COMPONENTS OF THE SPRINGER FIBER AND DOMINO TABLEAUX

THOMAS PIETRAHO

Abstract. Consider a complex classical semi-simple Lie group along with the set of its nilpotent coadjoint orbits. When the group is of type A, the set of orbital varieties contained in a given nilpotent orbit is described a set of standard Young tableaux. We parameterize both, the orbital varieties and the irreducible components of unipotent varieties in the other classical groups by sets of standard domino tableaux. The main tools are Spaltenstein's results on signed domino tableaux together with Garfinkle's operations on standard domino tableaux.

Key words: Orbit Method, Orbital Varieties, Domino Tableaux

Thomas Pietraho
Department of Mathematics
Bowdoin College
8600 College Station
Brunswick, ME 04011
E-mail: `tpietrah@bowdoin.edu`

Thomas Pietraho
Department of Mathematics
Bowdoin College
8600 College Station
Brunswick, ME 04011
E-mail: tpietrah@bowdoin.edu

Proposed running head: Springer Fiber and Domino Tableaux

1. Introduction

Let $\mathfrak{g}$ be a complex semisimple Lie algebra with adjoint group G and write $\mathcal{O}_f = G\cdot f$ for the coadjoint orbit of G through f in $\mathfrak{g}^*$. Fix a Borel subgroup B of G and let $\mathcal{F}$ be the flag variety G/B. For a unipotent element $u \in G$, $\mathcal{F}_u$ is the variety of flags in $\mathcal{F}$ fixed by the action of u. The orbit $\mathcal{O}_f$ has a natural G-invariant symplectic structure and the Kostant-Kirillov method seeks to attach representations of G to certain Lagrangian subvarieties of $\mathcal{O}_f$ (see [6], [9], and [10]). Of particular importance is the set of orbital varieties, Lagrangian subvarieties of $\mathcal{O}_f$ that are fixed by a given Borel subgroup of G.

A result of Spaltenstein identifies the set of orbital varieties for a given nilpotent orbit with the orbits of a finite group on the irreducible components of the corresponding unipotent variety [11]. The main purpose of this paper is to provide new parameterizations of both, the orbital varieties contained in a given nilpotent orbit, as well as the irreducible components of the unipotent variety $\text{Irr}(\mathcal{F}_u)$.

In the case of classical groups, nilpotent coadjoint orbits are classified by partitions. Because the number of orbital varieties contained in a given orbit is finite, one expects that both orbital varieties and the components of the unipotent variety should also admit combinatorial descriptions. This is most apparent when G is of type A.

Theorem ([12])**.** *Suppose that $G = GL_n(\mathbb{C})$ and the nilpotent orbit $\mathcal{O}_f$ corresponds to the partition λ of n. Then the orbital varieties contained in $\mathcal{O}_f$ as well as the set of components $\text{Irr}(\mathcal{F}_u)$ are both parameterized by the family of standard Young tableaux of shape λ.*

In the setting of other classical groups, a method similar to the one used to obtain the above can also be employed to describe both families of objects. However, the resulting parametrization by subsets of signed domino tableaux is somewhat cumbersome (see [12] and [15]). The following argument suggests a more appealing parameter set.

First we recall that the set of domino partitions indexes the unitary dual of W, the Weyl group of G. In types B_n and C_n, the elements of $\widehat{W}$ are parameterized by ordered pairs (d, f) of partitions such that $|d| + |f| = n$ [1]. In each case, the parameter set is in bijection with the set of domino partitions of $2n$ (type C_n) or $2n + 1$ (type B_n). Write S for this set and λ for a partition lying in S. The dimension of the representation given by λ is precisely the number of standard domino tableaux of shape λ. If we choose a unipotent representative $u_\lambda \in G$ in the conjugacy class corresponding to λ, then Springer's characterization of the representations $\widehat{W}$ in the top degree cohomology of $\mathcal{F}_u$ [13] indicates that

$$\#SDT(n) = \sum_{\lambda \in S} \dim H^{top}(\mathcal{F}_{u_\lambda}, \mathbb{C}) = \#\{\mathrm{Irr}(\mathcal{F}_{u_\lambda}) \mid \lambda \in S\}$$

This suggests that $\mathrm{Irr}(\mathcal{F}_u)$ should correspond to a set of standard domino tableaux in a natural way. Indeed, this is the case. The precise relationship between van Leeuwen's parameter set for $\mathrm{Irr}(\mathcal{F}_u)$ [15] and the set of domino tableaux can be described in terms of Garfinkle's notions of cycles and moving-through maps [2]. After defining the notion of a *distinguished* cycle for a cluster of dominos, we show that moving through sets of distinguished cycles of open and closed clusters in van Leeuwen's parameter set defines a bijection with the set of all domino tableaux of a given size.

Theorem 1.1. *Suppose that G is a complex classical simple Lie group not of type A. Then the collection of irreducible components of the unipotent varieties for G as the unipotent element ranges over all conjugacy classes is parameterized by $SDT(n)$, the set of standard domino tableaux of size n.*

The action of the finite group A_u on the irreducible components $\mathrm{Irr}(\mathcal{F}_u)$ is described in [15]. In the signed domino parametrization, it acts by changing the signs of open clusters. We exploit this to obtain a parametrization of orbital varieties by standard domino tableaux. This time, moving through the distinguished cycles of just the closed clusters in van Leeuwen's parameter set defines the required bijection. The result is a little simpler to state if we consider nilpotent orbits of the isometry group of a nondegenerate bilinear form, G_ϵ.

