ORBITAL VARIETIES AND UNIPOTENT REPRESENTATIONS

THOMAS PIETRAHO

Abstract. Using the notion of a Lagrangian covering, W. Graham and D. Vogan proposed a method of constructing representations from the coadjoint orbits for a complex semisimple Lie group G. When the coadjoint orbit $\mathcal{O}$ is nilpotent, a representation of G is attached to each orbital variety of $\mathcal{O}$ in this way. In the setting of classical groups, we show that whenever it is possible to carry out the Graham-Vogan construction for an orbital variety of a spherical $\mathcal{O}$, its infinitesimal character lies in a set of characters attached to $\mathcal{O}$ by W. M. McGovern. Furthermore, we show that it is possible to carry out the Graham-Vogan construction for a sufficient number of orbital varieties to account for all the infinitesimal characters in this set.

1. Introduction

Consider a complex semisimple Lie group G. The philosophy of the orbit method seeks to parameterize the unitary dual of G via orbits of the coadjoint action of G on the dual of its Lie algebra. The representations attached to nilpotent coadjoint orbits are the so-called unipotent representations of G. In [8], W. Graham and D. Vogan have proposed a general method of attaching a set of representations to each coadjoint orbit $\mathcal{O}$ of G. Very little is known about which representations of G actually arise in this way, but conjecturally, when $\mathcal{O}$ is nilpotent, they should coincide with the set of unipotent representations corresponding to $\mathcal{O}$. The goal of this paper is to shed some light on the situation, examining the Graham-Vogan construction for the family of spherical nilpotent orbits of G.

The idea behind the construction of [8] is to generalize the method of polarization. Instead of using Lagrangian foliations, it relies on the more general notion of a Lagrangian covering of a coadjoint orbit. When $\mathcal{O}$ is nilpotent, the main ingredients of a Lagrangian covering are certain Lagrangian submanifolds called *orbital varieties*. For each choice of orbital variety $\mathcal{V}$ and a choice of an *admissible orbit datum* π, the Graham-Vogan construction defines a subspace $V(\mathcal{V}, \pi)$ of sections of a bundle over $G/Q_{\mathcal{V}}$, where $Q_{\mathcal{V}} \subset G$ is the maximal subgroup of G stabilizing $\mathcal{V}$.

To examine which representations of G arise among the spaces $V(\mathcal{V}, \pi)$, we rely on a combinatorial description of orbital varieties in classical groups obtained by W. M. McGovern, [13], as well as the author, [18]. They are parametrized by standard Young tableaux in type A and standard domino tableaux in the other classical types. There are two advantages to this description. First, it is easy to determine $Q_{\mathcal{V}}$ from the tableau parameterizing $\mathcal{V}$. Second, the parametrization itself suggests a means of addressing our work inductively, setting up a framework for our calculations.

As our main goal is to determine how well the $V(\mathcal{V}, \pi)$ fit the role of unipotent representations, we would like to describe what reasonable conditions for this might be. There is at least one commonly accepted necessary criterion for a representation

V to be attached to an orbit $\mathcal{O}$. According to work of Borho and Brylinski, the variety $\mathcal{V}(\mathrm{Ann}(V)) \subset \mathfrak{g}^*$ associated to the annihilator of V in the universal enveloping algebra of $\mathfrak{g}$ is the closure of a single nilpotent orbit if V is irreducible. Thus, for a unipotent representation V arising from the nilpotent orbit $\mathcal{O}$, we should expect

$$\mathcal{V}(\mathrm{Ann}(V)) = \overline{\mathcal{O}}.$$

A classification of unitary representations of complex reductive Lie groups can be obtained from a construction which begins with a set of *special* unipotent representations first suggested by J. Arthur, see [1]. However, only *special* nilpotent orbits arise as associated varieties of special unipotent representations. To remedy this shortfall, McGovern has suggested extending the set of special unipotent representations to a set of q-unipotent representations whose associated varieties include all nilpotent orbits of G [12]. Included in his work is a description of the infinitesimal characters of q-unipotent representations for classical groups, suggesting a natural benchmark for examining the Graham-Vogan spaces. After incorporating certain geometric considerations into McGovern's list, we will define a set $IC^1(\mathcal{O})$ of infinitesimal characters attached to each nilpotent orbit $\mathcal{O}$. The set of infinitesimal characters of the representations attached to $\mathcal{O}$ should contain $IC^1(\mathcal{O})$. Our main result is that for spherical $\mathcal{O}$, this is exactly what happens. We paraphrase this as follows:

Theorems 4.18 and 4.19. *Let $\mathcal{O}$ be a spherical nilpotent orbit of a complex classical semisimple Lie group G of rank n and write $GV_{\mathcal{O}}$ for the set of representations $V(\mathcal{V},\pi)$ arising for some $\mathcal{V} \subset \mathcal{O}$. Let $\chi_{\mathcal{V}}$ be the infinitesimal character associated to $V(\mathcal{V},\pi)$. Then,*

(i) If $\mathcal{O}$ is rigid, then $IC^1(\mathcal{O}) = \{\chi_{\mathcal{V}} \mid V(\mathcal{V},\pi) \in GV_{\mathcal{O}}\}$,
(ii) If $\mathcal{O}$ is a model orbit and $n > 2$, then $IC^1(\mathcal{O}) \subset \{\chi_{\mathcal{V}} \mid V(\mathcal{V},\pi) \in GV_{\mathcal{O}}\}$.

The above theorems imply that, at least for spherical nilpotent orbits, the Graham-Vogan spaces are indeed candidates for unipotent representations. For larger non-spherical orbits, calculations indicate that the family of infinitesimal characters of representations in $GV_{\mathcal{O}}$ is too numerous to form the set of representations attached to $\mathcal{O}$. However, additional conditions on the closure $\overline{\mathcal{O}}$ not considered in the construction [8] should make it possible to restrict the resulting set of possible infinitesimal characters.

The paper is structured as follows. Section 2.1 presents a summary of the construction of the spaces $V(\mathcal{V},\pi)$ and details the combinatorics of nilpotent coadjoint orbits and orbital varieties in classical groups. Section 3 begins with an example which the rest of the paper is designed to mimic. A crucial assumption is that the stabilizing parabolic $Q_{\mathcal{V}}$ has a dense orbit in $\mathcal{V}$. We restrict our attention to nilpotent orbits all of whose orbital varieties enjoy this property and detail this restriction in terms of the combinatorics of Section 2.1. The section concludes with a description of the inductive process we will use in the rest of the paper. Finally, Section 4 addresses infinitesimal characters. We begin by detailing conditions under which it is possible to carry out the Graham-Vogan construction. After defining the desired set $IC^1(\mathcal{O})$ of infinitesimal characters that ought to be attached to the orbit $\mathcal{O}$, we compute which ones arise from spaces of the form $V(\mathcal{V},\pi)$.

2. Preliminaries

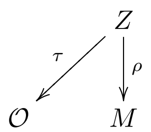
We begin this section with a brief outline of the Graham-Vogan construction. In the setting of classical groups, both nilpotent orbits and orbital varieties, on which this construction relies, admit combinatorial descriptions. We summarize these and list a few useful results.

2.1. **The Graham-Vogan Construction.** Let G be a semisimple Lie group and $\mathfrak{g}$ its Lie algebra. The coadjoint orbit through a point $f \in \mathfrak{g}^*$ is the set

$$\mathcal{O}_f = G \cdot f \cong G/G_f$$

where we write G_f for the isotropy subgroup of the coadjoint action of G based at f. The nondegeneracy of the Killing form permits us to identify the set of coadjoint orbits with the set of adjoint orbits. The Graham-Vogan construction of representations associated to a coadjoint orbit $\mathcal{O}$ is an extension of the method of polarizing a coadjoint orbit. We briefly recount this work, following [8]. It begins with the notion of a Lagrangian covering.

Definition 2.1. A *Lagrangian covering* of a symplectic manifold $\mathcal{O}$ is a pair (Z, M) of manifolds and smooth maps (τ, ρ)



such that the diagram is a double fibration and each fiber of ρ is a Lagrangian submanifold of $\mathcal{O}$.

Theorem 2.2 ([7])**.** *Let G be a complex reductive Lie group and $\mathcal{O}$ be a coadjoint orbit. Then there exists an equivariant Lagrangian covering of $\mathcal{O}$ where M is a partial flag variety for G.*

We are interested in this construction when $\mathcal{O}$ is a nilpotent coadjoint orbit. Fix a Borel subgroup B of G with unipotent radical N. Write $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{t} \oplus \mathfrak{n}$ for the corresponding triangular decomposition. Let us restrict our attention to nilpotent coadjoint orbits $\mathcal{O}$, and consider the set $\mathcal{O} \cap \mathfrak{n}$. This is a locally closed subset of $\mathfrak{n}$ and can be expressed as a union of its irreducible components.

Definition 2.3. Consider a nilpotent coadjoint orbit $\mathcal{O}$. Denote the set of irreducible components of the variety $\mathcal{O} \cap \mathfrak{n}$ by $\mathrm{Irr}(\mathcal{O} \cap \mathfrak{n})$. Each element of $\mathrm{Irr}(\mathcal{O} \cap \mathfrak{n})$ is an *orbital variety* for $\mathcal{O}$.

Proposition 2.4. *The set $Irr(\mathcal{O} \cap \mathfrak{n})$ is finite. Further, every orbital variety $\mathcal{V} \in Irr(\mathcal{O} \cap \mathfrak{n})$ has $\dim \mathcal{V} = \frac{1}{2} \dim \mathcal{O}$ and is a Lagrangian subvariety of $\mathcal{O}$.*

We will construct a distinct Lagrangian covering for each orbital variety contained in $\mathcal{O}$. Fix an orbital variety $\mathcal{V}$ and let $\mathcal{V}^0$ be its smooth part. Let $Q = Q_{\mathcal{V}} = \{q \in G \,|\, q \cdot \mathcal{V} = \mathcal{V}\}$. This is a parabolic subgroup of G since $\mathcal{V}$ is B-stable. Furthermore, define the manifold M by $M = \{g \cdot \mathcal{V} \,|\, g \in G\} \cong G/Q$. It is a partial flag variety for G.

Definition 2.5. For a subgroup $H \subset G$ and an H-space V, let $G \times_H V$ to be the set of equivalence classes in $G \times V$ with $(gh, v) \sim (g, h \cdot v)$ for $g \in G, h \in H$, and $v \in V$.

The manifold Z in the Lagrangian covering of $\mathcal{O}$ associated to the orbital variety $\mathcal{V}$ is now defined to be $Z = G \times_Q \mathcal{V}^0$. The map $\rho : Z \longrightarrow M$ arises from the projection of G onto G/Q. The action of G on $\mathcal{O}$ gives natural map $G \times \mathcal{V} \to \mathcal{O}$. It descends to an algebraic map $\tau : Z \longrightarrow \mathcal{O}$. We now have a Lagrangian covering:

$$\begin{array}{ccc} & & G \times_Q \mathcal{V}^0 \\ & \overset{\tau}{\swarrow} & \downarrow{\scriptstyle \rho} \\ G/G_f & & G/Q \end{array}$$

Because the diagram is a double fibration, we can identify fibers of ρ with subsets of $\mathcal{O}$. In fact, each fiber is Lagrangian in $\mathcal{O}$.

The next step is to construct a representation from this Lagrangian covering. Suppose that we have a G-equivariant line bundle $\mathcal{L}_M \to M$. We can again pull this bundle back along the fibration ρ, this time to obtain a bundle $\mathcal{L}_Z$.

$$\begin{array}{ccccc} & & G \times_Q \mathcal{V}^0 & \longleftarrow & \mathcal{L}_Z \\ & \overset{\tau}{\swarrow} & \downarrow{\scriptstyle \rho} & & \uparrow{\scriptstyle \rho^*} \\ \mathcal{O} & & M & \longleftarrow & \mathcal{L}_M \end{array}$$

Geometric quantization suggests that the representations attached to $\mathcal{O}$ should lie in the space of sections of $\mathcal{L}_M$, or in other words, in the space of sections of $\mathcal{L}_Z$ that are constant on the fibers of ρ. This is very similar to the situation arising in the polarization construction, as the fibers of ρ can again be identified with Lagrangian submanifolds of $\mathcal{O}$. As described in [8], however, the full set of sections of $\mathcal{L}_M$ is too large to quantize $\mathcal{O}$ and we pick out a subspace.

We relate only a general overview, and direct the reader to [8] itself for the relevant details. The main idea is to prune the full space of sections of $\mathcal{L}_M$, leaving ones which also come from an *admissible orbit datum* of $\mathcal{O}$.

To do this, one must first attach a geometric structure to each orbit datum. This is achieved by mimicking the construction of a Hermitian bundle that often arises in descriptions of geometric quantization of *integral* orbit data. The main difficulty then lies in finding a way of embedding the information from this bundle into the space of sections of $\mathcal{L}_M$.

Definition 2.6. An *admissible orbit datum* at $f \in \mathfrak{g}^*$ is a genuine irreducible unitary representation π of the metaplectic cover $\widetilde{G}_f$ satisfying

$$\pi(\exp Y) = \chi(f(Y))$$

for a fixed non-trivial character χ of $\mathbb{R}$.

Denote the metaplectic representation of $\widetilde{G}_f$ by τ_f and form the tensor product representation $\pi \otimes \tau_f$. While τ_f and π are genuine representations of $\widetilde{G}_f$, $\pi \otimes \tau_f$ in fact descends to a representation of G_f itself. This allows us to define a Hilbert bundle over the coadjoint orbit $\mathcal{O}$ by

$$\mathcal{S}_\pi = G \times_{G_f} (\pi \otimes \tau_f).$$

This is the *bundle of twisted symplectic spinors* on $\mathcal{O}$. The metaplectic representation τ_f of $\widetilde{G}_f$ decomposes into two irreducible and inequivalent representations τ_f^{odd} and τ_f^{even}. Write $\tau_f^{odd,\infty}$ and $\tau_f^{even,\infty}$ for the corresponding sets of smooth vectors.

This decomposition passes to the bundle $\mathcal{S}_\pi$ and the geometric structure attached to the admissible orbit datum π is the subbundle of $\mathcal{S}_\pi$ defined by

$$\mathcal{S}_\pi^{even,\infty} = G \times_{G_f} (\pi \otimes \tau_f^{even,\infty}).$$

Definition 2.7. Suppose that X is a symplectic manifold. The *bundle of infinitesimal Lagrangians* on X is a fiber bundle $\mathcal{B}(X)$ over X. The fiber over each point $x \in X$ is the set of Lagrangian subspaces of the tangent space at x of X, denoted by $\mathcal{B}(T_x X)$.

Definition 2.8. Let $\mathcal{O}$ be a coadjoint orbit, and consider a Lagrangian $\mathcal{V}$ in the tangent space $\mathfrak{g}/\mathfrak{g}_f$. Write $\mathcal{L}(\mathcal{V})$ for the line defined in [8, 7.4(c)] from the metaplectic representation τ_f. The admissible orbit datum π defines a G-equivariant vector bundle $\mathcal{V}_\pi$ on $\mathcal{B}(\mathcal{O})$ by letting the fiber at each $\mathcal{V}$ be $\mathcal{H}_\pi \otimes \mathcal{L}(\mathcal{V})$.

Theorem 2.9 ([8],[10])**.** *There exists a natural inclusion*

$$i : C^\infty(\mathcal{O}, \mathcal{S}_\pi^{even,\infty}) \hookrightarrow C^\infty(\mathcal{B}(\mathcal{O}), \mathcal{V}_\pi).$$

Next, we incorporate the bundle $\mathcal{V}_\pi$ over $\mathcal{B}(\mathcal{O})$ into the Lagrangian covering diagram. Define a map $\sigma : Z \to \mathcal{B}(\mathcal{O})$ as follows. Fix $z \in Z$. The definition of Lagrangian covering forces the fiber of ρ over $\rho(z) \in M$ to be a Lagrangian submanifold of $\mathcal{O}$ that contains $\tau(z)$. Hence its tangent space $T_{\tau(z)}(\rho^{-1}(\rho(z))$ is a Lagrangian subspace of $T_{\tau(z)}(\mathcal{O})$ and thus an element of $\mathcal{B}(\mathcal{O})$. Let

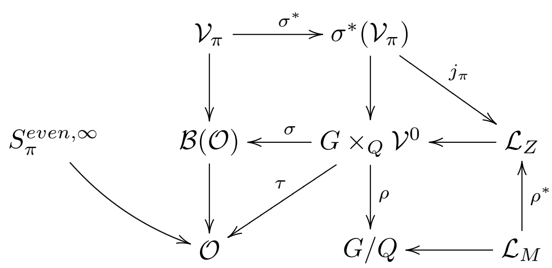
$$\sigma(z) = T_{\tau(z)}(\rho^{-1}(\rho(z)).$$

In this way, σ becomes a bundle map over $\mathcal{O}$.

