(P_k)}\, sign(P_k)\, sign(Q_k)\right) \cdot \prod_{k<l} (-1)^{inv(P_k,P_l)+inv(Q_k,Q_l)}
\end{aligned}$$

Since each $(-1)^{e(P_k)}\, sign(P_k)\, sign(Q_k)$ coincides with $sgn(w^{(k)})$ by Equation (1), we have

$$\begin{aligned}
\pi_i(w) &= (\zeta^i)^{\sum_{k=1}^{n} a_k} \cdot \prod_{k=0}^{r-1} sgn(w^{(k)}) \cdot \prod_{k<l} (-1)^{inv(P_k,P_l)+inv(Q_k,Q_l)} \\
&= (\zeta^i)^{\sum_{k=1}^{n} a_k} \cdot \prod_{k=0}^{r-1} sgn(w^{(k)}) \cdot \prod_{k<l} (-1)^{inv_P(w^{(k)},w^{(l)})+inv_Q(w^{(k)},w^{(l)})} \\
&= sgn_i(w)
\end{aligned}$$

where the second equality holds as a consequence of Corollary 3.2 and the final equality holds by Equation (3). □

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