Theorem 1.2. *Suppose that G is a complex classical simple Lie group not of type A and $\mathcal{O}$ is the nilpotent orbit of G_ϵ that corresponds to the partition λ . Then the set of orbital varieties contained in $\mathcal{O}$ is parameterized by the set of standard domino tableaux of shape λ.*

Parameterizations of orbital varieties by domino tableaux have been obtained in [8], by describing equivalence classes in the Weyl group of G, as well as in [14]. We will address the compatibility of these parameterizations with the one above in another paper.

In [10], this parametrization of orbital varieties is used to calculate infinitesimal characters of certain Graham-Vogan representations. The Graham-Vogan construction of representations associated to a coadjoint orbit is an extension of the method of polarizing a coadjoint orbit. Polarization relies on a construction Lagrangian foliations, which may not always exist. To amend this shortfall, [6] replaces Lagrangian foliations with Lagrangian coverings. By a theorem of V. Ginzburg, it is always possible to construct a Lagrangian covering of a coadjoint orbit. In fact, there is a unique one for each orbital variety contained in the orbit. For nilpotent orbits, the main ingredients of the Graham-Vogan construction are admissible orbit data and orbital varieties.

Our domino tableaux parametrization of orbital varieties facilitates the computation of a number of parameters required to calculate the infinitesimal characters of Graham-Vogan representations. For a given orbital variety, it is easy to extract information such as its maximal stabilizing parabolic as well as to construct certain basepoints from the corresponding domino tableau. For representations constructed from orbital varieties whose stabilizing parabolic has dense orbit, this information facilitates the computation of the the infinitesimal character.

2. Preliminaries

We first describe unipotent and orbital varieties, the relationship between them, and the combinatorial objects we will use in the rest of the paper.

2.1. **Unipotent and Orbital Varieties.**

Let G be a connected complex semisimple algebraic group, B a Borel subgroup fixed once and for all, and $\mathcal{F} = G/B$ the flag manifold of G. We consider the fixed point set $\mathcal{F}_u$ of a unipotent transformation u on $\mathcal{F}$. It has a natural structure of a projective algebraic variety, called the *unipotent variety*. We write $\mathrm{Irr}(\mathcal{F}_u)$ for the set of its irreducible components. The stabilizer G_u of u in G acts on $\mathcal{F}_u$ and gives an action of its component group $A_u = G_u/G_u^\circ$ on $\mathrm{Irr}(\mathcal{F}_u)$.

Now consider a nilpotent element f of the dual of the Lie algebra $\mathfrak{g}^*$ of G. Write $\mathcal{O}_f^{ad}$ for the orbit of f under the coadjoint action of G on $\mathfrak{g}^*$. Using the non-degeneracy of the Killing form, we can identify $\mathcal{O}_f^{ad}$ with a subset of $\mathfrak{g}$. If $\mathfrak{b}$ is the Lie algebra of B and $\mathfrak{n}$ its unipotent radical, then the set $\mathcal{O}_f^{ad} \cap \mathfrak{n}$ inherits the structure of a locally closed algebraic variety from the orbit $\mathcal{O}_f^{ad}$. Its components are Lagrangian submanifolds of $\mathcal{O}_f^{ad}$ and are known as *orbital varieties* [7]. There is a simple relationship between the set of orbital varieties contained in a given nilpotent orbit and the irreducible components of the corresponding unipotent variety. Suppose that the unipotent element u of G and the nilpotent element f of $\mathfrak{g}^*$ correspond to the same partition.

Theorem 2.1 ([11])**.** *There is a natural bijection*

$$\mathrm{Irr}(\mathcal{O}_f^{ad} \cap \mathfrak{n}) \longrightarrow \mathrm{Irr}(\mathcal{F}_u)/A_u$$

between the orbital varieties contained in the nilpotent orbit $\mathcal{O}_f^{ad}$ and the orbits of the finite group A_u on $Irr(\mathcal{F}_u)$.

The set of nilpotent orbits for a classical G admits a combinatorial description by partitions. Write $\mathcal{P}(n)$ for the set of partitions $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_k]$ of n, ordered so that $\lambda_i \geq \lambda_{i+1}$.

Theorem 2.2. *Nilpotent orbits in $\mathfrak{gl}_n$ are in one-to-one correspondence with the set $\mathcal{P}(n)$.*

The corresponding statement for the other classical groups is not much more difficult. To obtain slightly cleaner statements, we will state it in terms of the nilpotent orbits of the sightly larger isometry groups of nondegenerate bilinear forms. Let $\epsilon = \pm 1$, write $\epsilon_i = -\epsilon(-1)^i$ and consider a nondegenerate bilinear form on $\mathbb{C}^m$ satisfying $(x,y)_\epsilon = \epsilon(y,x)_\epsilon$ for all x and y. Let G_ϵ be the isometry group of this form and $\mathfrak{g}_\epsilon$ be its Lie algebra. Define a subset $\mathcal{P}_\epsilon(m)$ of $\mathcal{P}(m)$ as the partitions λ satisfying $\#\{j | \lambda_j = i\}$ is even for all i with $\epsilon_i = -1$. The classification of nilpotent orbits now takes the form:

Theorem 2.3 ([5])**.** *Let m be the dimension of the standard representation of G_ϵ. Nilpotent G_ϵ-orbits in $\mathfrak{g}_\epsilon$ are in one to one correspondence with the partitions of m contained in $\mathcal{P}_\epsilon(m)$.*

The nilpotent G_ϵ orbits in $\mathfrak{g}_\epsilon$ can be identified with the nilpotent orbits of the corresponding adjoint group with one exception. In type D, precisely two nilpotent orbits of the adjoint group correspond to every very even partition. We will write $\mathcal{O}_f$ for the G_ϵ-orbit through the nilpotent element f and $\mathcal{O}_\lambda$ for the G_ϵ-orbit that corresponds to the partition λ in this manner.

The group A_u is always finite, and in the setting of classical groups, it is always a two-group. More precisely:

Theorem 2.4 ([12](I.2.9))**.** *The group A_u is always trivial when G is of type A. In the other classical types, let B_λ be the set of the distinct parts λ_i of λ satisfying $(-1)^{\lambda_i} = -\epsilon$. Then A_u is a 2-group with $|B_\lambda|$ components.*

2.2. **Standard Tableaux.** A partition of of an integer m corresponds naturally to a Young diagram consisting of m squares. We call the partition underlying a Young diagram its *shape*. Recall the definitions of the sets of standard Young tableaux and standard domino tableaux from, for instance, [2]. We will write $SYT(\lambda)$ and $SDT(\lambda)$ respectively for the sets of Young and domino tableaux of shape λ. We refer to both objects generically as *standard tableaux* of shape λ, or $ST(\lambda)$, hoping that the precise meaning will be clear from the context. Also, we will write $ST(n)$ for the set of all standard tableaux with largest label n.

We view each standard tableau T as a set of ordered pairs (k, S_{ij}), denoting that the square in row i and column j of T is labelled by the integer k. When T is a domino tableau, the domino with label k, or $D(k,T)$, is a subset of T of the form $\{(k, S_{ij}), (k, S_{i+1,j})\}$ or

$\{(k, S_{ij}), (k, S_{i,j+1})\}$. We call these vertical and horizontal dominos, respectively. For convenience, we will refer to the set $\{(0, S_{11})\}$ as the zero domino when in type B. Whenever possible, we will omit labels of the squares and write S_{ij} for (k, S_{ij}). In that case, define *label* $S_{ij} = k$.

Definition 2.5. For a standard tableau T, let $T(k)$ denote the tableau formed by the squares of T with labels less than or equal to k. A domino tableau T is *admissible* of type $X = B$, C, or D, if the shape of each $T(k)$ is a partition of a nilpotent orbit of type X.

The dominos that appear within admissible tableaux fall into three categories. Following [15] , we call these types I^+, I^-, and N.

Definition 2.6. (1) In types B_n and D_n (respectively C_n), a vertical domino is of type I^+ if it lies in an odd (respectively even) numbered column.
(2) A vertical domino not of type I^+ is of type I^-.
(3) A horizontal domino is of type N if its left square lies in an even (respectively odd) numbered column.

Example 2.7. Suppose that G is of type C_n and consider the tableaux

$T =$
<table>
<tr><td rowspan="2">1</td><td rowspan="2">2</td><td rowspan="2">3</td><td rowspan="2">4</td><td colspan="2">5</td></tr>
<tr></tr>
</table>

$T' =$
<table>
<tr><td rowspan="2">1</td><td colspan="2">2</td><td colspan="2">3</td><td colspan="2">5</td></tr>
<tr><td colspan="2">4</td></tr>
</table>

Then T is admissible of type C but T' is not, since *shape* $T'(2) = [3, 1]$ is not the partition of a nilpotent orbit in type C. The dominos $D(1, T)$ and $D(3, T)$ are of type I^-, $D(2, T)$ and $D(4, T)$ are of type I^+, and $D(5, T)$ is of type N.

Clusters partition the set of dominos in an admissible standard domino tableau into subsets. We follow [15] and define them inductively. Hence suppose we already know the clusters of $T(k-1)$ and would like to known how $D(k, T)$ fits into the clusters of $T(k)$. Here is a summary:

Definition 2.8. In types B_n and C_n, let $cl(0)$ be the cluster containing $D(1, T)$.

(1) If $D(k, T) = \{S_{ij}, S_{i+1,j}\}$ and *type* $D = I^-$, then $D(k, T)$ joins the cluster of the domino containing $S_{i,j-1}$. If $j = 1$, then $D(k, T)$ joins $cl(0)$.

(2) If $D(k,T) = \{S_{ij}, S_{i+1,j}\}$ and *type* $D = I^+$ then $D(k,T)$ forms a singleton cluster in $T(k)$, unless $i \geq 2$ and $S_{i-1,j+1}$ is not in T. In the latter case, $D(k,T)$ joins the cluster of the domino containing $S_{i-1,j}$.
(3) Take $D(k,T) = \{S_{ij}, S_{i,j+1}\}$, so that *type* $D = N$. Let C_1 be the cluster of the domino containing $\{S_{i,j-1}\}$ but if $j = 1$, let $C_1 = cl(0)$. If $i \geq 2$ and $S_{i-1,j+2}$ is not in T, let C_2 be the cluster of the domino that containing $S_{i-1,j+1}$. If $C_1 = C_2$ or C_2 does not exist, the new cluster is $C_1 \cup D(k,T)$. If $C_1 \neq C_2$, the new cluster is $C_1 \cup C_2 \cup D(k,T)$.
(4) The clusters of $T(k-1)$ left unaffected by the above simply become clusters of $T(k)$.