We can pull back the bundle $\mathcal{V}_\pi$ along σ to a bundle $\sigma^*(\mathcal{V}_\pi)$ over Z. Smooth sections of $\mathcal{V}_\pi$ pull back to smooth sections of $\sigma^*(\mathcal{V}_\pi)$ and we have an injective map $\sigma^* \cdot i : C^\infty(\mathcal{O}, \mathcal{S}_\pi^{even,\infty}) \hookrightarrow C^\infty(Z, \sigma^*(\mathcal{V}_\pi))$. *Provided that* there is a G-equivariant vector bundle isomorphism $j_\pi : \sigma^*(\mathcal{V}_\pi) \to \rho^*(\mathcal{L}_M)$ we can define a smooth representation of G as:

$$V(\mathcal{V}, \pi) = \rho^*(C^\infty(M, \mathcal{L}_M)) \cap j_\pi(\sigma \cdot i(C^\infty(\mathcal{O}, \mathcal{S}_\pi^{even,\infty})))$$

If $\mathcal{L}_M$ is given by a representation γ of the parabolic subgroup Q, then $V(\mathcal{V}, \pi)$ lies in the space of smooth vectors of the degenerate principal series representation induced from γ. The entire construction may be summarized by the following diagram.



2.2. **Nilpotent Orbits in Classical Types.** The nilpotent coadjoint orbits of classical complex simple Lie groups are parametrized by partitions. When G is of type A, Jordan block sizes determine nilpotent adjoint orbits leading to a one-to-one correspondence between them and partitions. In classical groups not of type A, the parameterization has the same flavor, but not all partitions arise from Jordan block decompositions. We describe the details presently.

To be specific, let $\epsilon = \pm 1$ and take $<,>_\epsilon$ to be a non-degenerate bilinear form on $\mathbb{C}^m$ such that

$$< x, y >_\epsilon = \epsilon < y, x >_\epsilon \; \forall x, y \in \mathbb{C}^m.$$

Let G_ϵ be the isometry group of this form and write $\mathfrak{g}_\epsilon$ for its Lie algebra. If we set $\mathcal{P}_\epsilon(m)$ to be the set of partitions of m in which all even parts occur with even multiplicity when $\epsilon = 1$ and all odd parts occur with even multiplicity when $\epsilon = -1$, then the classification of nilpotent coadjoint orbits takes the form:

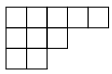
Theorem 2.10 ([6])**.** *The nilpotent coadjoint orbits of G_ϵ are in one-to-one correspondence with the set of partitions of $\mathcal{P}_\epsilon(m)$.*

When $\epsilon = -1$, G_ϵ is a group of type C; for $\epsilon = 1$ it is of type B when m is odd and type D when m is even. In types B and C, the adjoint orbits for the isometry group G_ϵ coincide with the adjoint orbits for the adjoint group G_{ad} of $\mathfrak{g}_\epsilon$. In type D, however, the adjoint group is $PSO(m/2, \mathbb{C})$ and every G_ϵ-orbit $\mathcal{O}$ that corresponds to a very even partition, that is, a partition with only even parts each of which appears with even multiplicity, is the union of two $PSO(m/2, \mathbb{C})$-orbits. We will denote them as $\mathcal{O}^I$ and $\mathcal{O}^{II}$.

In what follows, we will write $\mathcal{O}_\lambda$ for the nilpotent G_ϵ-orbit associated with the partition λ when the type of the underlying group is clear.

2.3. **Orbital Varieties in Classical Types.** While partitions were sufficient to describe the set of nilpotent orbits for a classical simple Lie group, somewhat more intricate combinatorial objects are necessary to describe the orbital varieties contained within each nilpotent orbit.

Definition 2.11. Let λ be a partition. A *Young diagram of shape* λ is a finite left-justified array of squares the length of whose ith row equals the ith part of λ.



Write $\mathbb{N}_n = \{1, 2, \dots, , n\}$. A *standard Young tableau of* λ is a Young diagram of shape $\lambda \dashv n$ whose squares are labeled by elements of $\mathbb{N}_n$ in such a way that each element of $\mathbb{N}_n$ labels exactly one square, and all labels increase along both rows and columns.

Definition 2.12. Let $r \in \mathbb{N}$ and λ be a partition of a positive integer m. A *standard domino tableau of rank r and shape* λ is a Young diagram of shape λ whose squares are labeled by elements of $\mathbb{N}_n \cup \{0\}$ in such a way that the integer 0 labels the square s_{ij} iff $i + j < r + 2$, each element of $\mathbb{N}_n$ labels exactly two adjacent squares, and all labels increase weakly along both rows and columns. We will write $SDT_r(\lambda)$ for the family of all domino tableaux of rank r and shape λ and $SDT_r(n)$ for the family of all domino tableaux of rank r which contain exactly n dominos.

Let $\mathfrak{b} \subset \mathfrak{g}$ be a Borel subalgebra, $\mathfrak{h} \subset \mathfrak{b}$ a Cartan subalgebra, and $\mathfrak{n}$ the nilradical so that $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$. Recall that for a nilpotent orbit $\mathcal{O}$, the irreducible components $\text{Irr}(\mathcal{O} \cap \mathfrak{n})$ are its orbital varieties. By [11], every orbital variety takes the form

$$V(w) = \overline{B(\mathfrak{n} \cap w^{-1}\mathfrak{n})} \cap \mathcal{O}$$

for some w in the Weyl group W. The set of Weyl group elements which map to the same orbital variety under this correspondence is known as a *geometric left cell.* In light of the Robinson-Schensted algorithms which associate elements of W with same-shape pairs of standard Young and domino tableaux (see [5]), the following results are somewhat natural:

Theorem 2.13 ([11]). *In type A, orbital varieties contained in the nilpotent orbit $\mathcal{O}_\lambda$ are parameterized by the set of standard Young tableaux of shape λ.*

Theorem 2.14 ([13],[18]). *In types C and D, orbital varieties contained in the nilpotent G_ϵ-orbit $\mathcal{O}_\lambda$ are parameterized by standard domino tableaux of rank zero and shape λ. In type B, orbital varieties contained in the nilpotent G_ϵ-orbit $\mathcal{O}_\lambda$ are parameterized by standard domino tableaux of rank one and shape λ.*

The construction of the Graham-Vogan space associated to an orbital variety $\mathcal{V}$ requires us to be able to explicitly identify its τ-invariant. We describe how to do this for an orbital variety $\mathcal{V}_T$ corresponding to a standard tableau T.

Let Δ be the set of roots in $\mathfrak{g}$, Δ^+ the set of positive roots and Π the set of simple roots. Write $\mathfrak{g} = \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_{-\alpha} \oplus \mathfrak{t} \oplus \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha$ for the triangular decomposition of $\mathfrak{g}$ and let $\mathfrak{b} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha$. Write W for the Weyl group, and let P_α be the standard parabolic subgroup with Lie algebra $\mathfrak{p}_\alpha = \mathfrak{b} \oplus \mathfrak{g}_{-\alpha}$. Following [11], for an element $w \in W$, an orbital variety $\mathcal{V}$, and a standard parabolic subgroup we define

$$\tau(P) = \{\alpha \in \Pi \mid P_\alpha \subset P\} \text{ and}$$
$$\tau(\mathcal{V}) = \{\alpha \in \Pi \mid P_\alpha(\mathcal{V}) = \mathcal{V}\}.$$

We would like to be able to read off $\tau(\mathcal{V})$ from the standard tableau parameterizing $\mathcal{V}$ as the maximal parabolic subgroup Q stabilizing $\mathcal{V}$ is precisely the standard parabolic subgroup satisfying $\tau(Q) = \tau(\mathcal{V})$.

Theorem 2.15 ([11]). *Consider an orbital variety $\mathcal{V}_T$ in type A that corresponds to the standard Young tableau T under the above parametrization. The simple root $\alpha_i \in \Pi$ lies in $\tau(\mathcal{V}_T)$ iff the square labeled i lies strictly higher in T than the square with label $i+1$.*

Theorem 2.16 ([18]). *Consider an orbital variety $\mathcal{V}_T$ in type B, C, or D that corresponds to the standard domino tableau T under the above parametrization. The simple root α_i lies in $\tau(\mathcal{V}_T)$ iff one of the following conditions is satisfied:*

(i) $i = 1$ and the domino with label 1 is vertical,
(ii) $i > 1$ and domino with label $i-1$ lies higher that the domino with label i in T.

Finally, we would like to define a map from the orbital varieties in types B, C, and D to orbital varieties in type A. Let $\mathfrak{g}$ be a classical complex Lie algebra of type $X_n = B_n$, C_n, or D_n and let $\mathfrak{n}$ be the unipotent part of $\mathfrak{b}$. There is a natural projection map π_A from $\mathfrak{n}$ to $\mathfrak{n}_A$, the corresponding unipotent part in type A_{n-1}. Let $\mathcal{O}$ be a nilpotent orbit of type X_n. The image of an orbital variety for $\mathcal{O}$ under π_A is always an orbital variety for some nilpotent orbit $\mathcal{P}$ of type A. In fact, if $\mathcal{P}$ arises in this way, then *all* of its orbital varieties lie in the image of π_A for $\mathcal{O}$. To describe this in terms of the underlying combinatorics, we need a result of Carre and Leclerc.

Theorem 2.17 ([3]). *There is a bijection*

$$SDT(\lambda) \xrightarrow[(\pi_1,\pi_2)]{} \amalg_\nu Yam_2(\lambda,\nu) \times SYT(\nu).$$

where $Yam_2(\lambda, \nu)$ is the family of Yamanouchi domino tableaux of shape λ and evaluation μ.

The bijection itself is an algorithm that takes a tableau T and modifies it successively until its column reading becomes a Yamanouchi word. The standard Young tableau records the sequence of moves. We are interested only in the second coordinate of this map.

Definition 2.18. Define a map

$$\pi_A : SDT(n) \longrightarrow SYT(n)$$

by $\pi_A(T) = \pi_2(T)$ where π_2 is the second component of the Carre-Leclerc map. We also denote by π_A the map induced on orbital varieties obtained by identifying T with $\mathcal{V}_T$.

3. Restriction to Spherical Orbital Varieties

Armed with a description of the orbital varieties contained in a given nilpotent orbit as well as the corresponding τ-invariants, we now begin to describe the Graham-Vogan representations attached to a nilpotent orbit in the setting of classical groups. We begin by illustrating our method with an example, which is sufficiently naïve to quickly describe our approach.

3.1. **Model Example.** We will calculate the infinitesimal character associated to $V(\mathcal{V}, \pi)$ constructed from a particular orbital variety in type C. Suppose $G = Sp(8)$ and realize the Lie algebra $\mathfrak{g}$ as a set of 8×8 matrices of the form

$$\mathfrak{sp}(8) = \left\{ \mathfrak{m}(A, B, C) = \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \mid A, B, C \in M_4(\mathbb{C}) \text{ and } B, C \in Sym_4(\mathbb{C}) \right\}.$$

Let $\mathcal{O}$ be the nilpotent coadjoint orbit in $\mathfrak{g}^*$ corresponding to the partition $[2^3, 1^2]$. It has dimension 18. There are four orbital varieties contained in $\mathcal{O}$, corresponding to the domino tableaux:

Let $\mathcal{V}$ be the orbital variety corresponding to the first domino tableau. Then $\dim \mathcal{V} = \frac{1}{2} \dim \mathcal{O} = 9$. As a representative, we take $f = \mathfrak{m}(A, B, 0)$ with

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

To describe the Graham-Vogan space for $\mathcal{V}$, we need to compute the following parameters: an admissible orbit datum $(\pi, \mathcal{H}_\pi)$, $\mathcal{V}^o$, the smooth part of $\mathcal{V}$, the stabilizing parabolic $Q_\mathcal{V} \subset G$, a smooth representation (γ, W_γ) of $Q_\mathcal{V}$, and a G-equivariant isomorphism of vector bundles j_π, where notation is as in Section 2.1

Write G_f for the isotropy subgroup of f and $\mathfrak{g}_f$ for its Lie algebra. As G is complex, the metaplectic cover $\widetilde{G}_f$ is isomorphic to $G_f \times \mathbb{Z}/2\mathbb{Z}$. We choose one admissible orbit datum; it is trivial on G_f° and acts by the non-trivial character on $\mathbb{Z}/2\mathbb{Z}$. The orbital variety $\mathcal{V}$ is smooth so that in the notation of the first section, $\mathcal{V}^o = \mathcal{V}$. From Theorem 2.16, we find that the stabilizer of $\mathcal{V}$ is the standard parabolic subgroup $Q_\mathcal{V}$ with Levi factor isomorphic to $GL(2) \times GL(2)$. One can quickly check that, in this case, both $Q_{\mathcal{V}}$ and the standard Borel subgroup B act with dense orbit on $\mathcal{V}$.

This observation simplifies calculations, as it allows us to replace the Lagrangian covering $G \times_{Q_{\mathcal{V}}} \mathcal{V}$ by G/Q_f, where $Q_f = Q_{\mathcal{V}} \cap G_f$ and $\mathcal{V}$ contains $Q_{\mathcal{V}}/Q_f$ as a dense subset. We note that B/B_f is also dense in $\mathcal{V}$. The equivariant line bundle $\tau^* \mathcal{V}_\pi$ is induced by a character α of B_f. It is given by the square root of the absolute value of the real determinant of B_f acting on the tangent space $\mathfrak{b}/\mathfrak{b}_\mathfrak{f}$ of $\mathcal{V}$ at f. This is

$$\alpha \begin{pmatrix} A & * \\ 0 & A^{t^{-1}} \end{pmatrix} = |\, t_1^3\, t_3^6 \,|^{-1} \text{ , where } A = \begin{pmatrix} t_1 & * & * & * \\ 0 & t_3 & * & * \\ 0 & 0 & t_3 & * \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Because we are looking for a map j_π, we would like to find a character γ of B whose restriction to B_f is α. Such a character is given by

$$\gamma \begin{pmatrix} A & * \\ 0 & A^{t^{-1}} \end{pmatrix} = |\, t_1\, t_2\, t_3\, t_4 \,|^{-3} \text{ , where } A = \begin{pmatrix} t_1 & * & * & * \\ 0 & t_2 & * & * \\ 0 & 0 & t_3 & * \\ 0 & 0 & 0 & t_4 \end{pmatrix}.$$

The character γ extends uniquely from B to $Q_{\mathcal{V}}$. Let the half-density bundle on $G/Q_{\mathcal{V}}$ be given by the character $\rho_{Q_{\mathcal{V}}}$ and define another character γ' on $Q_{\mathcal{V}}$ to equal $\gamma \otimes \rho_{Q_{\mathcal{V}}}^{-1}$. Then

$$V(\mathcal{O}, \mathcal{V}, \pi, \gamma, j_{\gamma,\pi}) \subset Ind_Q^G(\gamma').$$

Hence the infinitesimal character that we associated to the representation space $V(\mathcal{V}, \pi)$ equals $-(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}) + \rho = (\frac{5}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2})$, where $\rho = (4, 3, 2, 1)$ and equality is up to Weyl group action. This is precisely the unique infinitesimal character attached to the orbit $\mathcal{O}_{[2^3,1^2]}$ by McGovern in [12]. Similar calculations for the other orbital varieties in this orbit yield the same infinitesimal character.

A significant simplification in this example came from the fact that the parabolic subgroup $Q_{\mathcal{V}}$ acted with dense orbit on $\mathcal{V}$. It made it easy to find the isomorphism j_π. Unfortunately, this is not always the case.

Example 3.1 ([15]). Let $G = SL_9$ and let

$$\mathrm{T}= \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 6 & 9 \\ \hline 4 & 5 & 8 \\ \cline{1-3} 7 \\ \cline{1-1} \end{array}$$

Then $\mathcal{V}_T$, the orbital variety in $\mathcal{O}_{[5,3,1]}$ corresponding to T has dimension 31. However, $\dim Q_{\mathcal{V}} \cdot f \leq 30$ for all $f \in \mathcal{V}$.

This example can be extended to produce other instances where $Q_{\mathcal{V}}$ does not act with dense orbit on $\mathcal{V}$ both, in larger groups of type A as well in groups of other classical types. We will restrict our attention to a class of nilpotent orbits all of whose orbital varieties do admit a dense orbit of their stabilizing parabolic, but note that every nilpotent orbit contains at least one orbital variety with this property.

3.2. **Spherical Orbital Varieties and Orbits of S-type.** We would like to use the methods of our model example to calculate the infinitesimal character associated to $V(\mathcal{V}, \pi)$ for as many nilpotent orbits as feasible. The main assumption required is that the stabilizer of an orbital variety has a dense orbit in that variety. Such orbital varieties are called of *S-type*, as are the nilpotent orbits *all* of whose orbital varieties satisfy this condition. Among classical groups, there is a class of *small* nilpotent orbits that are of S-type. We first describe this set and then place it among other important nilpotent coadjoint orbits.

Let G be a complex simple Lie group and B a Borel subgroup. We will say that a nilpotent coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$ is *spherical* iff it contains an open B-orbit. The work of D. Panyushev provides a concise description of spherical nilpotent orbits contained in classical groups.

Theorem 3.2 ([16]). *A nilpotent orbit $\mathcal{O}_\lambda$ of a complex classical simple Lie group that is parametrized by the partition λ is spherical if λ is of the form:*

(i) $[2^b, 1^c]$ *in type* A,
(ii) $[3^a, 2^{2b}, 1^c]$ *with* $a \leq 1$ *in type* B,
(iii) $[2^b, 1^{2c}]$ *in type* C, *and*
(iv) $[3^a, 2^{2b}, 1^c]$ *with* $a \leq 1$ *in type* D.

A few properties characterize spherical orbits. First, they are precisely the orbits which contain a representative that is a sum a root vectors corresponding to orthogonal simple roots ([14] and [17]). Furthermore, for complex simply-connected semisimple Lie groups, there is an orbit for which the G-module structure of $R(\mathcal{O})$, the coordinate ring of regular functions, has all multiplicities either 0 or 1 [14]. The largest such orbit is called the *model orbit.* Spherical orbits may be characterized as those nilpotent coadjoint orbits contained in the closure of the model orbit.