Definition 2.9. A cluster is *open* if it contains domino of type I^+ or N along its right edge and is not $cl(0)$. A cluster that is neither $cl(0)$ nor open is *closed*. Denote the set of open clusters of T by $OC(T)$ and the set of closed clusters as $CC(T)$. For a cluster $\mathcal{C}$, let $I_{\mathcal{C}}$ be the domino in $\mathcal{C}$ with the smallest label and take S_{ij} as its left and uppermost square. For X equal to B or C, we say that $\mathcal{C}$ is an X-cluster iff $i+j$ is odd. For X equal to D or D' (see [4] for definition), we say that $\mathcal{C}$ is an X-cluster iff $i+j$ is even.

This definition differs from [15] as we do not call $cl(0)$ an open cluster.

Example 2.10. Using the domino tableaux from Example 2.7, if G is of type C, then T has three clusters: $\{1\}, \{2,3\}$, and $\{4,5\}$; the first is $cl(0)$, the second is closed, and the third is open. The tableau T' consists of one cluster.

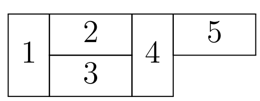
The open clusters of T correspond to the parts of λ contained in B_λ, the set parameterizing the $\mathbb{Z}_2$ factors of A_λ. As the latter set parameterizes the $\mathbb{Z}_2$ factors of A_λ, we will ultimately use open clusters to describe the action of A_λ on the irreducible components of $\mathcal{F}_u$. To be more precise, define a map

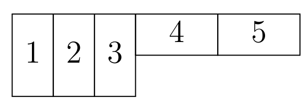
$$b_T : B_\lambda \longrightarrow OC(T) \cup cl(0).$$

For $r \in B_\lambda$, let $b_T(r)$ be the cluster that contains a domino ending a row of length r in T. This map is well-defined: any two dominos that end two rows of the same length belong to the same cluster; furthermore, such a cluster is always open or it is $cl(0)$. The map b_T is also onto $OC(T)$, but it is not one-to-one as T may have fewer open clusters than $|B_\lambda|$.

We also recall the notions of a cycle in a domino tableau and moving through such a cycle, as defined in [2]. We will think of cycles as both, subsets of dominos of T, as well as just sets of their labels. Write $MT(D(k,T),T)$ for the image of the domino $D(k,T)$ under the moving through map and $MT(k,T)$ for the image of T under moving through the cycle containing the label k. If U is a set of cycles of T that can be moved through independent of one another, we will further abuse notation by writing $MT(U,T)$ for the tableau obtained by moving through all the cycles in U. Recall the definition of X-fixed and X-variable squares for $X = B$, C, D, or D' [2]. Under the moving through map, the labels of the fixed squares are preserved while those of variable ones may change. We will call a cycle whose fixed squares are X-fixed an X-*cycle*. Note also that the B- and C-cycles as well as the D- and D'-cycles in a given tableau T coincide.

Example 2.11. Consider the domino tableaux T and T' from Example 2.7. The C-cycles in T are {1}, {2,3}, and {4,5} while those in T' are {1} and {2,3,4,5}. We have

$MT(2,T) =$ 

$MT(4,T) =$ 

The D-cycles in T are {1,2}, {3,4}, and {5}, while there is only one in T', mainly {1,2,3,4,5}.

3. Signed Domino Tableaux Parameterizations

The irreducible components of the unipotent variety $\mathcal{F}_u$ for classical G were described by N. Spaltenstein in [12]. We summarize this parametrization as interpreted by M.A. van Leeuwen [15]. Its advantage lies in a particularly translucent realization of the action of A_u on $\mathrm{Irr}(\mathcal{F}_u)$.

3.1. **Equivalence Classes of Signed Domino Tableaux.**

Let m be the rank of G. Fix a unipotent element $u \in G$ and let λ be the partition of the corresponding nilpotent orbit. We define a map

$$\mathcal{F}_u \longrightarrow ST(\lambda)$$

by the following procedure. Fix a flag $F = 0 \subset F_1 \subset F_2 \subset \ldots \in \mathcal{F}_u$ and let λ' be the shape of the Jordan form of the unipotent operator $u^{\downarrow}$ induced by u on the space $F^{\downarrow}$ defined as F/F_1 in type A and $F_1^{\perp}/F_1$ in the other classical types. The difference between the Young diagrams of λ and λ' is one square in type A and a domino in the other classical types. By assigning the label m to the set $\lambda \setminus \lambda'$ and repeating the procedure with the triple (F, u, m) replaced by $(F^{\downarrow}, u^{\downarrow}, m-1)$, we obtain a standard tableau of shape λ.

Theorem 3.1. *When G is of type A, the this construction defines a surjection onto $SYT(\lambda)$ that separates points of* $\text{Irr}(\mathcal{F}_u)$*. That is, it defines a bijection*

$$\text{Irr}(\mathcal{F}_u) \longrightarrow SYT(\lambda_u).$$

Corollary 3.2. *When G is of type A, the orbital varieties* $\text{Irr}(\mathcal{O}_\lambda \cap \mathfrak{n})$ *are parameterized by the set $SYT(\lambda)$.*

In the other classical types, any domino tableau in the image of the above map is admissible. Admissible tableaux, however, do not fully separate the components of $\mathcal{F}_u$. If two flags give rise to different domino tableaux in this way, they lie in different components of $\mathcal{F}_u$. However, the converse is not true. The inverse image $\mathcal{F}_{u,T}$ of a given admissible tableau T under this identification is in general not connected. Nevertheless, the irreducible components of $\mathcal{F}_{u,T}$ are precisely its connected components [15](3.2.3). Accounting for this disconnectedness yields a parametrization of $\text{Irr}(\mathcal{F}_u)$.

Definition 3.3. A signed domino tableau T of shape λ is an admissible domino of shape λ together with a choice of sign for each domino of type I^+. The set of signed domino tableaux is denoted $\Sigma DT(\lambda)$.

The set $\Sigma DT(\lambda)$ is too large to parameterize $\text{Irr}(\mathcal{F}_u)$ and we follow [15] in defining equivalence classes.

Definition 3.4. Write $|T|$ for the standard domino tableau underlying a signed domino tableau T. If $T, T' \in \Sigma DT(\lambda)$, let $T \sim_{op,cl} T'$ iff $|T| = |T'|$ and the products of signs in all corresponding open and closed clusters of T and T' agree. Denote the equivalence classes by $\Sigma DT_{op,cl}(\lambda)$. Define the set $\Sigma DT_{cl}(\lambda)$ similarly. We represent the elements of $\Sigma DT_{op,cl}(\lambda)$ and $\Sigma DT_{cl}(\lambda)$ as admissible tableaux with a choice of sign for each of the appropriate clusters.

3.2. **Parametrization Map.** There is a considerable amount of freedom in how a bijection between $\Sigma DT_{op,cl}(\lambda)$ and $\mathrm{Irr}(\mathcal{F}_u)$ can be defined. In fact, it is possible to choose the bijection in such a way that a specific element of $\Sigma DT_{op,cl}(\lambda)$ with underlying tableau $|T|$ is mapped to any chosen component of $\mathcal{F}_{u,|T|}$. We follow [15] and define a particular choice. A similar construction appears in [12](II.6).

The main step requires constructing certain flags F_T for $T \in \Sigma DT(\lambda)$ that will lie in $\mathcal{F}_{u,|T|}$. They will be build up from *special lines* which we now need to define. We begin by recalling the notion of a $\mathbb{C}[u]$-module from [12](II.6) for a unipotent u. Essentially, these are finite-dimensional modules over the polynomial ring $\mathbb{C}[u-1]$ together with a bilinear form b on which $u-1$ acts nilpotently and b is fixed by the action of u. For a $\mathbb{C}[u]$-module N, we will write $J(N)$ for the partition of the nilpotent orbit corresponding to u.

We construct a few basic $\mathbb{C}[u]$-modules. Let M_j be $\mathbb{C}^j$ with an action of $u-1$ defined by $(u-1)\cdot e_1 = 0$ and $(u-1)\cdot e_i = e_{i-1}$ for $i > 1$ on the basis elements $\{e_i\}$. Note that $J(M_j) = j$. The bilinear form b_{M_j} can be defined inductively. Let $b_{M_1}(e_1, e_1) = 1$. Suppose that M_{j-2} is already defined. The form b_{M_j} is then determined by the conditions that M_j is non-degenerate, and that the isomorphism $\langle e_1\rangle^{\perp}/\langle e_1\rangle \to M_{j-2}$ sending the coset of e_i to the coset of e_{i-1} becomes a $\mathbb{C}[u]$-module isomorphism. In this case, define the special line in M_j to be $\langle e_1\rangle$.

Note the $M_j \times M_j$ is also a $\mathbb{C}[u]$-module. Define two special lines l_+ as $\langle (e_1, ie_1)\rangle$ and l_- as $\langle (e_1, -ie_1)\rangle$ where i is a fixed square root of negative one.

Define modules $M_{j,j}$ as submodules of $M_{j+1} \times M_{j+1}$ given by $l_+^{\perp}/l_+$. The corresponding special line is the image of $l_+ \oplus l_-/l_+$.

Now let λ be a partition in $\mathcal{P}_\epsilon(m)$. Its Young diagram can be partitioned in a unique way in to rows of length j with $\epsilon_j = 1$ and pairs of adjacent rows of length j with $\epsilon_j = -1$. We define a module M_λ as a product of M_j for each $\epsilon_j = 1$ and $M_{j,j}$ for each pair of rows with $\epsilon_j = -1$. The special lines in M_λ will correspond to the dominos at the periphery of λ. Let D be such a domino and define the special line in M_λ *belonging to* D to be the special line in the appropriate summand of M_λ. When D is of type I^+, this leaves us the choice between l_+ and l_-, so we choose $l_{sign(D)}$.