Theorem 3.3 ([14]). *Let $\epsilon = 0$ or 1. In each of the classical types, the model orbit is the largest spherical nilpotent orbit and is parametrized by the following partition:*

(i) $[2^n, 1^\epsilon]$ *in type* $A_{2n+\epsilon-1}$,
(ii) $[3, 2^{4m-2\epsilon}, 1^{2\epsilon}]$ *in type* $B_{2(2m-\epsilon)+1}$,
(iii) $[2^n]$ *in type* C_{2n}, *and*
(iv) $[3, 2^{2m-2}, 1^{1+2\epsilon}]$ *in type* $D_{2(2m+\epsilon)}$.

Following A. Melnikov in [15], we will say that an orbital variety $\mathcal{V} \subset \mathcal{O}$ is of *S-type* iff it admits a dense orbit of its maximal stabilizing parabolic $Q_\mathcal{V}$ and extend the terminology to nilpotent coadjoint orbits all of whose orbital varieties are of S-type. The following proposition is a consequence of the dimension argument in Corollary 3.17.

Proposition 3.4. *In the setting of complex classical simple Lie groups, all spherical nilpotent orbits are of S-type.*

Although we will restrict our attention to spherical nilpotent orbits, for completeness, we provide a partial description of the S-type orbits in groups of type A. The result is incomplete, as it fails to resolve the status of a number of nilpotent orbits.

Theorem 3.5 ([15]). *A nilpotent orbit $\mathcal{O}_\lambda$ in type A is of S-type whenever λ satisfies one of the following:*

(i) $\lambda > (n-4, 4)$,
(ii) $\lambda = (\lambda_1, \lambda_2, 1, \dots, 1)$ *with* $\lambda_2 \leq 2$,
(iii) $\lambda = (2, \dots)$ *where* $\lambda_i \leq 2$ *for all* i.

If we suppose that $n \geq 13$, the partition λ has $\lambda_2 > 2$, and $(5, 3, 1, \dots) \leq \lambda \leq (n-4, 4)$ in the usual partial order on partitions, then the orbit $\mathcal{O}_\lambda$ in type A_{n-1} is not of S-type.

We finish this section by listing how spherical orbits fit among two other important classes of nilpotent orbits. A nilpotent orbit is *rigid* if it is not induced from any proper parabolic subalgebra. It is *special* if it is in the range of a particular order-reversing map d, see for instance [4](6.3.7) for a characterization. The following propositions are immediate consequences of the results in [4].

Proposition 3.6. *All nilpotent orbits are special in type A. In the other classical types, the nilpotent orbit $\mathcal{O}_\lambda$ that is parametrized by the partition λ is spherical and special if λ is of the form:*

(i) $[3, 2^{2b}, 1^c]$ *or* $[1^c]$ *in type* B,
(ii) $[2^{2b}, 1^{2c}]$ *or* $[2^b]$ *in type* C,
(iii) $[2^{2b}, 1^c]$ *or*$[3, 1^c]$ *in type* D.

Proposition 3.7. *All non-zero orbits are not rigid in type A. In the other classical types, the nilpotent orbit $\mathcal{O}_\lambda$ that is parametrized by the partition λ is spherical and non-rigid if λ is of the form:*

(i) $[3, 1^{2c}]$ *or* $[2^{2b}, 1^2]$ *in type* B,
(ii) $[2^2, 1^{2c}]$ *or* $[2^{2c}]$ *in type* C, *and*
(iii) $[3, 1^c]$ *or* $[2^{2c}]$ *in type* D.

3.3. **Basepoints in Orbital Varieties.** From the previous section, we know that each spherical orbital variety $\mathcal{V}$ contains a point whose orbit under the Borel subgroup is dense in $\mathcal{V}$. We would like a simple expression for some such point to simplify the forthcoming calculations. For orbital varieties within classical nilpotent orbits, such an expression can be easily read off from the standard tableau corresponding to $\mathcal{V}$.

In type A, such a basepoint is essentially defined in [15]. We provide a slightly more general construction and extend the result to other classical types. The main tool for the latter is the surjection π_A from domino tableaux onto standard Young tableaux defined in [3]. It induces a map on the level of orbital varieties that helps us define the basepoint in the "type A component" of each orbital variety.

3.3.1. *Type A.* Consider a spherical nilpotent orbit $\mathcal{O}$ and let $\mathcal{V}_T \subset \mathcal{O}$ be the orbital variety associated to the standard Young tableau $T \in YT(n)$. Let T^i denote the set of labels contained in the i-th column of T, so that in our case $T^i = \emptyset$ if $i > 2$. We will define a point f_T contained in $\mathcal{V}_T$ whose orbit under the Borel subgroup is dense in $\mathcal{V}_T$. For $x \in \mathbb{N}$, let $\tilde{x} = n + 1 - x$ and let us adopt the notation from [9, IV.1], writing $E_{e_i - e_j}$ for the matrix with the ij-entry equal to one and zero otherwise.

Proposition 3.8. *Let $\phi : T^2 \longrightarrow T^1$ be an injection with the property that $\phi(k) < k$ for all $k \in T^2$. Such a map always exists, and furthermore, the point*

$$f_T = \sum_{k \in T^2} E_{e_{\tilde{k}} - e_{\widetilde{\phi(k)}}}$$

is contained in the variety $\mathcal{V}_T$.

Proof. The fact that a map ϕ always exists is clear by inspection. A spherical nilpotent orbit in type A is uniquely determined by the rank of its elements. For each f_T defined above, $f_T^2 = 0$, so it lies in *some* spherical orbital variety. That it lies precisely in $\mathcal{V}_T$ follows from induction and the above rank condition. □

This definition includes Melnikov's construction as a special case. More precisely, it is always possible to choose ϕ in such a way so that $\phi(k) = k - 1$ whenever $\alpha_{\widetilde{k-1}} \notin \tau(T)$. In this incarnation, f_T is a *minimal representative* of $\mathcal{V}_T$ in the sense described below. Let $f \in \mathfrak{n}$ and for its root space decomposition, let us write

$$f = \sum_{\epsilon \in \Delta^+} c_\epsilon(f) E_\epsilon.$$

Definition 3.9. An element $f \in \mathcal{V}$ is a *representative* of $\mathcal{V}$ if f does not belong to any other orbital varieties. A representative f of $\mathcal{V}$ is *minimal* if

(i) each $c_\epsilon(f) \in \mathbb{Z}$,
(ii) for every $\alpha_i \notin \tau(\mathcal{V})$, $c_{\alpha_i}(f) \neq 0$,
(iii) If $g \in \mathcal{V}$ also satisfies the above, the the number of non-zero $c_\epsilon(g)$ will be greater than or equal to the number of non-zero $c_\epsilon(f)$.

We would like the basepoints we choose to be minimal representatives. In type A, we have already seen that this is always possible and in further work we would like f_T to be close to satisfying this condition.

Example 3.10. Consider the orbital variety $\mathcal{V}_T$ associated with the standard Young tableau

$$\mathrm{T} = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1} 5 \\ \cline{1-1} \end{array}$$

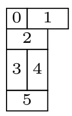
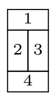
The points $f_1 = E_{e_1 - e_2}$ and $f_2 = E_{e_1 - e_3}$ both lie in $\mathcal{V}_T$ and both are f_T for some choice of injection ϕ by Proposition 3.8. However, only f_1 is a minimal representative of $\mathcal{V}_T$.

3.3.2. *Other Classical Types.* Let $X = B$, C, or D, and let $\mathcal{V}_T$ be the orbital variety in a spherical nilpotent orbit of type X associated with the standard domino tableau T. In search of a suitable basepoint, we first define a matrix M_T^X from the horizontal dominos of T. Let N^T to be the set of labels of the horizontal dominos in T and define S^T be the subset of N^T whose underlying dominos intersect the first column of T. If M is a family of sets of integers, let M° denote the union of all integers contained in elements of M. Write $T(m)$ for the domino tableau consisting of the first m dominos of T and $D(m)$ for the domino with label m. We now inductively define a set N_1^T of *pairs* of labels in T by $N_1^\varnothing = \varnothing$ and

$$N_1^T = \begin{cases} N_1^{T(n-1)} \cup \{\{k, n\}\} & \text{if } D(n) \in S^T \setminus (N_1^{T(n-1)})^\circ \\ & \text{ and if } X = C,\ k = n-1, \\ N_1^{T(n-1)} & \text{otherwise.} \end{cases}$$

Finally, let $N_2^T = S^T \setminus (N_1^T)^\circ$ and $N_3^T = N^T \setminus ((N_1^T)^\circ \cup N_2^T)$. Note that N_3^T is always empty in type C while N_2^T is always empty in types B and D.

Example 3.11. Suppose T and U are the following domino tableaux and note that U can be viewed as a domino tableau of type C as well as D:

$T =$  $U =$  .

For the tableau T of type B, $S^T = \{2,5\}$, $N_1^T = \{\{2,5\}\}$, $N_2^T = \varnothing$, and $N_3^T = \{1\}$. For the tableau U, $S^U = \{1,4\}$, $N_1^U = \varnothing$, $N_2^U = \{1,4\}$, and $N_3^U = \varnothing$ when it is viewed as a domino tableau of type C, and $S^U = \{1,4\}$, $N_1^U = \{\{1,4\}\}$, $N_2^U = \varnothing$, and $N_3^U = \varnothing$ when it is viewed as a domino tableau of type D.

As in the previous section, we will adopt the notation for simple roots from [9, IV.1], write E_α for a basis vector for the root space $\mathfrak{g}_\alpha$ of a simple root α and take $T_i \in \mathfrak{t}_i$ where $\mathfrak{t} = \oplus_{i \leq rank\, G} \mathfrak{t}_i$. Let

$$M_T^X = \sum_{\{i,j\} \in N_1^T} E_{e_{\tilde{i}} + e_{\tilde{j}}} + \sum_{\alpha \in U_T^X} E_\alpha,$$

where U_T^X is the set of roots defined by

$$U_T^X = \begin{cases} \{e_{\widetilde{k-1}} + e_{\tilde{k}} \,|\, k \in N_3^T\} & \text{if } X = D, \\ \{2e_{\tilde{k}} \,|\, N_2^T\} & \text{if } X = C, \text{ and} \\ \{e_{\tilde{k}} \,|\, k \in N_3^T\} \cup \{e_{\tilde{3}} \,|\, 2 \in T^3 \text{ and } 3 \in T^2\} & \text{if } X = B. \end{cases}$$

Definition 3.12. For $X = B, C$, or D, and a domino tableau T, let

$$f_T^X = f_{\pi_A(T)} + M_T^X$$

where $f_{\pi_A(T)}$ is a minimal representative of the type A orbital variety $\mathcal{V}_{\pi_A(T)}$ interpreted naturally as lying inside the Lie algebra of type X.

Proposition 3.13. *The point f_T^X is a minimal representative of $\mathcal{V}_T$.*

Proof. The proof is a little simpler if we use an alternate parametrization of orbital varieties which uses the set of admissible domino tableaux with signed closed clusters ΣDT_{cl}. We refer the reader to [18] for the details, and let Φ be the bijection between the two parameter sets defined therein.

Define $T' = \Phi^{-1}(T) \in \Sigma DT_{cl}(shape\, T)$. We first show that $f_T^X \in \mathcal{V}_S$, where $S = \Phi(T^*)$ and $T^* \in \Sigma DT_{cl}(shape\, T)$ has the same underlying domino tableau as T'. We then show that T' and T^* must share the set of closed clusters with positive sign, which implies that $S = T$ by the definition of Φ. This verifies that f_T^X is a representative of $\mathcal{V}_T$. Minimality of f_T^X may then be checked by inspection.

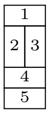
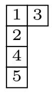
We would like to show that for all $k \leq n$, $f_{T(k)}^X \in \mathcal{O}_{\text{shape } T'(k)}$. By induction, it is enough to verify this for $k = n-1$. Note that for spherical orbits, the partition of the orbit containing a nilpotent element f is completely determined by rank f and rank f^2. The above statement can be now verified by inspecting the definition of f_T^X and comparing rank $f_{T(n-1)}^X$ and rank $(f_{T(n-1)}^X)^2$ with rank f_T^X and rank $(f_T^X)^2$. In this way, we have $f_T^X \in \mathcal{V}_S$, where $S = \Phi(T^*)$ and T^* is some tableau in $\Sigma DT_{cl}(shape\, T)$ sharing its underlying tableau with T'.

Now note that if $\mathcal{C}$ is a closed cluster of T' or T^*, then because the orbit $\mathcal{O}_{shape\, T}$ is spherical, the initial cycle $I_\mathcal{C}$ through $\mathcal{C}$ must have the form $I_\mathcal{C} = \{i, i+1, \ldots, j\}$. Theorem 2.16 implies that the simple root $\alpha_{\tilde{i}} \in \tau(T)$ iff there is a closed cluster $C \in \mathcal{C}^+$ with $I_\mathcal{C} = \{i, i+1, \ldots, j\}$ for some j. Further note that if $C \in \mathcal{C}^+$, then $E_{e_i + e_j}$ appears in the expansion of f_T^X with non-zero coefficient while $E_{e_i - e_j}$ has coefficient zero. Similarly, if $C \in \mathcal{C}^-$, then $E_{e_i - e_j}$ appears in the expansion of f_T^X with non-zero coefficient while $E_{e_i - e_j}$ has coefficient zero. But this forces $\Phi(T^*)$ to have the same τ-invariant as $\Phi(T')$, which implies that $\Phi(T^*) = \Phi(T')$. Hence f_T^X is a representative of $\mathcal{V}_T$. $\square$

Lemma 3.14. *Consider an orbital variety $\mathcal{V}_T$ in a spherical nilpotent orbit of classical type that corresponds to the standard tableau T, and let $Q = Q_{\mathcal{V}_T} \supset B$ be the maximal parabolic stabilizing it. Then the orbits $B \cdot f_T^X$ and $Q \cdot f_T^X$ are dense in $\mathcal{V}_T$.*

Proof. For the result in in type A, see [15, 4.13]. The present result follows by induction from Corollary 3.17 below. $\square$

Example 3.15. Let $X = C$ and consider the orbital variety $\mathcal{V}_T$ associated with the domino tableau

T=  . Then the Young tableau $\pi_A(T) =$  .

We have $N = \{1, 4, 5\}$, $N_1 = \{1\}$ and $N_2 = N \setminus N_2$. Finally, the basepoint $f_T^C = \begin{pmatrix} A & M \\ 0 & -A^t \end{pmatrix}$, where

$$A = f_{\pi_A(T)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

3.4. **Induction.** Our calculation of infinitesimal characters of Graham-Vogan representations attached to the orbital variety $\mathcal{V}_T$ will proceed by a type of induction on the standard tableau T. As in our model example, we would like to describe the action of Q_f on the space $\mathfrak{q}/\mathfrak{q}_\mathfrak{f}$. Because we consider only spherical orbits, it necessary only to describe this action on $\mathfrak{b}/\mathfrak{b}_\mathfrak{f}$. In this section, we describe it, verifying Lemma 3.14 in the process.

Fix a standard tableau T of a given classical type and write $\mathcal{V}_T$ for the orbital variety corresponding to it. Ideally, we would like to be able extract information about $\mathcal{V}_T$ from $\mathcal{V}_{T(n-1)}$ and in this manner set up a type of induction. However, the standard domino tableau $T(n-1)$ does not always correspond to an orbital variety of the same classical type as $\mathcal{V}_T$, so in order for induction to make sense, we have to be careful. To this effect, we define a standard tableau $T^\downarrow$ by

$$T^\downarrow = \begin{cases} MT(\mathcal{C}, T(n-1)) & X = B \text{ or } D, \text{ type } \mathcal{V}_{T(n-1)} \neq X \text{ and} \\ & \mathcal{C} \text{ the cycle in } T(n-1) \text{ through } n-1, \\ T(n-2) & X = C,\ D(n) \text{ and } D(n-1) \text{ are horizontal,} \\ & \text{while } D(n-2) \text{ is not,} \\ T(n-1) & \text{otherwise.} \end{cases}$$

The notion of a cycle and the moving-through map MT are defined in [5, §5]. With this definition, shape $T^\downarrow$ and shape T are partitions of the same classical type. Therefore, we are able to associate an orbital variety $\mathcal{V}_{T^\downarrow}$ of the same type as $\mathcal{V}_T$ to the standard tableau $T^\downarrow$. We will write $f^\downarrow$ for $f^X_{T^\downarrow}$, and $\mathfrak{b}^\downarrow, \mathfrak{q}^\downarrow$, and $\mathfrak{g}^\downarrow$ for the Lie algebras corresponding to $\mathcal{V}_{T^\downarrow}$.

As in our model example, we would like to describe the action of Q_f on the space $U = \mathfrak{b}/\mathfrak{b}_{f_T}$. If we think inductively, however, we can break this task down into a study of the quotients $U_n = (\mathfrak{b}/\mathfrak{b}_{f_T})/(\mathfrak{b}^\downarrow/\mathfrak{b}^\downarrow_{f^\downarrow})$. It will be often convenient to divide our work into cases that arise from an inductive construction of of the representative f_T^X. Let ι be the natural inclusion map of the Lie algebra of type X of rank $n-1$ to the one of rank n. The cases are distinguished by the possible forms of the difference $f_T^X - \iota(f_{T^\downarrow}^X)$. We describe the possibilities along with what they imply on the level of tableaux.

(C1) When $f_T^X = \iota(f_{T^\downarrow}^X)$, the domino $T \setminus T^\downarrow$ lies entirely in the first column of T.

(C2) When $f_T^X = \iota(f_{T^\downarrow}^X) + E_{e_1 - e_{\tilde{\phi}(n)}}$, the domino $T \setminus T^\downarrow$ lies entirely in the second column of T.

(N1) When $f_T^X = \iota(f_{T^\downarrow}^X) + E_{e_1 + e_{\tilde{k}}}$ and $X = B$ or D, then this is the case when $T^\downarrow \neq T(n-1)$ and $\{k, k+1, \ldots, n-1\}$ is a cycle in $T(n-1)$. If $X = C$ and $\tilde{k} \neq 2$, then $k = n-1$ and $T^\downarrow = T(n-2)$.

(N2) When $f_T^X = \iota(f_{T^\downarrow}^X) + E_{2e_1}$, then $X = C$ and $T^\downarrow = T(n-1)$.

(N3) When $f_T^X = \iota(f_{T^\downarrow}^X) + E_{e_1 - e_2} + E_{e_1}$, we have $X = B$ and $T \setminus T^\downarrow$ is a horizontal domino that intersects the third column of T. When $f_T = \iota(f_{T^\downarrow}) + E_{e_1 - e_2} + E_{e_1 + e_2}$, we have $X = D$ and again $T \setminus T^\downarrow$ is a horizontal domino that intersects the third column of T.

(*) When $f_T^X = \iota(f_{T^\downarrow}^X) + E_{e_1}$, we have $X = B$ and $T \setminus T^\downarrow = D(3) \in T^2$ while $D(2) \in N_3^T$.

Lemma 3.16. *Consider a standard tableau T, the orbital variety $\mathcal{V}_T$, and representative f_T or f_T^X. Let ϕ be the injection from Propositions 3.8 and 3.13 used to define this representative. In each of the above cases, the space U_n is:*

(C1) *$U_n = \bigoplus_{T^2} \mathfrak{g}_{e_1 - e_{\tilde{i}}}$ in type A. In the other classical types, let $N = (N_1^T)^\circ \cup N_3^T \cup \{3\}$ in type B, $N = (N_1^T)^\circ \cup N_3^T$ in type D and $N = (N_1^T)^\circ \cup N_2^T$ in type C. Then*

$$U_n = \bigoplus_{N} \mathfrak{g}_{e_1 - e_{\tilde{i}}} \oplus \bigoplus_{(\pi_A(T))^2} (\mathfrak{g}_{e_1 - e_{\tilde{i}}} \oplus \mathfrak{g}_{e_1 + e_{\widetilde{\phi(i)}}})$$

(C2) *Define V and W by*

$$V = \bigoplus_{\substack{j > \widetilde{\phi(n)} \\ \tilde{j} \notin N^T \cup (\pi_A(T))^2}} \mathfrak{g}_{e_{\tilde{\phi}(n)} - e_j} \oplus \bigoplus_{\substack{(\pi_A(T(n-1)))^2 \\ \phi(i) > \phi(n)}} \mathfrak{g}_{e_1 - e_{\tilde{i}}} \oplus \mathfrak{t}_1$$

$$W = \bigoplus_{\substack{j \neq \tilde{\phi}(n) \cup 1 \\ j \neq \tilde{\phi}(i), \tilde{i} \in (\pi_A(T^\downarrow))^2}} \mathfrak{g}_{e_{\tilde{\phi}(n)} + e_j}.$$

Then $U_n = V$ in type A and $U_n = V \oplus W \oplus \mathfrak{g}_{e_1} \oplus_{N_3^T = \varnothing} \mathfrak{g}_{e_{\tilde{\phi}(n)}}$ in type B. In type C,

$$U_n = V \oplus W \oplus \mathfrak{g}_{2e_{\tilde{\phi}(n)}} \oplus \mathfrak{g}_{e_1 + e_{\tilde{\phi}(n)}},$$

while in type D,

$$U_n = V \oplus W \oplus_{N_3^T} (\mathfrak{g}_{e_1 - e_{\tilde{i}}} \oplus \mathfrak{g}_{e_1 - e_{\widetilde{i-1}}}).$$

(N1) *Let $N_1^T = \{\{k, n\}\}$ and write $N = (N_1^T)^\circ$. Then*

$$U_n = \bigoplus_{N} \mathfrak{g}_{e_1 - e_{\tilde{j}}} \oplus \bigoplus_{(\pi_A(T^\downarrow))^2} (\mathfrak{g}_{e_1 - e_{\tilde{i}}} \oplus \mathfrak{g}_{e_1 + e_{\tilde{\phi}(i)}}) \oplus \mathfrak{t}_1$$

(N2) *This case arises only in type C. Write $N = N^{T^\downarrow} \setminus \{1\}$. Then*

$$U_n = \bigoplus_N (\mathfrak{g}_{e_1 - e_{\tilde{i}}} \oplus \mathfrak{g}_{e_2 - e_{\tilde{i}}}) \oplus \bigoplus_{(\pi_A(T^\downarrow))^2} (\mathfrak{g}_{e_1 - e_{\tilde{i}}} \oplus \mathfrak{g}_{e_2 - e_{\tilde{i}}} \oplus \mathfrak{g}_{e_1 + \tilde{\phi}(i)} \oplus \mathfrak{g}_{e_2 + \tilde{\phi}(i)}) \oplus \mathfrak{t}_1.$$

(N3) *This case arises only in types B and D. In the former case,*

$$U_n = \bigoplus_{j>2} (\mathfrak{g}_{e_2 - e_j} \oplus \mathfrak{g}_{e_2 + e_j}) \oplus \mathfrak{g}_{e_2} \oplus \mathfrak{t}_1 \oplus \mathfrak{t}_2.$$

In the latter, it is

$$U_n = \bigoplus_{j>2} (\mathfrak{g}_{e_2 - e_j} \oplus \mathfrak{g}_{e_2 + e_j}) \oplus \mathfrak{t}_1 \oplus \mathfrak{t}_2.$$

(*) *In this special case, $U_3 = \mathfrak{g}_{e_1 - e_2} \oplus \mathfrak{g}_{e_1 + e_3} \oplus \mathfrak{g}_{e_2}$.*

Proof. Form a decomposition $\mathfrak{b} = \mathfrak{b}_1 \oplus \iota(\mathfrak{b}^\downarrow)$ that is compatible with the root space decomposition. For $B \in \mathfrak{b}$, write $B = B_1 + B_2$ with $B_1 \in \mathfrak{b}_1$ and $B_2 \in \iota(\mathfrak{b}^\downarrow)$. We will write f_T for f_T^X. Note that $B \in \mathfrak{b}_{f_T}$ if and only if

$$[B, f_T] = 0. \tag{1}$$

To describe U_n, we assume that B_2 lies in $\iota(\mathfrak{b}_{f_{T^\downarrow}})$, i.e. that

$$[B_2, \iota(f_{T^\downarrow})] = 0. \tag{2}$$

We would like to know what additional conditions on B are necessary to make sure that it satisfies (1). If we write

$$B = \sum_{\alpha \in \Delta^+} c_\alpha E_\alpha + \sum_{i \leq n} c_i T_i,$$

then (1) imposes linear conditions on the coefficients in the expansion of B. If we choose a representative α or i within each linear condition and denote the set of representatives by P, then

$$\mathfrak{b}/\mathfrak{b}_f \simeq \bigoplus_{\alpha \in P} \mathfrak{g}_\alpha \oplus \bigoplus_{i \in P} \mathfrak{t}_i.$$

The natural action of Q_{f_T} has the same determinant on both spaces. Note that we only need to include representatives for linear conditions that do not already arise as conditions for (2). We carry out this plan by describing the set of representatives in each of the cases.

Case (C1). In this case, $f_T = \iota(f_{T^\downarrow})$. Condition (1) boils down to

$$[B_1, \iota(f_{T^\downarrow})] = 0. \tag{3}$$

Write $B_1 = \sum_S c_\alpha E_\alpha + c_1 T_1$. If we expand the left hand side of (3) in terms of root space coordinates, the resulting linear conditions imposed by (3) all take the form $c_\gamma = 0$ for γ in some set Ω. The quotient U_n then takes the form $\bigoplus_\Omega \mathfrak{g}_\alpha$. Deciphering (3) explicitly leads to the description in the statement of the lemma.

Case (C2). In this case, $f_T = \iota(f_{T^\downarrow}) + E_{e_1 - e_{\tilde{\phi}(n)}}$. Equation (1) reduces to

$$[B_1, \iota(f_{T^\downarrow})] + [B_1, E_{e_1 - e_{\tilde{\phi}(n)}}] + [B_2, E_{e_1 - e_{\tilde{\phi}(n)}}] = 0. \tag{4}$$

We can again write $B_1 = \sum_S c_\alpha E_\alpha + c_1 T_1$ and expand (4) in terms of root space coordinates. For each linear condition on the coefficients obtained from (4), we select as representative the largest root γ such that c_γ appears within the linear equation. If, however, c_i also appears within a linear condition, we select the coefficient i instead. When we account for linear conditions that already appear in (2), we obtain the description of U_n in the statement of the lemma.

Case (N1). In this case, $f_T = \iota(f_{T^\downarrow}) + E_{e_1+e_{\tilde{k}}}$. Equation (1) reduces to

$$[B_1, \iota(f_{T^\downarrow})] + [B_1, E_{e_1+e_{\tilde{k}}}] + [B_2, E_{e_1+e_{\tilde{k}}}] = 0. \tag{5}$$

In types B and D, the method of case (C2) can be used verbatim, we only have to account for the different linear conditions imposed by (5). When $X = C$, we merely have to account for the different definition of $T^\downarrow$ in this case by letting $B_1 = \sum_S c_\alpha E_\alpha + c_1 T_1 + c_2 T_2$ for the appropriate set S.

Case (N2). In this case, $f_T = \iota(f_{T^\downarrow}) + E_{e_1+e_2}$. Equation (1) reduces to $[B_1, \iota(f_{T^\downarrow})] + [B_1, E_{e_1+e_2}] + [B_2, E_{e_1+e_2}] = 0$ and the method of case (C2) can again be used verbatim to describe U_n.

Case (N3). In type B, $f_T = \iota(f_{T^\downarrow}) + E_{e_1-e_2} + E_{e_1}$, while in type D, $f_T = \iota(f_{T^\downarrow}) + E_{e_1-e_2} + E_{e_1+e_2}$. In both cases, $f_{T^\downarrow} = 0$ and $\mathfrak{b}_{f_{T^\downarrow}} = \mathfrak{b}^\downarrow$. Hence equation (1) reduces to $[B, E_{e_1-e_2}] + [B, E_{e_1+e_2}] = 0$ in type B and $[B, E_{e_1-e_2}] + [B, E_{e_1+e_2}] = 0$ in type D.

Case (*). In this case, $f_T = \iota(f_{T^\downarrow}) + E_{e_1}$.Equation (1) reduces to $[B_1, \iota(f_{T^\downarrow})] + [B_1, E_{e_1}] + [B_2, E_{e_1}] = 0$ and the method of case (C2) can again be used verbatim to describe U_n. □

Corollary 3.17. *For a standard Young or domino tableau T,*

$$\dim U_n = \dim \mathcal{V}_T - \dim \mathcal{V}_{T^\downarrow}.$$

Proof. We can compute $\dim \mathcal{V}_T - \dim \mathcal{V}_{T^\downarrow}$ from the formula for the dimension of a nilpotent orbit, see [4]. Let $[\lambda_1, \cdots \lambda_p]$ be the dual partition to *shape* T. In each of the cases, $\dim \mathcal{V}_T - \dim \mathcal{V}_{T^\downarrow}$ equals

$$\frac{1}{2}(\dim \mathcal{O}_{shape\,T} - \dim \mathcal{O}_{shape\,T^\downarrow}) = \begin{cases} \lambda_2 + \lambda_3 & \text{Case (C1)} \\ \lambda_1 & \text{Case (C2) and } X = A \\ \lambda_1 - 1 + \lambda_3 & \text{Case (C2) and } X = B \text{ or } D \\ \lambda_1 + 1 & \text{Case (C2) and } X = C \\ \lambda_1 & \text{Case (N1) and } X = B \text{ or } C \\ \lambda_1 - 1 & \text{Case (N1) and } X = D \\ 2\lambda_1 - 1 & \text{Case (N2)} \\ \lambda_1 & \text{Cases (N3) and (*)} \end{cases}$$

One can now check these are exactly the dimensions of the corresponding spaces U_n. We detail the calculation in case (C2) for groups of type C. The other cases are similar. Recall from [18] the two types of vertical dominos that arise within a domino tableau, and denote by I^- and I^+ the set of labels of the dominos of that type that are contained in the tableau T. From our description of U_n, we obtain:

$$\begin{aligned} \dim U_n = \; & |\{j < \phi(n) \,|\, j \notin N^T \cup (\pi_A(T))^2\}| + |\{i \in (\pi_A(T))^2 \,|\, \phi(i) > \phi(n)\}| \\ + \; & |\{j \,|\, j \neq \phi(i) \text{ for } i \in (\pi_A(T))^2, j \neq n\}| + 3 \\ = \; & \{j \in I^- \,|\, j \neq \phi(n), \text{ and if } j > \phi(n), \text{ then } j \in Im\,\phi\} \\ + \; & \{j \in I^- \cup N^T \,|\, j \neq n\} = (|I^-| - 1) + (|I^-| + |N^T| - 1) + 3 \\ = \; & 2|I^-| + |N^T| + 1 = \lambda_1 + 1, \end{aligned}$$

as claimed.

□

3.5. **The Trace of the Adjoint Action.** Let $\mathfrak{t}_f$ be a maximal torus inside the Lie algebra $\mathfrak{q}_f$. Again, we abbreviate f_T^X as f_T. It is easy to check that $\mathfrak{q}_{f_T} \cap \mathfrak{t}$ is a maximal torus in $\mathfrak{q}_{f_T}$. The inductive procedure of the previous section provides a description of the coordinates of $\mathfrak{t}_f$. The trace of the adjoint action of $\mathfrak{t}_f$ on $\mathfrak{q}/\mathfrak{q}_f$ can then be calculated as a sum of the traces of the actions of the quotient spaces U_m for $m \leq n$. In keeping with the inductive philosophy of this section, we compute this trace on the space U_n, separating each of the inductive cases.

Proposition 3.18. *Let $f = f_T$ and write $a \in \mathfrak{t}$ as $a = \sum_{1 \leq i \leq n} a_i \, \mathfrak{t}_i$. Then a lies in $\mathfrak{t}_f$ iff $\sum_{2 \leq i \leq n} a_i \, \mathfrak{t}_i$ lies in the torus $\iota(\mathfrak{t})_{\iota(f^{\downarrow})}$ and additionally*

(i) $a_1 = a_{\tilde{\phi}(n)}$ in case (C2),
(ii) $a_1 = -a_{\tilde{k}}$ in case (N1)*, where $\{k, n\}$ is a pair in N_1^T,*
(iii) $a_1 = 0$ in cases (N2), (N3)*, as well as* (*).

Proof. This follows immediately from the inductive description of f_T. □

Proposition 3.19. *Let p be the partition corresponding to the nilpotent orbit passing through f_T and let $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_m]$ be its dual partition. The trace of the adjoint action of $\mathfrak{t}_f$ on the quotient U_n is listed below:*

(C1) *The trace is $-\lambda_2 \, a_1 + \sum_{i \in T^2} a_{\bar{\imath}}$ in type A and $-(\lambda_2 + \lambda_3) a_1$ otherwise.*
(C2) *The trace is $-\lambda_1 \, a_1 + \sum_{i \in T^1} a_{\bar{\imath}}$ in type A and $(-\lambda_1 - c) a_1$ otherwise, with $c = 2$ in type C and $c = -2 + \lambda_3$ in types B and D when $N_3^T = \emptyset$.*
(N1) *The trace is $-(\lambda_1 - c) a_1$ in types B and D and zero in type C. The constant c is defined as in* (C2).
(N2) *The trace is* 0.
(N3) *The trace is $-\lambda_1 a_1$.*
(*) *The trace is $-2a_1 - a_3$.*

Proof. We use the description of the quotient U_n in Lemma 3.16 together with Proposition 3.18. In type A, determining the trace is simply a matter of reading off the coordinates. We provide the calculations in Case (C1) for the other classical types which is only a little more subtle. The other cases are similar.

Case (C1). By reading off the coordinates, we find that the trace is

$$\sum_{(\pi_A(T))^2} (-2a_1 + a_{\bar{\imath}} - a_{\tilde{\phi}(i)}) + \sum_{i \in N^T} (-a_1 - a_{\bar{\imath}}) \, \big(+ (-a_1 + a_{\bar{3}}) \big)$$

where the final parenthetical expression appears iff some sub-tableau $T(m)$ of T lies in case (*). Applying Proposition 3.18 reduces the above to

$$-2 \, |(\pi_A(T))^2| \, a_1 - \, |N^T| \, a_1 \, (-1) = -(\lambda_2 + \lambda_3) a_1,$$

as claimed. □

For future use, let us define the vector $(c_n, c_{n-1}, \ldots, c_1)$ by letting c_i equal the number of times the term a_i appears in the expression for the trace of the adjoint action on $\bigoplus_{i \leq n} U_i$ described by Proposition 3.19.