If l is a special line in M_λ that belongs to a domino D, and λ' is the partition with D removed, then there is a canonical isomorphism $l^\perp/l \rightarrow M_{\lambda'}$. When the sign of D is negative, we use the automorphism mapping l_- to l_+ to transform the canonical isomorphism $l_+^\perp/l_+ \rightarrow M_{\lambda'}$ to an isomorphism $l_-^\perp/l_- \rightarrow M_{\lambda'}$. We write $F \simeq F'$ when a flag F in $l^\perp/l$ corresponds in this manner to a flag $F' \in M_{\lambda'}$.

Finally, we are ready to define F_T. This is done inductively by requiring that for all $k \leq m$:

(1) $(F_{T(k)})_1$ is the special line belonging $D(k, T(k))$, and
(2) $(F_{T(k)})_1^\perp/(F_{T(k)})_1 \simeq F_{T(k-1)}$

A easy enumeration of cases shows that two such flags F_T and $F_{T'}$ lie in the same component of $\mathcal{F}_{u,|T|}$ whenever $T \sim_{op,cl} T'$. This allows us to define a map Γ_u from $\Sigma DT(\lambda)$ to the components of $\mathcal{F}_{u,|T|}$ by sending the equivalence class of T to the unique component containing F_T.

We describe an action of A_u on $\Sigma DT_{op,cl}(\lambda_u)$. For $r \in B_\lambda$, let $b_T(r)$ be the cluster that contains a domino ending a row of length r in T. Let ξ_r act trivially if $b_T(r) = cl(0)$ and by changing the sign of the open cluster $b_T(r)$ otherwise. For each $r \in B_\lambda$, let g_r denote the generator of the corresponding $\mathbb{Z}_2$ factor of A_u. One can now define the action of g_r on $\Sigma DT_{op,cl}(\lambda_u)$ by $g_r[T] = \xi_r[T]$.

Theorem 3.5 ([15])**.** *Suppose that G is a classical group not of type A and u is a unipotent element of G corresponding to the partition λ. The map Γ_u defines an A_u-equivariant bijection between the components* $\mathrm{Irr}(\mathcal{F}_u)$ *and* $\Sigma DT_{op,cl}(\lambda)$.

Since A_u acts by changing the signs of the open clusters of $\Sigma DT_{op,cl}(\lambda)$, it is simple to parameterize the A_u orbits on $\mathrm{Irr}(\mathcal{F}_u)$.

Corollary 3.6. *Suppose that G is a classical group not of type A and $\mathcal{O}'_\lambda$ is the nilpotent orbit corresponding to the partition λ . The orbital varieties* $\mathrm{Irr}(\mathcal{O}_\lambda \cap \mathfrak{n})$ *are parameterized by* $\Sigma DT_{cl}(\lambda)$.

4. Domino Tableaux Parameterizations

We show how to index the components $\mathrm{Irr}(\mathcal{F}_u)$ and $\mathrm{Irr}(\mathcal{O}_\lambda \cap \mathfrak{n})$ by families of standard tableaux. In type A, this is Theorem 1. For the other classical types, we define maps from domino tableaux with signed clusters to the set of standard domino tableaux by applying Garfinkle's moving through map to certain distinguished cycles.

4.1. **Definition of Bijections.** Consider an X-cluster $\mathcal{C}$ and let I_C be the domino in $\mathcal{C}$ with the smallest label. Let $\mathcal{Y}_\mathcal{C}$ be the X-cycle through I_C. We call it the *initial* cycle of the cluster $\mathcal{C}$.

Proposition 4.1. *A cluster of an admissible domino tableau T that is either open or closed contains its initial cycle.*

We defer the proof to another section. Armed with this fact, we can propose a map

$$\Phi : \Sigma DT_{op,cl}(n) \longrightarrow SDT(n)$$

by moving through the distinguished cycles of all open and closed clusters with positive sign. More explicitly, for a tableau $T \in \Sigma DT_{op,cl}$, let $C^+(T)$ denote the set of open and closed clusters of T labelled by a $(+)$ and let $\sigma(T) = \{\mathcal{Y}_\mathcal{C} \mid \mathcal{C} \in C^+(T)\}$ be the set of their distinguished cycles. Write $|T|$ for the standard domino tableau underlying T. We define

$$\Phi(T) = MT(\sigma(T), |T|).$$

Lemma 4.2. *The map $\Phi : \Sigma DT_{op,cl}(n) \longrightarrow SDT(n)$ is a bijection. We can view the set $\Sigma DT_{cl}(n)$ as a subset of $\Sigma DT_{op,cl}(n)$ by assigning a negative sign to each unsigned open cluster of a domino tableau in $\Sigma DT_{cl}(n)$. Restricted to $\Sigma DT_{cl}(n)$, Φ preserves the shapes of tableaux and defines a bijection $\Phi : \Sigma DT_{cl}(\lambda) \longrightarrow SDT(\lambda)$ for each λ a shape of a nilpotent orbit.*

Proof. We check that Φ is well-defined, that its image lies in $SDT(n)$, and then construct its inverse. We first need to know that the definition of Φ does not depend on which order we move through the cycles in $\sigma(T)$. It is enough to check that if $\mathcal{Y}_\mathcal{C}$ and $\mathcal{Y}_{\mathcal{C}'} \in \sigma(T)$, then $\mathcal{Y}_{\mathcal{C}'}$ is also lies in $\sigma(MT(|T|, \mathcal{Y}_\mathcal{C}))$. While this statement is not true for arbitrary cycles, in our setting, this is Lemma 4.4.