4. Infinitesimal Characters

Armed with the constructions of the previous section, we are ready to examine the Graham-Vogan construction. We restrict our work to those representations that arise from spherical orbital varieties, when the corresponding Lagrangian coverings are quotients of G.

Let $\mathcal{O}$ be a spherical nilpotent orbit of a classical simple Lie group G, fix a Borel subgroup B, and consider the orbital variety $\mathcal{V} = \mathcal{V}_T \subset \mathcal{O}$ that corresponds to the standard Young or domino tableau T by the parametrization of Theorems 2.13 and 2.14. Write Q for its stabilizer in G described by Theorem 2.16. We will write $f \in \mathcal{V}$ for the point f_T in type A as well as the point f_T^X in the other classical types defined in §3.3 via the map ϕ of Proposition 3.8. Let $Q_f \subset Q$ be its stabilizer. Lemma 3.14 implies that the $Q \cdot f$ and $B \cdot f$ are dense in $\mathcal{V}$.

4.1. **Characters, Weights, and Extensions.** A crucial step in the Graham-Vogan construction of the space $V(\mathcal{V}, \pi)$ relies on the existence of a map j_π, where we use the notation of §2.1. It is by no means clear that such a map exists. The goal of this section is to describe a condition for its existence which we will use in §4.3. Graham and Vogan's construction examines the character α of Q_f given by the square root of the absolute value of the real determinant of Q_f acting on the tangent space $\mathfrak{q}/\mathfrak{q}_f$ of $\mathcal{V}$ at f. A homomorphism j_π exists iff there is a representation γ of Q such that $\gamma|_{Q_f} \supset \alpha$. If γ is character, then j_π will be an isomorphism.

We begin by examining the weight w_α of α. First note that α is a real character. Recall the vector $(c_n, \dots, c_1)$ defined at the end of the previous section. Splitting the weight of α into holomophic and anti-holomorphic parts, we obtain:

$$w_\alpha = (\tfrac{c_n}{2}, \tfrac{c_{n-1}}{2}, \dots, \tfrac{c_1}{2})(\tfrac{c_n}{2}, \tfrac{c_{n-1}}{2}, \dots, \tfrac{c_1}{2}).$$

We interpret a weight of Q_f as an equivalence class of weights of Q so the above is just a representative of such an equivalence class. To answer the existence question for γ, we examine its corresponding weights. Suppose that γ is a real character. Write the Levi subalgebra $\mathfrak{l}$ as a sum of reductive parts as $\bigoplus_{i \leq s} \mathfrak{g}(l_i)$. A real character γ of L takes the form

$$(\dagger) \qquad \gamma(A) = \prod_{i \leq s} |\det A_i|^{\alpha_i} = \prod_{i \leq s} (\det A_i)^{\frac{\alpha_i}{2}} \overline{(\det A_i)}^{\frac{\alpha_i}{2}}$$

where $A \in L$, $\alpha_i \in \mathbb{R}$ and A_i is the restriction of A to the ith reductive part of L. It has weight

$$w_\gamma = (\tfrac{\alpha_n}{2}, \tfrac{\alpha_{n-1}}{2}, \dots, \tfrac{\alpha_1}{2})(\tfrac{\alpha_n}{2}, \tfrac{\alpha_{n-1}}{2}, \dots, \tfrac{\alpha_1}{2}).$$

We would like to know conditions under which w_γ lies in the same equivalence class of weights of Q as w_α. If we write $w_\gamma = w_\alpha + \epsilon$ for some weight ϵ, then in the case of spherical orbits Proposition 3.18 implies that this occurs iff

- $\epsilon_i + \epsilon_j = 0$ whenever $i = \phi(j)$,
- $\epsilon_i - \epsilon_j = 0$ whenever $\{i, j\} \in N_1^T$, and
- $\epsilon_i = 0$ for all $i \notin N^T \cup T^2 \cup \phi(T^2)$.

Denote the set of weights w_γ that satisfy the above conditions by $HW_r(w_\alpha)$ and write $HW_r^1(w_\alpha)$ for the subset of weights which correspond to a real character γ of Q. We would like to understand the relationship between these two sets. First, let us define some notation.

For a parabolic subgroup Q of G, we group the coordinates that correspond to the same reductive part of its Levi L by setting them off with an additional set of parentheses. If

$$\mathfrak{l} = \bigoplus_{i \leq s} \mathfrak{g}(l_i) \qquad \text{and} \qquad \mathfrak{g}(l_j) \cap \mathfrak{t} = \bigoplus_{c_i \leq j \leq d_i} \mathbb{C}T_j,$$

then we will write a weight a as

$$a = ((a_n \; a_{n-1} \; \dots a_{d_1})(a_{c_2} \; \dots \; a_{d_2}) \dots (a_{c_k} \dots a_{d_k}) \dots (a_{d_l} \dots a_1)).$$

Proposition 4.1. *A weight $a \in HW_r(w)$ lies in $HW_r^1(w)$ iff all coefficients corresponding to a given reductive part of the Levi of Q are the same. That is, iff*

$$a_{c_k} = a_{c_k+1} = \dots a_{d_k} \text{ for all } 1 \leq k \leq l.$$

Proof. Suppose that a satisfies the above hypothesis. Then a character of Q with weight a is given by a product of exponents of absolute values of determinants of the reductive parts of L. The exponent of the determinant of the part corresponding to $\{c_k, c_k + 1, \dots, d_k\}$ is given by twice their common value, as per the description of real characters in (†). □

Now suppose that γ is an arbitrary character of Q that restricts to the real character α on Q_f. Then γ takes the form $\gamma = \chi \cdot \gamma'$ where γ' is a real character such that $\gamma'|_{Q_f} = \alpha$, and χ is a unitary character such that $\chi|_{Q_f} = 1$. In particular, this means that $\chi|_{T_f} = 1$. If we write $A \in T$ as $\sum a_i T_i$, then

$$\chi(A) = \prod_{i \leq n} \left(\tfrac{a_i}{|a_i|} \right)^{\beta_i} = \prod_{i \leq n} a_i^{\frac{\beta_i}{2}} (\overline{a_i})^{-\frac{\beta_i}{2}}.$$

It has weight

$$w_\chi = (\tfrac{\beta_n}{2}, \tfrac{\beta_{n-1}}{2}, \dots, \tfrac{\beta_1}{2})(-\tfrac{\beta_n}{2}, -\tfrac{\beta_{n-1}}{2}, \dots, -\tfrac{\beta_1}{2}).$$

The character χ restricts to the identity on Q_f iff w_χ lies in the equivalence class of 0 of weights of Q. Again by Proposition 3.18, this occurs iff

- $\beta_i + \beta_j = 0$ whenever $i = \phi(j)$,
- $\beta_i - \beta_j = 0$ whenever $\{i, j\} \in N_1^T$, and
- $\beta_i = 0$ for all $i \notin N^T \cup T^2 \cup \phi(T^2)$.

Furthermore, because χ is a unitary character of L, its entries also satisfy the conditions of Proposition 4.1. For a weight w_γ, write w_γ^h and w_γ^a for its holomorphic and anti-holomorphic parts. Note that w_γ will be in the same equivalence class as w_α iff $w_\gamma^h = (d_n, d_{n-1}, \dots, d_1)$ satisfies:

- $d_i + d_j = c_i + c_j$ whenever $i = \phi(j)$,
- $d_i - d_j = c_i - c_j$ whenever $\{i, j\} \in N_1^T$, and
- $d_i = c_i$ for all $i \notin N^T \cup T^2 \cup \phi(T^2)$.

Furthermore, such w_γ will be a weight of a character of Q iff the d_i satisfy the conditions of Proposition 4.1.

Definition 4.2. Let w be the weight of a one-dimensional representation of Q_f and define $HW(w)$ to be the set of weights of representations of Q that restrict to w on the torus $\mathfrak{t}_f$. Furthermore, let $HW^1(w)$ be the set of weights in $HW(w)$ that correspond to weights of characters of Q.

The arguments of this section reduce the question of extending a character α of Q_f to a description of the set $HW^1(w_\alpha)$. Propositions 3.19 and 4.1 calculate the weight w_α and a character γ that restricts to α on Q_f exists whenever $HW^1(w_\alpha)$ is non-empty.

4.2. **The Infinitesimal Characters** $IC^1(\mathcal{O})$**.** The goal of this section is to describe a set of infinitesimal characters that ought to be attached to an arbitrary nilpotent orbit of a classical simple Lie group. We follow the work of W. M. McGovern [12].

A classification of unitary representations of complex reductive Lie groups can be obtained from a construction that begins with a set of special unipotent representations first suggested by Arthur [1]. However, only special nilpotent orbits arise as associated varieties of special unipotent representations. To remedy this shortfall, [12] suggests extending this set to a set of q-unipotent representations. We first recall McGovern's description of the infinitesimal characters of q-unipotent representations for classical groups. However, not all q-unipotent infinitesimal characters obtained by his method can reasonably correspond to representations attached to nilpotent orbits. After describing this phenomenon more closely, we prune the set of q-unipotent infinitesimal characters to a set that should be attached to nilpotent orbits.

4.2.1. *Infinitesimal Characters of q-unipotent Representations.* We reproduce the procedure from [12] for attaching infinitesimal characters to nilpotent orbits. Given a nilpotent orbit $\mathcal{O}$, we first describe a way of producing an element $h_{\mathcal{O}}$ in a Cartan subalgebra of $\mathfrak{g}$.

Proposition 4.3. *For each nilpotent element $f \in \mathfrak{g}$, there is a homomorphism $\psi : \mathfrak{sl}_2 \longrightarrow \mathfrak{g}$ that maps the element $E_{e_1-e_2}$ onto f. If the nilpotent orbit $\mathcal{O}$ through f corresponds to the partition $[p_1, \ldots, p_l]$, then $h_{\mathcal{O}} = \psi(T_1 - T_2)$ has eigenvalues:*

$$p_1 - 1, p_1 - 3, \ldots, -(p_1 - 1), p_2 - 1, \ldots, p_l - 1, \ldots, -(p_l - 1).$$

We can describe the element $h_{\mathcal{O}}$ more precisely in terms of its coordinates.

(i) If $\mathfrak{g}$ is of type A, then the coordinates of $h_{\mathcal{O}}$, regarded as an element of a Cartan subalgebra of $\mathfrak{g}$, are its eigenvalues in non-increasing order.

(ii) If $\mathfrak{g}$ is of type B, C, or D, embed it in some $\mathfrak{sl}(n)$ via the standard representation.

a. Suppose the partition of $\mathcal{O}$ has the numeral I *or none at all. Also suppose that* 0 *occurs as an eigenvalue of the matrix $h_{\mathcal{O}}$ with multiplicity k. Then the coordinates of $h_{\mathcal{O}}$ are its positive eigenvalues together with $[k/2]$ zeros, arranged in non-increasing order.*

b. If the numeral of $\mathcal{O}$ is II, *then the coordinates of $h_{\mathcal{O}}$ are obtained in a similar manner, except that the final coordinate is replaced by its negative.*

Definition 4.4 ([2])**.** An irreducible representation of G is *special unipotent* if its annihilator is of the form $J_{max}(\lambda_{\mathcal{O}})$ for $\lambda_{\mathcal{O}} = \frac{1}{2}h_{\mathcal{O}}$.

In each of the classical types except for type B, let n be the dimension d of the standard representation of ${}^LG$. In type B, let $n = d + 1$.

Definition 4.5 ([12])**.** Let $\mathcal{U}$ be a nilpotent orbit in $\mathfrak{sl}(n)$ and $\lambda_{\mathcal{U}} = \frac{1}{2}h_{\mathcal{U}}$. Let $\lambda'_{\mathcal{U}}$ be any $SL(n)$-conjugate of $\lambda_{\mathcal{U}}$ lying inside a Cartan subalgebra of ${}^L\mathfrak{g}$. When regarded as an infinitesimal character of $\mathfrak{g}$, $\lambda'_{\mathcal{U}}$ is called *q-unipotent*.

It remains to attach a nilpotent orbit in $\mathfrak{g}^*$ to each of the q-unipotent infinitesimal characters. The philosophy of the orbit method dictates that this is the open orbit $\mathcal{O}$ contained in the associated variety of $U(\mathfrak{g})/J_{max}(\lambda'_\mathcal{U})$. We describe it presently:

Theorem 4.6 ([12])**.** *Suppose that the orbit $\mathcal{U} \subset \mathfrak{sl}(n)^*$ corresponds to the partition p. The open orbit $\mathcal{O}$ in the associated variety of $U(\mathfrak{g})/J_{max}(\lambda'_\mathcal{U})$ has partition:*

(i) p^t *in type A,*
(ii) $(p^t)_B$ *in type B,*
(iii) $(l(p^t))_C$ *in type C,*
(iv) $(p^t)_D$ *in type D, except when p is very even, in which case $\mathcal{O}$ depends on the choice of $\lambda_\mathcal{U}$ and can be either (p^t, I) or (p^t, II).*

The maps p_X are the X-collapses of the partition p and $l(p)$ is the partition obtained from p by subtracting 1 from its smallest term. By letting $M(\mathcal{U}) = \mathcal{O}$, we can define a map

$$M : \text{nilpotent orbits in } \mathfrak{sl}(n) \longrightarrow \text{nilpotent orbits in } \mathfrak{g}.$$

We will interpret M as a map on partitions. We also adopt some notation for a q-unipotent infinitesimal character by associating it with the partition of the type A orbit that is used to compute it. For example, the orbit $\mathcal{U}_{[4^2,1]}$ lies in the preimage $M^{-1}(\mathcal{O}_{[2^4]})$ of the type C orbit with partition $[2^4]$. Then $\lambda'_\mathcal{U} = (\frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2})$ which we write as $\lambda'_\mathcal{U} = [4^2, 1]$. This expression is unique as long as the classical type of the orbit $\mathcal{O}$ is specified. In the case of very even orbit in type D, we take this to mean that the infinitesimal character with all nonnegative terms is attached to the orbit with numeral I and the infinitesimal character with one negative term is attached to the orbit with numeral II.

According to our present philosophy, the q-unipotent infinitesimal characters that are attached to the nilpotent coadjoint orbit $\mathcal{O}$ for a classical $\mathfrak{g}$ consist of

$$IC(\mathcal{O}) = \{\lambda'_\mathcal{U} \mid \mathcal{U} \in M^{-1}(\mathcal{O})\}.$$

We describe this set explicitly for spherical nilpotent orbits.

Proposition 4.7. *Let $\mathcal{O} = \mathcal{O}_p$ be a spherical nilpotent orbit in a classical Lie algebra $\mathfrak{g}$. In type A, $IC(\mathcal{O}_p) = \{p^t\}$. In the other classical types, the set $IC(\mathcal{O}_p)$ is as follows:*

Type B

p	$IC(\mathcal{O}_p)$
$[2^{2k}, 1^{2n-4k+1}]$ $k \neq \frac{n}{2}$	$\{[2n-2k+1, 2k], [2n-2k, 2k+1]\}$
$[2^{2k}, 1^{2n-4k+1}]$ *n even*	$\{[n+1, n]\}$
$[3, 1^{2n-2}]$	$\{[2n-1, 1^2], [2n-2, 2, 1], [2n-2, 1^3]\}$
$[3, 2^{2k}, 1^{2n-4k-2}]$ $k \neq \frac{n-1}{2}, 0$	$\{[2(n-k)-1-\epsilon, 2k+1+\epsilon, 1] \mid \epsilon = 0, 1\}$
$[3, 2^{n-1}]$ *n odd*	$\{[n^2, 1]\}$

Type C

p	$IC(\mathcal{O}_p)$
$[1^{2n}]$	$\{[2n+1]\}$
$[2, 1^{2n-2}]$	$\{[2n, 1]\}$
$[2^k, 1^{2n-2k}]$ $k \neq 1$ *or n*	$\{[2n-k+1, k], [2n-k+1, k-1, 1]\}$
$[2^n]$	$\{[n+1, n], [n^2, 1], [n+1, n-1, 1]\}$

Type D

p	$IC(\mathcal{O}_p)$
$[2^{2k}, 1^{2n-4k}]$ $k \neq \frac{n}{2}$	$\{[2n-2k, 2k], [2n-2k-1, 2k+1]\}$
$[2^n]$ n *even*	$\{[n^2]\}$
$[3, 1^{2n-3}]$	$\{[2n-2, 1^2], [2n-3, 2, 1], [2n-3, 1^3]\}$
$[3, 2^{2k}, 1^{2n-4k-3}]$ $k \neq \frac{n-2}{2}$	$\{[2(n-k-1)-\epsilon, 2k+1+\epsilon, 1] \mid \epsilon = 0, 1\}$
$[3, 2^{n-2}, 1]$ n *even*	$\{[n, n-1, 1]\}$.