The image of Φ indeed lies in $SDT(n)$. That $\Phi(T)$ is itself a domino tableau follows from the fact that moving through any cycle of $|T|$ yields a domino tableau. Hence $\Phi(T) \in SDT(n)$ and if $T \in \Sigma DT_{cl}(\lambda)$ then $\Phi(T) \in SDT(\lambda)$ since in this case Φ moves through only closed cycles.

The definition of a cluster forces the initial domino $I_{\mathcal{C}}$ of every closed cluster to be of type I^+. By the definition of moving through, the image of $MT(I_{\mathcal{C}}, T)$ in $MT(\mathcal{Y}_{\mathcal{C}}, T)$ is inadmissible, i.e. it is a horizontal domino not of type N. In general, all the inadmissible dominos in $\Phi(T)$ appear within the image of distinguished cycles under moving through. Furthermore, the lowest- numbered domino within each cycle is the image of the initial domino of some distinguished cycle. With this observation, we can construct the inverse of Φ. We define a map

$$\Psi : \Phi(\Sigma DT_{op,cl}(n)) \longrightarrow \Sigma DT_{op,cl}(n)$$

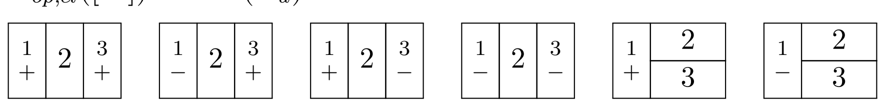
that satisfies $\Psi \circ \Phi = \text{Identity}$. Let $\iota(\Phi(T))$ be the set of cycles in $\Phi(T)$ that contain inadmissible dominos. We define $\Psi(\Phi(T)) = MT(\Phi(T), \iota(\Phi(T)))$. By the above discussion, $\iota(\Phi(T))$ contains precisely the images of cycles in $\sigma(T)$. Hence

$$\Psi(\Phi(T)) = MT(\Phi(T), \iota(\Phi(T))) = MT(MT(|T|, \sigma(T))) = T$$

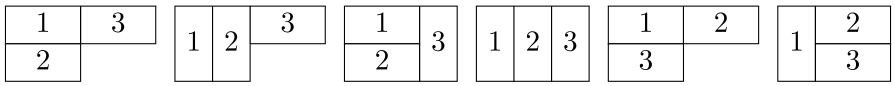
as desired. Thus Φ is a bijection onto its image in $SDT(n)$ and restricted to $\Sigma DT_{cl}(\lambda)$, it is a bijection with its image in $SDT(\lambda)$. As we already know that the sets $\Sigma DT_{cl}(\lambda)$ and $SDT(\lambda)$ both parameterize the same set of orbital varieties, and that $\Sigma DT_{op,cl}(n)$ and $SDT(n)$ both parameterize the same set of irreducible components of unipotent varieties, Φ must provide bijections between these two sets. $\square$

Theorems 1.1 and 1.2 are immediate consequences.

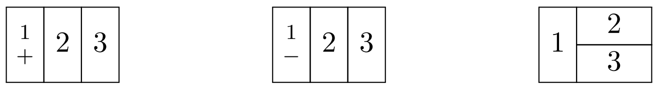
Example 4.3. Let G be of type D and suppose that both u and $\mathcal{O}_\lambda$ correspond to the partition $\lambda = [3^2]$. The van Leeuwen parameter set $\Sigma DT_{op,cl}([3^2])$ for $\text{Irr}(\mathcal{F}_u)$ is:



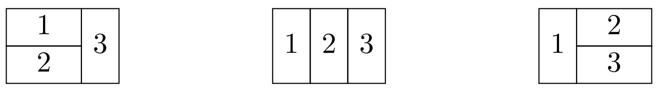
The image of $\Sigma DT_{op,cl}([3^2])$ under Φ is the following set of standard domino tableaux. We write the image of a given tableau in the same relative position. Note that this parameter set for $\mathrm{Irr}(\mathcal{F}_u)$ consists of all tableaux of shapes $[3^2]$ and $[4,2]$.



The van Leeuwen parameter set $\Sigma DT_{cl}([3^2])$ for the orbital varieties contained in $\mathcal{O}_\lambda$ is:



Its image under Φ is the set of all domino tableaux of shape $[3^2]$. Again, we write the image of a tableau in the same relative position.

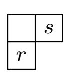


4.2. **Independence of Moving Through Initial Cycles.**

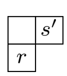
Lemma 4.4. *Consider open or closed clusters* $\mathcal{C}$ *and* $\mathcal{C}'$ *and their initial cycles* $\mathcal{Y}_{\mathcal{C}}$ *and* $\mathcal{Y}_{\mathcal{C}'}$*. Then* $\mathcal{Y}_{\mathcal{C}}$ *is again a cycle in* $MT(|T|, \mathcal{Y}_{\mathcal{C}'})$*.*

Proof. If $\mathcal{C}$ and $\mathcal{C}'$ are clusters of the same type, then so are their initial cycles and the lemma is [2](1.5.29). Otherwise, without loss of generality, take $\mathcal{C}$ to be a C-cluster and $\mathcal{C}'$ to be a D-cluster. As the proof in the other cases is similar, we can also assume that $\mathcal{Y}_{\mathcal{C}}$ is C-boxed while $\mathcal{Y}_{\mathcal{C}'}$ is D-boxed.