Proof. The proof is much simpler than the statement. It consists of understanding the above map and analyzing all the possibilities. The details are left to the interested reader. □

Unfortunately, even among this list, there already appear orbits $\mathcal{U}$ whose associated q-unipotent infinitesimal characters $\lambda'_{\mathcal{U}}$ cannot be attached to the nilpotent orbit $\mathcal{O} \subset \mathfrak{g}^*$ in any reasonable way. To explain this, write U for the spherical q-unipotent bimodule $U(\mathfrak{g})/J_{max}(\lambda'_{\mathcal{U}})$ and define the $m_{\lambda'_{\mathcal{U}}}$ to be the multiplicity of the associated variety $\mathcal{V}(U)$ in the characteristic cycle $Ch(U)$. The orbit method dictates that in order for U to correspond to a cover of a nilpotent coadjoint orbit $\mathcal{O}$, $m_{\lambda'_{\mathcal{U}}}$ cannot exceed the order of the fundamental group of $\mathcal{O}$. That is, U should not be too large to meaningfully correspond to $\mathcal{O}$. It turns out that for certain $\mathcal{U}$, this unfortunately does occur. Examples of this phenomenon arise already among spherical nilpotent orbits.

Example 4.8. Let $\mathcal{U}$ be the nilpotent orbit corresponding to the partition $[6, 3]$ in $\mathfrak{sl}(9)$. Fix the type of the Lie algebra $\mathfrak{g}$ to be C. Then $M(\mathcal{U}) = \mathcal{O}_{[2^3,1^2]} \subset \mathfrak{sp}(8)^*$. Furthermore, $\lambda'_{\mathcal{U}} = (\frac{5}{2}, \frac{3}{2}, 1, \frac{1}{2})$. However, $m_{\lambda'_{\mathcal{U}}} = 4$ while $|\pi_1(\mathcal{O}_{[2^3,1^2]})| = 2$. According to the above philosophy, $m_{\lambda'_{\mathcal{U}}}$ should not be the infinitesimal character of a unipotent representation attached to $\mathcal{O}_{[2^3,1^2]}$. In fact, this is also true for any $\mathcal{U}$ with partition of the form $[2n-k+1, k]$.

There are similar examples in the other classical groups not of type A. Therefore, in order to find the set of the infinitesimal characters of representations attached to the nilpotent orbit $\mathcal{O}$, we have to prune the set $IC(\mathcal{O})$.

4.2.2. *Pruning of* $IC(\mathcal{O})$. We would like to exclude the infinitesimal characters which arise from those nilpotent orbits for which $m_{\lambda'_{\mathcal{U}}} > |\pi_1(\mathcal{O})|$. Write a partition p as $[p_1, p_2, ..., p_l]$ and define

$a =$ number of distinct odd p_i,
$b =$ number of distinct even nonzero p_i, and
$c = gcd(p_i)$.

Proposition 4.9 ([4])**.** *Let* $\mathcal{O} = \mathcal{O}_p$ *be an orbit in a classical simple Lie algebra* $\mathfrak{g}$. *Then the order of the fundamental group* $\pi_1(\mathcal{O}_p)$ *is:*

(i) c *in type* A*,*
(ii) 2^a *in type* B *if* p *is rather odd and* 2^{a-1} *otherwise,*
(iii) 2^b *in type* C*,*
(iv) $2 \cdot 2^{\max(0,a-1)}$ *in type* D *if* p *is rather odd and* $2^{\max(0,a-1)}$ *otherwise.*

We follow [12] in determining the multiplicity $m_{\lambda'_{\mathcal{U}}}$. The process is a bit complex and requires notation incompatible with some used here, so rather than referring the reader to the above, we replicate the relevant parts here using new notation.

Definition 4.10. Let $M(\mathcal{U}) = \mathcal{O}$, and suppose that $\mathcal{U}$ corresponds to the partition p. In each of the classical types $X = B, C$, and D, we define two numbers μ and ν.

- When $X = D$, let $q = p_{odd} = (q_1^{\lambda_1}, \dots, q_t^{\lambda_t})$ and break it up into chunks as follows. Starting from the left, each chunk takes on one of the forms: $(q_i^{\lambda_i}, q_{i+1}^{\lambda_{i+1}})$ with both λ_i and λ_{i+1} odd; $(q_i^{\lambda_i})$ with λ_i even; or $(q_i^{\lambda_i})$ with λ_i odd and λ_{i+1} even. Let ν be the number of chunks of the first two types. The number μ is defined the same way but with $q = [(p_{even})_D]_{odd}$.
- When $X = B$, break up p_{odd} into chunks as in type D. Let ν_1 be the number of chunks of the first type. Let c be the leftmost chunk of the third type and let ν_2 be the number of chunks of the second type to the right of c, plus one. If no c exists, let $\nu_2 = 0$. Finally, let $\nu = \nu_1 + \nu_2$. The number μ is defined the same way but with $([r(p_{even})]_B)_{odd}$.
- When $X = C$, define ν in the same way as in type B. To define μ, replicate its definition in type D but with the partition $[(p_{even})_D]_{odd}$.

Finally, in each of the cases let $\nu^* = \max(0, \nu - 1)$ and $\mu^* = \max(0, \mu - 1)$.

Definition 4.11. To start, write the coordinates of the infinitesimal character $\lambda'_{\mathcal{U}}$ as $((\frac{i}{2})^{r_i}, \dots, (\frac{1}{2})^{r_1}, 0^{r_0})$. If $\lambda'_{\mathcal{U}}$ contains the coordinate $-\frac{1}{2}$, write this as an additional $\frac{1}{2}$. In type B, define the following numbers:

- $\kappa =$ number of even positive i with r_i odd and r_{i-1} even,
- $\kappa_1 =$ number of even positive i with r_i odd, r_{i-1} even, and either $r_{i-2} > r_i$ with $i > 2$, or $r_0 > \frac{1}{2} r_2$,
- $\kappa_2 =$ number of even positive i with r_i odd, r_{i-1} even positive, and the largest integer j with the following property is even: for even m, $i \leq m \leq j$, r_m is odd, while for odd m in the same range, r_m is positive even.

In type D, first let i_0 be the smallest odd integer i with r_i odd if one exists. Otherwise, let $i_0 = \infty$. Then define:

- $\kappa =$ number odd i with r_i odd and either r_{i-1} even or $i = i_0$,
- $\kappa_1 =$ number of odd $i > i_0$ with r_i odd, r_{i-1} even, and either $r_{i-2} > r_i$,
- $\kappa_2 =$ number of odd $i.i_0$ with r_i odd, r_{i-1} even positive, and the largest integer j with the following property is odd: for even m, $i \leq m \leq j$, r_m is positive even, while for odd m in the same range , r_m is odd.

In type C, the definition is a bit longer. Define a string of integers $i, \dots j$ to be *relevant* if $j > i \geq 0$, for $i < m \leq j$, r_m is odd, either $i > 0$ and r_i is odd, or $i = 0$ and $r_i = \frac{1}{2}(r_{i+2} - 1)$, and the string is maximal subject to the above. Now let

- E_S be the set of positive even integers i in S such that $r_i > 1$ and $i > 2$, or $r_{i-1} \neq 1$,
- F_S be the set of odd integers i in S with $r_i > 1$, and
- $\kappa'_S = \max(|E_S \cup F_S| - (length(S) - 2), 0)$.

We can now list the relevant strings as $S_1, \dots, S_r$ in such a way that the ones with $\kappa'_S = 2$ come first, followed by the ones with $\kappa'_S = 1$, and then the ones with $\kappa'_S = 0$. Enumerate the integers in $\cup_S E_S$ as $i_1, \dots i_s$ in such a way that the ones in S_1 come first, etc. Now let $\kappa(i_a) = 1$ iff $a \leq \nu^*$ and 0 otherwise. Also let $\kappa(j_b) = 1$ iff $b \leq \mu^*$ and 0 otherwise. Finally, for each relevant string S, we can define

- $\kappa_S = \sum_{i_a \in E_S} \kappa(i_a) + \sum_{j_b \in F_S} \kappa(j_b)$.

We are now ready to describe the multiplicity $m_{\lambda'_{\mathcal{U}}}$. Let $n_B = 2\kappa - \min(\nu^*, \kappa_1) - \min(\mu^*, \kappa_2)$, $n_C = \sum_S \max(length(S) - 2 - \kappa_S, 0)$, and $n_D = 2\kappa - \min(\mu^*, \kappa_1) - \min(\nu^*, \kappa_2) + \kappa_3$.

Proposition 4.12 ([12]). *Consider the type A nilpotent orbit $\mathcal{U} = \mathcal{U}_q$. With notation as above, $m_{\lambda'_{\mathcal{U}}}$ equals 1 in type A, 2^{n_B} in type B, 2^{n_C} in type C, and $2^{max(n_D-2,0)}$ in type D.*

Corollary 4.13. *Consider a spherical nilpotent orbit $\mathcal{O}$ and let $M(\mathcal{U}_p) = \mathcal{O}$. Then*

(i) $n_B = 2\kappa$ except when $p = [2n-1, 1^2]$, or $[2(n-k)-1, 2k+1, 1]$, in which case it equals $2\kappa - 1$
(ii) $n_C = \kappa - 1$ when q has the form $[2n-k+1, k]$, and is 0 otherwise,
(iii) $n_D = 2\kappa$.

Proof. In type B, both μ and ν are less than 2, except when $p = [2n-1, 1^2]$, $[2(n-k)-1, 2k+1, 1]$, $[2n-2, 2, 1]$, or $[2n-2k-2, 2k+2, 1]$. In the case of the former two, $\min(\nu^*, \kappa_1) = 1$, and in the case of all four, $\min(\mu^*, \kappa_2) = 0$. For spherical orbits of type D, both μ and ν are less than 2. Finally, in type C, relevant strings of length greater than 2 arise only when $p = [2n-k+1, k]$. □

We are now ready to state a second approximation to the set of infinitesimal characters that should appear as infinitesimal characters of representations attached to spherical nilpotent coadjoint orbits. For a given nilpotent orbit $\mathcal{O}$ of a given classical type, this is the set of characters of the form $\lambda'_{\mathcal{U}}$ with $M(\mathcal{U}) = \mathcal{O}$ that also satisfy the condition

$$m_{\lambda'_{\mathcal{U}}} \leq |\pi_1(\mathcal{O})|.$$

We will denote this set by $IC^1(\mathcal{O})$ and compute it in the next proposition.

Proposition 4.14. *Let $\mathcal{O}_p$ be a spherical nilpotent orbit in a classical Lie algebra $\mathfrak{g}$ that corresponds to the partition p. The set $IC^1(\mathcal{O}_p)$ of infinitesimal characters attached to $\mathcal{O}_p$ by the above procedure is $\{p^t\}$ in type A and as in the following tables for the other classical types:*

Type B

p	$IC^1(\mathcal{O}_p)$
$[2^{2k}, 1^{2n-4k+1}]$	$\{[2n-2k, 2k+1]\}$
$[3, 1^{2n-2}] n \neq 2$	$\{[2n-2, 2, 1], [2n-2, 1^3]\}$
$[3, 1^2]$	$\{[2^2, 1], [2, 1^3], [3, 1^2]\}$
$[3, 2^{2k}, 1^{2n-4k-2}]\ k \neq \frac{n-1}{2}, 0$	$\{[2n-2k-2, 2k+2, 1]\}$
$[3, 2^{n-1}]$	$\{[n^2, 1]\}$

Type C

p	$IC^1(\mathcal{O}_p)$
$[1^{2n}]$	$\{[2n+1]\}$
$[2^k, 1^{2n-2k}]\ k \neq 2$	$\{[2n-k+1, k-1, 1]\}$
$[2^2, 1^{2n-4}]$	$\{[2n-1, 1^2], [2n-1, 2]\}$
$[2^n] n \neq 2$	$\{[n^2, 1], [n+1, n-1, 1]\}$;
$[2^2]$	$\{[2^2, 1], [3, 1^2], [3, 2]\}$;

Type D

p	$IC^1(\mathcal{O}_p)$
$[2^{2k}, 1^{2n-4k}]$ $k \neq \frac{n}{2}$	$\{[2n-2k-1, 2k+1]\}$
$[2^n]$	$\{[n^2]\}$
$[3, 1^{2n-3}]$	$\{[2n-3, 2, 1], [2n-3, 1^3]\}$
$[3, 2^{2k}, 1^{2n-4k-3}]$	$\{[2n-2k-3, 2k+2, 1]\}$

4.3. **Infinitesimal Characters of $V(\mathcal{V}, \pi)$.** Recall the character α, defined as the square root of the absolute value of the real determinant of the Q_f action on $\mathfrak{q}/\mathfrak{q}_f$ used to define $V(\mathcal{V}, \pi)$. Suppose that α extends to a character γ on Q. According to §4.1, such an extension exists whenever the set $HW^1(w_\alpha)$ is not empty. The first goal of this section is to decide whether and when this occurs. This is important as the construction of $V(\mathcal{V}, \pi)$ relies on the existence of a bundle isomorphism $j_{\gamma,\pi}$ defined in §2.1. In the setting of spherical nilpotent orbits, $j_{\gamma,\pi}$ exists precisely when there is a character γ of the parabolic Q which restricts to α on Q_f.

The second goal of the section is to decide how well the infinitesimal characters of $V(\mathcal{V}, \pi)$ fit within those that ought to be attached to the nilpotent orbit $\mathcal{O}$. Suppose that the half-density bundle on G/Q is given by the character $\rho_{G/Q}$, and define $\gamma' = \gamma \otimes \rho_{G/Q}^{-1}$. The space $V(\mathcal{V}, \pi)$ is then a subset of $Ind_Q^G(\gamma')$. If we write w_γ for the character of γ and ρ for the half-sum of the positive roots of G, then the associated infinitesimal character is $\chi_\gamma = w_\gamma + \rho$. One expects that χ_γ should be a character attached to $\mathcal{O}$ in §4.2, that is, it should lie in the set $IC^1(\mathcal{O})$. We begin with a short list of examples of what is *not* true.

4.3.1. *A Few Examples.* First, we show that it is not always possible to find a character γ of Q that restricts to α on Q_f. This occurs already in type A for the minimal orbit in rank 5.

Example 4.15. Let $\mathfrak{g} = \mathfrak{gl}_5$ and consider the orbital variety $\mathcal{V}_T$ associated to the standard Young tableau

$$T = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 5 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

The basepoint $f = f_T = E_{e_2 - e_3}$ constructed in Definition 3.12 has dense B-orbit by Lemma 3.14. The τ-invariant of T and hence that of $\mathcal{V}_T$ can be gleaned from Theorem 2.16 and equals $\{e_1 - e_2, e_3 - e_4, e_4 - e_5\}$. If Q is the parabolic stabilizing $\mathcal{V}_T$ and L is its Levi subgroup, the τ-invariant forces $\mathfrak{l} = \mathfrak{gl}_2 \times \mathfrak{gl}_3$. We can now compute the weight of the square root of the absolute value of the determinant of the Q_f action on $\mathfrak{q}/\mathfrak{q}_f$. According to the inductive procedure of Proposition 3.19, the weight of α is $w_\alpha = -\mathfrak{t}_1 - \mathfrak{t}_3 + \mathfrak{t}_4 + \mathfrak{t}_5$ which we write as $w_\alpha = ((-1, 0), (-1, 1, 1))$ by grouping terms that correspond to the same reductive part of the Levi. The set of weights that restrict to w_α consists of the one-parameter family

$$HW(w_\alpha) = \{w_\alpha + \epsilon = ((-1, \epsilon_1), (-1 - \epsilon_1, 1, 1))\}.$$

The weight $w_\alpha + \epsilon$ corresponds to a one-dimensional representation of Q iff $-1 = \epsilon$ and $-1 - \epsilon = 1$. This is not possible, implying that $HW^1(w_\alpha) = \emptyset$.

One can reasonably expect that the property $HW^1(w_\alpha) = \emptyset$ is preserved by induction on tableau. This too is false.

Example 4.16. Let $\mathfrak{g} = \mathfrak{gl}_6$ and consider

$$S = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 6 \\ \hline 3 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array}$$

Then S contains T as a subtableau. Following the procedure of the previous example, we find that $w_\alpha = ((-2), (0, 0), (0, 1, 1))$ which extends to a two-parameter set of weights of the form $HW(w_\alpha) = \{w_\alpha + \epsilon = ((-2 - \epsilon_1), (\epsilon_1, \epsilon_2), (-\epsilon_2, 1, 1))\}$. The weight $w_\alpha + \epsilon$ corresponds to a one-dimensional representation γ of Q whenever $\epsilon_1 = \epsilon_2 = -1$, so that $HW^1(w_\alpha) \neq \emptyset$. In fact,

$$w_\gamma = w_\alpha + (1, -1, -1, 1, 0, 0) = (\tfrac{3}{2}, \tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{3}{2}) \in IC^1(\mathcal{O}_{[2^2,1^2]}).$$

One can also hope that if there does exists a character γ that restricts to α, then $\chi_\gamma \in IC^1(\mathcal{O})$. Unfortunately, this also fails.

Example 4.17. Let $\mathfrak{g} = \mathfrak{gl}_6$ and consider the tableau

$$\mathrm{T} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & 6 \\ \hline\end{array}$$

The Levi of the parabolic stabilizing $\mathcal{V}_T$ is $\mathfrak{l} = \mathfrak{gl}_2 \oplus \mathfrak{gl}_2 \oplus \mathfrak{gl}_2$. Proposition 3.19 implies that $w_\alpha = ((-\frac{3}{2}, -\frac{3}{2}), (\frac{1}{2}, -\frac{1}{2}), (\frac{3}{2}, \frac{3}{2}))$. The set of weights that restrict to α on $\mathfrak{t}_f$ is the three-parameter family

$$HW(w_\alpha) = \left\{w(\epsilon_1, \epsilon_2, \epsilon_3) = ((-\tfrac{3+\epsilon_1}{2}, -\tfrac{3+\epsilon_2}{2}),\ (\tfrac{1+\epsilon_2}{2}, -\tfrac{1+\epsilon_3}{2}), (\tfrac{3+\epsilon_3}{2}, \tfrac{3+\epsilon_1}{2}))\right\}.$$

For $w(\epsilon_1, \epsilon_2, \epsilon_3)$ to lie in $HW^1(w_\alpha)$, we must have $\epsilon_1 = \epsilon_2 = \epsilon_3$ and $1+\epsilon_2 = -1-\epsilon_3$. This forces $\epsilon_1 = \epsilon_2 = \epsilon_3 = -1$. Hence

$$HW^1(w_\alpha) = \{w(-1, -1, -1) = ((-1, -1), (0, 0), (1, 1))\}$$

which corresponds to the character of the parabolic Q given by

$$\gamma\begin{pmatrix} A_1 & * & * \\ 0 & A_2 & * \\ 0 & 0 & A_3 \end{pmatrix} = \left(|A_1|^{-2}|A_3|^2\right)^{\frac{1}{2}}$$

The infinitesimal character of $Ind_Q^G(\gamma \otimes \rho_{G/Q}^{-1})$ is then $\chi_\gamma = (\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2})$. But χ_γ does not lie in $IC^1(\mathcal{O}_{[3,3]}) = \{(1\ 1\ 0\ 0 - 1\ -1)\}$. In fact, $\chi_\gamma \in IC^1(\mathcal{O}_{[4,2]})$!

4.3.2. *Exhaustion of* $IC^1(\mathcal{O})$. We address the question of when it is possible to extend the character α of Q_f to a character of Q, and whether the set of such extensions for a given orbit provides enough candidates whose associated infinitesimal characters exhaust $IC^1(\mathcal{O})$. Example 4.15 shows that it is certainly not always possible find an extension γ of α for every orbital variety $\mathcal{V} \subset \mathcal{O}$. However, there exists at least one orbital variety within each orbit whose associated α does admit such an extension. Furthermore, there exists a sufficient number of such orbital varieties in $\mathcal{O}$ to account for all infinitesimal characters in $IC^1(\mathcal{O})$.

Theorem 4.18. *Let $\mathcal{O}$ be a rigid spherical orbit or a model orbit with $n > 2$ for a classical simple Lie group of rank n. For every $\chi \in IC^1(\mathcal{O})$, there exists an orbital variety $\mathcal{V} \subset \mathcal{O}$ for which $\alpha_\mathcal{V}$ extends to a character γ of Q, and $\chi_\gamma = \chi$.*

Proof. Consider $\mathcal{O}$ as above. It is always possible to construct a unique standard tableau $T_\mathcal{O}$ satisfying the following:

(i) There exists an integer k such that $\forall i \leq k$, $i \in T^1$ and $i \notin T^2$,

(ii) k is maximal among all standard tableaux of shape equal to the partition corresponding to $\mathcal{O}$.

When $\mathcal{O}_{shape\ T_{\mathcal{O}}}$ is a very even orbit in type D with Roman numeral II, define a tableau T_{II} by requiring that $\{n-1, n\} = N_1^{T_{II}}$, T_{II}^1 consist of odd numbers, and T_{II}^2 consist of even ones.

The desired orbital variety is $\mathcal{V}_{T_{\mathcal{O}}}$ (or $\mathcal{V}_{T_{II}}$). The Levi of the stabilizing subgroup of $\mathcal{V}_{T_{\mathcal{O}}}$ has exactly two reductive components. We first examine the case where the largest element of the partition p corresponding to $\mathcal{O}$ is 2 and the Roman numeral associated to $\mathcal{O}$, if any, is I. Let $[\lambda_1, \lambda_2]$ be the partition dual to p. Then $w_\alpha = ((c_1, c_1, \dots c_1), (c_2, c_2, \dots c_2))$ where

$$(c_1, c_2) = \begin{cases} (-\lambda_1, -\lambda_2) & \text{in type } A, \\ (-\lambda_1 + 2, 0) & \text{in types } B \text{ and } D, \text{ and} \\ (-\lambda_1 - 2, 0) & \text{in type } C. \end{cases}$$

The elements of $HW(w_\alpha)$ have the form $w(\epsilon_1, \epsilon_2, \dots, \epsilon_s) = ((c_1 - \epsilon_1, c_1 - \epsilon_2, \dots c_1 - \epsilon_s), (c_2, \dots, c_2 + \epsilon_2, c_2 + \epsilon_1))$ if $\mathcal{O}$ is rigid, $((\epsilon_1, c_1 - \epsilon_2, \dots c_1 - \epsilon_s), (c_2 + \epsilon_s, \dots, c_2 + \epsilon_2, c_2 + \epsilon_1))$ if $p = [2^n]$ and n is odd in type C, and $((c_1 - \epsilon_1, c_1 - \epsilon_2, \dots c_1 - \epsilon_s), (c_2 + \epsilon_s, \dots, c_2 + \epsilon_2, c_2 + \epsilon_1))$ otherwise.

In the first case, $w(\epsilon_1, \epsilon_2, \dots, \epsilon_s) \in HW^1(w_\alpha)$ iff $\epsilon_i = 0$ for all i. In the third case, $w(\epsilon_1, \epsilon_2, \dots, \epsilon_s) \in HW^1(w_\alpha)$ iff $\epsilon_i = \epsilon_j$ for all i and j. This produces a one-parameter family of weights that depends on the common value of the $\epsilon_i = \epsilon$. In the second case, $w(\epsilon_1, \epsilon_2, \dots, \epsilon_s) \in HW^1(w_\alpha)$ iff $\epsilon_i = \epsilon_j$ for $i, j \geq 2$ and $\epsilon_1 = c_1 - \epsilon_2$. This again yields a one-parameter family of weights that depends on the common value of the $\epsilon_i = \epsilon$ with $i \geq 2$. Therefore, an orbital variety always exists for which $\alpha_{\mathcal{V}}$ extends to a character γ of Q.

It is also easy to check that the χ_γ exhaust $IC^1(\mathcal{O})$. In the first case above, that is, whenever $\mathcal{O}$ is rigid, $|IC^1(\mathcal{O})| = 1$ and a comparison with Proposition 4.14 shows that $\{w(0, 0, \dots 0) + \rho\} = IC^1(\mathcal{O})$. In the third case above in types B and D, $|IC^1(\mathcal{O})| = 1$ again and with $\epsilon = 0$, $\{w(0, 0, \dots 0) + \rho\} = IC^1(\mathcal{O})$. In type C when $n > 2$, $|IC^1(\mathcal{O})| = 2$. Note that $w(-1, -1, \dots -1) \neq w(0, 0, \dots 0)$ and it is an easy check that $\{w(-1, -1, \dots -1) + \rho, w(0, 0, \dots 0) + \rho\} = IC^1(\mathcal{O})$. The second case is similar. Now consider the case when $\mathcal{O}$ is very even in type D with numeral II and examine the orbital variety $\mathcal{V}_{T_{II}}$. We find that

$$w_\alpha = \tfrac{1}{2}((-2n+2, -2n+2, -2n+4), (-2n+4, -2n+6), \dots, (-4, -2), (-2, 0), (0)).$$

The elements in $HW(w_\alpha)$ have the form $w(\beta, \epsilon_1, \dots, \epsilon_s) = \frac{1}{2}(-2n + 2 + \beta, -2n + 2 + \beta, -2n + 4 - \epsilon_1), (-2n + 4 + \epsilon_1, \dots (-2 + \epsilon_{s-1}, -\epsilon_s), (\epsilon_s))$. The set of elements in $HW^1(w_\alpha)$ is a one-parameter family, consisting of $w(\epsilon_s) = \frac{1}{2}((\dots (-4 + \epsilon_s, -4 + \epsilon_s), (-\epsilon_s, -\epsilon_s), (\epsilon_s))$. When n is odd, let $\epsilon_s = -1$ and when n is even, let $\epsilon_s = 3$. Inductively, it is now easy to show that

$$w(\epsilon_s) + \rho = (\tfrac{n-1}{2}, \tfrac{n-1}{2}, \tfrac{n-3}{2}, \tfrac{n-3}{2}, \dots, 3, 3, 1, -1)$$

which again accounts for $IC^1(\mathcal{O})$.

Now assume that p has largest part 3 and contains parts of size 2. Then according to Proposition 3.19, $w_\alpha = ((c_1, c_1, \dots, c_1 - 1), (c_2, c_2, \dots c_2))$ where $(c_1, c_2) = (-\lambda_1 + 1, 0)$ in both types B and D. The elements in $HW(w_\alpha)$ have the form

$$w(\epsilon_1, \dots \epsilon_{s+2}) = ((c_1 - \epsilon_1, \dots c_1 - \epsilon_s, c_1 - \epsilon_{s+1}), (c_2 + \epsilon_{s+2}, c_2 + \epsilon_s, \dots c_2 + \epsilon_1))$$

According to Proposition 4.1, $w(\epsilon_1, \ldots \epsilon_{s+2}) \in HW^1(w_\alpha)$ iff $\epsilon_i = 0$ for all $i \leq s$ and $s+2$, and $\epsilon_{s+1} = -1$. Furthermore, $\{w(0,0,\ldots,0,-1,0)+\rho)\} = IC^1(\mathcal{O})$.

Now if p has no parts of size two and $n > 2$, then $\mathcal{O}$ is neither rigid nor model, but the same result holds. We find that $w_\alpha = ((c_1)(c_2, \ldots c_2))$ and the form of the elements in $HW(w_\alpha)$ is $w(\epsilon_1, \epsilon_2) = ((c_1 - \epsilon_1)(\epsilon_2, 0, 0 \ldots 0))$ where $c_1 = -\lambda_1$. Now $w(\epsilon_1, \epsilon_2)$ lies in $HW^1(\mathcal{O})$ iff $\epsilon_2 = 0$. It is an easy check that $\{w(1,0)+\rho, w(0,0)+\rho\} = IC^1(\mathcal{O})$. This finishes the proof of the theorem.

□

4.3.3. *Inclusion in $IC^1(\mathcal{O})$.* The phenomenon of Example 4.17 fortunately occurs only among certain model spherical orbits. For all other spherical orbits, the infinitesimal character χ_γ, if defined, does indeed lie in $IC^1(\mathcal{O})$.

Theorem 4.19. *Let $\mathcal{O}$ be a rigid, non-model spherical nilpotent orbit for a classical simple Lie group and consider an orbital variety $\mathcal{V} \subset \mathcal{O}$ with stabilizer Q. Suppose that there exists a character γ of Q which restricts to the character α on Q_f defined as the absolute value on the real determinant of its action on $\mathfrak{q}/\mathfrak{q}_f$. Then $\chi_\gamma \in IC^1(\mathcal{O})$.*

We begin with an example illustrating our approach.

Example 4.20. Let $\mathfrak{g} = \mathfrak{gl}_7$ and let $\mathcal{O}_{[4,3]}$ be the nilpotent orbit corresponding to the partition $[4,3]$. Consider the standard Young tableau

$$T = \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & 6 \\ \hline 7 \\ \cline{1-1} \end{array}$$

The orbital variety $\mathcal{V}_T$ has stabilizer Q with Levi L whose Lie algebra is $\mathfrak{l} = \mathfrak{gl}_4 \oplus \mathfrak{gl}_3$. We would like to know that if γ is a character of Q which restricts to α on Q_f, then $w_\gamma + \rho$ lies in $IC^1(\mathcal{O})$. By Proposition 3.19 and the analysis of Section 5.1, $w_\alpha = ((-\frac{3}{2}, -1, -1, -1), (\frac{3}{2}, \frac{3}{2}, \frac{3}{2}))$, and

$$HW(w_\alpha) = \{w(\epsilon_1, \epsilon_2, \epsilon_3) = ((-\tfrac{3}{2}, -\tfrac{2+\epsilon_1}{2}, -\tfrac{2+\epsilon_2}{2}, -\tfrac{2+\epsilon_3}{2}), (\tfrac{3+\epsilon_3}{2}, \tfrac{3+\epsilon_2}{2}, \tfrac{3+\epsilon_1}{2})).$$

Hence $w(\epsilon_1, \epsilon_2, \epsilon_3) \in HW^1(w_\alpha)$ iff $\epsilon_i = 1$ for all i. Therefore $w_\gamma = w(1,1,1)$ and

$$w_\gamma + \rho = (\tfrac{3}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{3}{2}, 1, 0, -1) \in IC^1(\mathcal{O}_{[4,3]}),$$

as desired. Now note that $w_\alpha + \rho = w(0,0,0) + \rho = (\frac{3}{2}, 1, 0, -1, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2})$. While w_α does not correspond to a character of Q, $w_\alpha + \rho$ is nevertheless a permutation of $w_\gamma + \rho$ and also lies in $IC^1(\mathcal{O}_{[4,3]})$. This observation suggests an approach to our problem. We will prove:

Lemma 4.21. *Suppose that we are in the setting of Theorem 4.19. Then there exists a weight w_β such that $w_\beta + \rho$ lies in $IC^1(\mathcal{O})$ as well as the Weyl group orbit of $w_\gamma + \rho$.*

The lemma implies that $w_\gamma + \rho \in IC^1(\mathcal{O})$, proving Theorem 4.19. Its proof will occupy the remainder of this section.

As in our examples, an element of $HW(w_\alpha)$ can be written as $w(\epsilon_1, \epsilon_2, \ldots \epsilon_s) = (b_n \; b_{n-1} \; \ldots b_2 \; b_1)$. Each entry b_i may be a constant or depend on a single independent variable and at most two entries can depend on the same independent variable. We can divide the entries of b into disjoint maximal strings of entries of the form $b_l, b_{l-1}, \ldots b_k$ which satisfy:

- b_l and b_k both depend on the same independent variable, and
- there is no pair (l',k') such that $b_{l'}$ and $b_{k'}$ both depend on the same independent variable and $l' > l$ and $k' < k$.

For such a maximal string, call $I = (k, k+1, \ldots, l)$ a *dependent interval* of b. It is an easy consequence of Proposition 3.18 that if i lies in a dependent interval, the entry b_i is not constant. If $i \in I$ and b_i depends on the variable ϵ_{N_i}, we will say that ϵ_{N_i} *corresponds* to I. Note that each ϵ_i for $i \leq s$ corresponds to one and only one dependent interval I. If for all $i \leq \frac{l-k+1}{2}$ the entries b_{l-i} and b_{k-i} depend on ϵ_{N_i} for some $N_i \leq s$, we will say that I is *simple*.

For each simple dependent interval $I = (k \ldots l)$, define a permutation

$$\sigma_I = \prod_{i < \frac{l-k}{2}} (l-i \quad \tfrac{k+l}{2}+i)$$

as a product of transpositions; σ_I simply interchanges the first half of the entries of I with the second half, preserving the relative order of elements within each. By hypothesis, we know that there exists a character γ of Q which restricts to α on Q_f. Hence there exists a constant c_i for each variable ϵ_i such that $w(c_1, c_2, \ldots) \in HW(w_\alpha)$ that equals w_γ. If there exists a $c_i \neq 0$ that corresponds to the dependent interval I, we say that I is *non-zero.*

Example 4.22. Maintain the setting of Example 4.20. There is a unique dependent interval $I = (1, \ldots, 6)$ corresponding to the entries

$$(b_6\ b_5\ b_4\ b_3\ b_2\ b_1) = (-\tfrac{2+\epsilon_1}{2}, -\tfrac{2+\epsilon_2}{2}, -\tfrac{2+\epsilon_3}{2}, \tfrac{3+\epsilon_3}{2}, \tfrac{3+\epsilon_2}{2}, \tfrac{3+\epsilon_1}{2}).$$

In fact, I is simple and $\sigma_I = (3\ 6)\ (5\ 2)\ (4\ 1)$. Now note that if we write $w_\alpha + \rho$ as $(c_7\ c_6 \ldots c_1)$, then $w_\gamma + \rho = (c_{\sigma(7)}\ c_{\sigma(6)} \ldots c_{\sigma(1)})$. Hence, at least in this case, we have produced a method of describing the permutation relating $w_\gamma + \rho$ and $w_\alpha + \rho$.

We are ready to define w_β and state a proposition outlining the remainder of the proof of Lemma 4.21.

Definition 4.23. We define w_β inductively. For $\delta = \alpha$ or β, let $v_\delta = w_\delta - \iota(w_\delta^\downarrow)$. Further, let

$$v_\beta = v_\alpha + \begin{cases} -(n+2)T_1 & \text{in case (N1) where } X = C \\ -(n+1)T_1 & \text{in case (N2) where } X = C \\ \quad T_1 & \text{in case (N3) where } X = B \text{ or } D, \text{ and} \\ \quad T_3 & \text{in case (*) where } X = B. \end{cases}$$

Proposition 4.24. *If all non-zero dependent intervals in $HW(w_\alpha)$ are simple, then*

(i) $w_\beta + \rho \in IC^1(\mathcal{O})$,

(ii) $w_\gamma + \rho = \sigma(w_\beta + \rho)$, *where σ is the product of the σ_I taken over all non-zero simple dependent intervals I and acts by permuting the order of the entries of the weights.*

Furthermore,

(iii) a non-zero non-simple dependent interval cannot exist under the hypotheses of Theorem 4.19.

We will verify the present proposition using a sequence of four lemmas. First, note that Proposition 3.18 implies that $w_\beta \in HW(w_\alpha)$. Because of the hypotheses of Theorem 4.19, we know that there is a weight $w_\gamma \in HW^1(w_\alpha)$. If we write a general element of $HW(\alpha)$ as $w(\epsilon_1, \dots \epsilon_s) = (b_n, b_{n-1}, \dots, b_1)$, then there exists constants $c_1, \dots c_s$ such that $w(c_1, \dots c_s) = w_\gamma$. Because $\mathcal{O}$ is rigid, there exists at least one entry b_p which is constant. Note that it does not belong to any dependent interval. We will prove:

Lemma 4.25. *If b_p is adjacent to a non-zero non-simple dependent interval, then there are no constants $c_1, \dots c_s$ such that $w(c_1, \dots c_s) \in HW^1(w_\alpha)$.*

Lemma 4.26. *If $I_1 = (k_1, \dots l_1)$ is a non-zero non-simple dependent interval that is adjacent to a simple dependent interval $I_2 = (k_2, \dots l_2)$, then there are no constants $c_1, \dots c_s$ such that $w(c_1, \dots c_s) \in HW^1(w_\alpha)$.*

Lemma 4.27. *If b_p is adjacent to a non-zero simple dependent interval $I = (k, \dots l)$, then $\sigma_I\left((w_\beta + \rho)_l \dots (w_\beta + \rho)_k\right) = \left((w_\gamma + \rho)_l \dots (w_\gamma + \rho)_k\right)$.*

Lemma 4.28. *If $I_1 = (k_1, \dots l_1)$ is a non-zero simple dependent interval that is adjacent to either a simple dependent interval or a zero non-simple dependent interval $I_2 = (k_2, \dots l_2)$, then $\sigma_I\left((w_\beta + \rho)_{l_1} \dots (w_\beta + \rho)_{k_1}\right) = \left((w_\gamma + \rho)_{l_1} \dots (w_\gamma + \rho)_{k_1}\right)$.*

Assuming these, we first prove the proposition.

Proof of Proposition 4.24(iii). If $w(\epsilon_1, \dots \epsilon_s)$ contains a non-zero non-simple dependent interval, it must contain at least one that is adjacent to either a simple dependent interval or a constant. Lemmas 4.25 and 4.26 then provide a contradiction, proving Proposition 4.24(iii).

□

Proof of Proposition 4.24(ii). Proposition 4.24(iii) shows that $w(\epsilon_1, \dots \epsilon_s)$ consists solely of simple dependent intervals and constants. For an integer i that either lies in a zero dependent interval or whose corresponding entry is a constant, we know that $(w_\gamma + \rho)_i = (w_\beta + \rho)_i$. If, however, i lies in a non-zero dependent interval, Lemmas 4.27 and 4.28 show that $(w_\gamma + \rho)_i = (w_\beta + \rho)_{\sigma(i)}$, which implies Proposition 4.24(ii).

□

Proof of Proposition 4.24(i). We would like to show that $w_\beta \in IC^1(\mathcal{O})$. Let $S = \{IC^1(\mathcal{O}) - \iota(IC^1(\mathcal{O}^\downarrow))\}$ and define $w = w_\beta + \rho - \iota(w_\beta^\downarrow - \rho^\downarrow)$. It is easy to verify the lemma for small n. By induction, it is enough to show that $w \in S$. The proof in type C includes all the essential elements of the general proof, and is particularly easy to state. We detail each inductive case.

- (C1) Proposition 3.19 implies that $w = (\lambda_1, 0, \dots, 0)$. Recall the abbreviated character notation of §4.2. Note that $k = \lambda_2$ and that the difference $w = [2n - k + 1, k - 1, 1] - [2n - 2 - k + 1, k - 1, 1]$ always lies in the one or two element set S.
- (C2) This time, $w = (\lambda_2, 0, \dots, 0)$. Again using the notation of §4.2, we find that $w = [2n - k + 1, k - 1, 1] - [2n - k + 1, k - 3, 1]$ always lies in S.
- (N1) Here, $w = (n - 2, 0, \dots, 0)$. Using the notation of §4.2, $w = [n + 1, n - 1, 1] - [n + 1, n - 3, 1]$, which lies in S by Proposition 4.14.
- (N2) Here, $w = (n - 1, 0, \dots, 0)$. Using the notation of §4.2, $w = [n^2, 1] - [n, n - 2, 1]$, which lies in S by Proposition 4.14.

This accounts for all the cases that arise in type C. For the other classical types, the proof requires the same inductive verification except in one instance. When the partition corresponding to $\mathcal{O}$ has no parts of size 1, then $w_\beta \notin IC^1(\mathcal{O})$. This is not a contradition, as $\mathcal{O}$ is not rigid, but it does complicate the induction step. If $\mathcal{W}$ is an orbital variety such that $\mathcal{W}^\downarrow \subset \mathcal{O}$, then the associated w_β again lies in $IC^1(\mathcal{O})$. This proves Proposition 4.24(i).

□

Finally, we address the lemmas.

Proof of Lemma 4.25. Write $I = [k, k+1, \dots l]$ for the non-zero non-simple dependent interval adjacent to b_p, and further assume that $p = k-1$. The proof for the other possibility is symmetric. We utilize notation suggested by Proposition 4.1, separating each interval along the break points of the underlying Levi. The entries of I must have the form:

$$b_l, b_{l-1}, \dots b_{m_1}), (b_{m_1-1}, \dots, b_{m_2}) \dots (b_{m_q-1}, \dots b_k.$$

We examine two possibilities. Either $l, k-1 \in T^2 \setminus T^1$ and $k \in T^1 \setminus T^2$, or $\{l, k\} \in N_1^T$. Consider the first case. The entries of I must then have the form

$$a_l - \epsilon, a_{l-1} - \epsilon, \dots, a_{m_1} - \epsilon), (a_{m_1-1} + \epsilon, \dots a_{m_2} - \epsilon), \dots, (a_{m_q-1} + \epsilon, \dots a_k + \epsilon$$

for some ϵ since they must correspond to a weight in $HW^1(w_\alpha)$. Because all the entries grouped within parentheses must equal each other, according to Proposition 4.1 this gives us the conditions $a_k + \epsilon = a_{k-1} a_{m_i+1} + \epsilon = a_{m_{i+1}} - \epsilon$ for all $i < q$, which translate to

$$\text{(a)} \qquad \epsilon = a_{k-1} - a_k = \tfrac{a_{m_{i+1}} - a_{m_i+1}}{2} \qquad \text{for all } i < q.$$

We would like to show that these conditions are impossible to satisfy. Proposition 3.19 and Definition 4.23 give us a description of each of the a_i. We restrict the proof to type C, which contains all the elements of the general proof.

Let $[\lambda_1(i), \lambda_2(i)]$ be the partition dual to *shape* $T(i)$. Proposition 3.19 implies that $a_{k-1} = -\lambda_1(k-1) + 2$, $a_k = -\lambda_2(k-1) + 2$, $a_{m_2} = -\lambda_1(m_2) + 2$, and $a_{m_1+1} = -\lambda_2(m_1+1)$. Equations (a) translate to

$$\text{(b)} \qquad \epsilon = -\lambda_1(k-1) + \lambda_2(k-1) = \tfrac{-\lambda_1(m_2) + 2 + \lambda_2(m_1+1)}{2}.$$

However, $\lambda_1(k-1) - \lambda_2(k-1) = \lambda_1(l) - \lambda_2(l)$ because I is a dependent interval. Furthermore, the form of the entries in I implies that $\lambda_2(l) > \lambda_2(m_1+1)$ and $\lambda_1(l) < \lambda_1(m_2)$. But this implies that it is impossible to satisfy (b) and we cannot find constants c_i so that $w(c_1, \dots c_s) \in HW^1(w_\alpha)$. The only difference in proof for the other classical types are the precise values for the a_i.

Now suppose we are in the second case and that $\{k, l\} \in N_1^T$. The entries corresponding to the interval I must have the form

$$a_l + \beta, a_{l-1} - \epsilon, \dots, a_{m_1} - \epsilon), (a_{m_1-1} + \epsilon, \dots a_{m_2} - \epsilon), \dots, (a_{m_q-1} + \epsilon, \dots a_{k+1} + \epsilon, a_k + \beta.$$

Because I is non-simple, this means that the interval $\{k+1, \dots, l-1\}$ cannot be simple either. This time, we need to solve the equations

$$\epsilon = a_k + \beta - a_{k+1} = a_{l-1} - a_l - \beta = \tfrac{a_{m_{i+1}} - a_{m_i+1}}{2}$$

First, we find that $\beta = \frac{(a_{l-1}-a_l)+(a_{k+1}-a_k)}{2}$. This means that we still need to solve

(c) $$\epsilon = \frac{(a_{l-1}-a_l)-(a_{k+1}-a_k)}{2} = \frac{a_{m_{i+1}}-a_{m_i+1}}{2}$$

By an analysis similar to the above divided into each classical type, (c) again cannot be satisfied and Lemma 4.25 holds.

□

Proof of Lemma 4.26. If I_2 is a *zero* interval, then the proof is identical to the proof of Lemma 4.25, as the only property we needed was the expression for the term a_{k_1-1}, which is the same in the zero case. Now assume that I_1 is to the left of I_2 in the coordinate expression for w_γ of this section; the other possibility has a symmetric proof. There are again two cases in the proof. First assume that $\{k,n\} \notin N_1^T$. The two intervals must then have the form

$$a_{l_1}-\epsilon, a_{l_1-1}-\epsilon, \ldots, a_{m_1}-\epsilon), (a_{m_1-1}+\epsilon, \ldots a_{m_2}-\epsilon), \ldots, (a_{m_q-1}+\epsilon, \ldots a_{k_2}+\epsilon$$

and $$a_{l_2}-\mu, a_{l_2-1}-\mu, \ldots, a_{m'}-\mu), (a_{m'-1}+\mu, \ldots, a_{k_2}+\mu$$

with the additional restriction that $a_{l_2}-\mu = a_{k_1}+\epsilon$. Write ρ in coordinates as $(\rho_n, \rho_{n-1}, \ldots, \rho_1)$. The proof of Lemmas 4.27 and 4.28 imply that either $\mu = 0$, or $\mu = a_{l_2} - a_{m'} + \rho_{l_2} - \rho_{m'}$. The first possibility was considered above. As for the second, following the outline of the proof of Lemma 4.25, we would like to solve the equations

(d) $$\epsilon = a_{l_2} - \mu - a_k = \frac{a_{m_{i+1}}-a_{m_i+1}}{2} \qquad \text{for all } i < q.$$

In each of the classical types, Proposition 3.19 gives us values for the a_i, and we can similarly give an explicit description of ρ. In a manner similar to the proof of Lemma 4.25, we can now show that a solution to (d) does not exists. A similar analysis works for the case when $\{k,l\} \in N_1^T$ and Lemma 4.26 holds.

□

Proof of Lemma 4.27. Assume that $b_p = b_{k-1}$ as the proof for the other possibility in symmetric. The entries of I must have the form

$$b_l, b_{l-1}, \ldots b_m), (b_{m-1}, \ldots b_k.$$

As in the proof of Lemma 4.25, there are two possibilities. Either $l, k-1 \in T^2 \setminus T^1$ and $k \in T^1 \setminus T^2$, or $\{l,k\} \in N_1^T$. We examine the first case. The second in analogous. Write ρ in coordinates as $(\rho_n, \ldots, \rho_1)$. The entries of w_γ have the form

$$a_l-\epsilon, a_{l-1}-\epsilon, \ldots, a_m-\epsilon), (a_{m-1}+\epsilon, \ldots, a_k+\epsilon$$

where entries grouped by parentheses must equal since $w_\gamma \in HW^1(w_\alpha)$. This condition further forces $a_{k-1} = a_k + \epsilon$, or in other words,

(e) $$\epsilon = a_k - a_{k-1}$$

After examining the definition of the permutation σ_I, we need to verify that $a_{l+i} - \epsilon + \rho_{l+1} = a_{m+i} + \rho_{m+i}$ holds for all $i < (l-k)/2$, which will imply Lemma 4.27. We first consider type A. First of all, $\rho_{l+i} = n+1-2(l+i)$, hence we would like to know whether the equality $a_{l_i} - \epsilon + n + 1 - 2(l+i) = a_{m+i} + n + 1 - 2(m+i)$ holds. Proposition 3.19 implies that $a_{l+i} = a_l$ and $a_{m+i} = a_k$ for all of the above i and the above equation becomes $a_l - a_k + k - l + 1 = \epsilon$. This is possible iff this equation is compatible with (e). To verify this, we note that repeated application of Proposition 3.19 implies $a_{k-1} = -\lambda_1(k) + (\lambda_1 - \lambda_1(k)) = \lambda_1 - 2\lambda_1(k)$ which also equals $a_l + (k - l + 1)$. This implies that $a_l - a_k + (k - l + 1) = a_{k-1} - a_k$, and thus Lemma 4.27 holds in type A. The proof for groups of other types is analogous, only complicated by the appearance of horizontal dominos. However, dominos falling in cases (N2) or (N3) do not affect the dependent intervals because of Proposition 3.18. Case (N1) is dealt with precisely as in the proof of Lemma 4.25.

□

Proof of Lemma 4.28. If I_2 is a *zero* dependent interval, then the proof is identical to the proof of Lemma 4.27. We would like to show that in fact, if I_1 is a non-zero simple dependent interval, then I_2 must be a zero dependent interval. We can assume that I_1 is to the left of I_2 in the coordinate notation we have grown accustomed to. As in Lemma 4.27, the interval I_1 has the form

$$a_{l_1} - \epsilon, a_{l_1 - 1} - \epsilon, \dots, a_m - \epsilon), (a_{m-1} + \epsilon, \dots, a_{k_1} + \epsilon$$

while the interval I_2 has the form

$$a_{l_2} - \mu, a_{l_2 - 1} - \mu, \dots, a_{m'} - \mu), (a_{m'-1} + \mu, \dots, a_{k_2} + \mu$$

with the additional constraint that $l_2 - 1 = k_2$. We would like to show that $\mu = 0$. Because $w_\gamma \in HW(w_\alpha)$, we know that $a_{l_2} - \mu = a_{k_l} + \epsilon$. But our proof of Lemma 4.27 implies that in fact, $a_{l_2} = a_{k_l} + \epsilon$, forcing μ to be zero, implying Lemma 4.28.

□

References

[1] D. Barbasch, The unitary dual for complex classical Lie groups. *Invent. Math.* 96 (1989), no. 1, 103–176.

[2] D. Barbasch and D. Vogan, Primitive ideals and orbital integrals in complex classical groups. *Math. Ann.* 259 (1982), no. 2, 153–199.

[3] C. Carre and B. Leclerc, Splitting the Square of a Schur Function into its Symmetric and Antisymmetric Parts, *J. Algebraic Combin.* 4 (1995), no. 3, 201–231.

[4] D. Collingwood and W. M. McGovern. *Nilpotent orbits in semisimple Lie algebras.* Van Nostrand Reinhold Mathematics Series. Van Nostrand Reinhold Co., New York, 1993.

[5] D. Garfinkle, On the classification of primitive ideals for complex classical Lie algebras. I. *Compositio Math.* 75 (1990), no. 2, 135–169.

[6] M. Gerstenhaber, Dominance over the classical groups. *Ann. of Math.* (2) 74 1961 532–569.

[7] V. Ginsburg, $\mathfrak{G}$-modules, Springer's representations and bivariant Chern classes, *Adv. in Math*, 61 (1986), 1-48.

[8] W. Graham and D. Vogan, Geometric quantization for nilpotent coadjoint orbits, *Geometry and representation theory of real and p-adic groups*, J. Tirao, D. Vogan, and J. Wolf Birkhäuser, Boston-Basel-Berlin, 1998 .

[9] A. Knapp. *Representation Theory of Semisimple Lie Groups*, Princeton Mathematical Series, 36. Princeton University Press, Princeton, NJ, 1986.

[10] B. Kostant, Symplectic Spinors, 139-152 in *Geometria Simplettica e Fisica Matematica.* Symposia Mathematica XIV, Istituto Nazionale di Alta Matematica. Academic Press, London and New York, 1974.

[11] A. Joseph, On the characteristic polynomials of orbital varieties. *Ann. Sci. Ecole Norm. Sup.* (4) 22 (1989), no. 4, 569–603.

[12] W. M. McGovern, Completely prime maximal ideals and quantization, *Memoirs of the AMS American Mathematical Society*, Providence, Rhode Island , 1994

[13] W. M. McGovern, On the Spaltenstein-Steinberg Map for Classical Lie Algebras, *Comm. Algebra*, 27 (1999), 2979–2993.

[14] W. M. McGovern, Rings of regular functions on nilpotent orbits. II. Model algebras and orbits. *Comm. Algebra* 22 (1994), no. 3, 765–772

[15] A. Melnikov, Orbital Varieties in $\mathfrak{sl}_n$ and the Smith conjecture, *Journal of Algebra*, 200 (1998), 1-31.

[16] D. Panyushev, Complexity and nilpotent orbits. *Manuscripta Math.* 83 (1994), no. 3-4, 223–237.
[17] D. Panyushev, On spherical nilpotent orbits and beyond. *Ann. Inst. Fourier* (Grenoble) 49 (1999), no. 5, 1453–1476.
[18] T. Pietraho, Components of the Springer fiber and domino tableaux, *J. Algebra*272 (2004), no. 2, 711–729.