Suppose that the dominos $D(r) \in \mathcal{Y}_{\mathcal{C}}$ and $D(s) \in \mathcal{Y}_{\mathcal{C}'}$ lie in relative positions compatible with the diagram

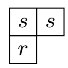


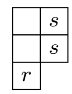
where the box labelled by r is fixed. The same squares in $MT(|T|, \mathcal{Y}_{\mathcal{C}'})$ have the labels



for some s'.

To prove the lemma, we need to show that $s < r$ implies $s' < r$ and $s > r$ implies $s' > r$. Since our choice of r and s was arbitrary, this will show that $\mathcal{Y}_{\mathcal{C}}$ remains a cycle. There are two possibilities for the domino $D(s)$. It is either horizontal or vertical and must occupy the following squares:

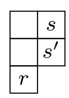




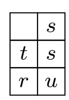
Case (i) Case (ii)

Case (i). In this case, $s < r$ always. Garfinkle's rules for moving through imply that $MT(|T|, D(r)) \cap \mathcal{C}' \neq \varnothing$. This is a contradiction since we know by hypothesis that $\mathcal{Y}_{\mathcal{C}} \neq \mathcal{Y}_{\mathcal{C}'}$. Hence this case does not occur.

Case (ii). First suppose $s > r$. Then the squares within $MT(|T|, \mathcal{Y}_{\mathcal{C}'})$ must look like

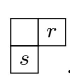


for some $s' \neq s$. Since the tableau $MT(\mathcal{Y}_{\mathcal{C}'}, T)$ is standard, this requires that $s' > s$ implying $s' > r$ which is what we desired. Now suppose $s < r$ and suppose the squares in our diagram look like



As in Case (i), we find that $D(t) \notin \mathcal{C}'$. Since $D(t) \in \mathcal{C}$, type $D(s) = I^+$ implies type $D(t) = I^-$, type $D(r) = I^-$, and type $D(u) = I^+$. Otherwise, the rules defining clusters would force s to lie in the cluster $\mathcal{C}$. Now $D(u)$ lies in the initial cycle of a closed cluster of same type as $\mathcal{C}'$. Since it lies on the periphery and its type is I^+, then its top square must be fixed. In particular, $D(u) \notin \mathcal{C}$. But $s < r$ implies $MT(D(r)) \cap D(u) \neq \varnothing$. This is a contradiction, implying that this case does not arise.

To finish the proof, we must examine the possibility that $D(s)$ and $D(r)$ lie in the relative positions described by



.

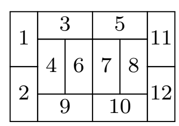
This case is completely analogous and we omit the proof. □

This lemma shows that the image of moving though a subset of distinguished cycles is independent of the order in which these cycles are moved though. Note, however, that a similar result is not true for subsets of arbitrary cycles.

4.3. **Nested Clusters and the Periphery of a Cluster.** We aim to show that closed and open clusters contain their distinguished cycles. The proof has two parts. First, we show that $\mathcal{Y}_{\mathcal{C}}$ is contained in a larger set of clusters $\overline{\mathcal{C}}$, defined as the union of $\mathcal{C}$ with all of its *nested* clusters. Then, we show that $\mathcal{Y}_{\mathcal{C}}$ intersects each of the nested clusters trivially.

Let $\mathcal{C}$ be a cluster of a tableau T and denote by $row_k\, T = \{S_{k,j} | \, j \geq 0\}$ the kth row of T. Define $col_k\, T$ similarly. If $row_k\, T \cap \mathcal{C} \neq \emptyset$, let $\inf_k \mathcal{C} = \inf\{j | \; S_{k,j} \in row_k\, T \cap \mathcal{C}\}$ and $\sup_k \mathcal{C} = \sup\{j \mid S_{k,j} \in row_k T \cap \mathcal{C}\}$.

Example 4.5. Consider the following tableau of type D. It has two closed clusters given by the sets $\mathcal{C} = \{1, 2, 3, 4, 5, 8, 9, 10, 11, 12\}$ and $\mathcal{C}' = \{6, 7\}$.



$\mathcal{C}$ is a D-cluster while $\mathcal{C}'$ is a B-cluster. $\mathcal{Y}_{\mathcal{C}}$ is then a D-cycle and consists of the dominos in the set $\{1, 3, 5, 11, 12, 10, 9, 2\}$. T has two other D-cycles, $\{4, 6\}$ and $\{7, 8\}$. Both intersect $\mathcal{C}$, but are not contained within it. The B-cycle $\mathcal{Y}_{\mathcal{C}'}$ equals $\{6, 7\}$ and is contained in $\mathcal{C}'$. Hence an X-cluster may not contain all the X-cycles through its dominos. However, it always contains its initial cycle. Also notice that $\mathcal{C}$ completely surrounds $\mathcal{C}'$. We call such interior clusters *nested.*

Nested clusters complicate the description of clusters. To simplify our initial results, we would like to consider the set formed by a cluster together with all of its nested clusters. To be more precise:

Definition 4.6. Let $\mathcal{C}'$ be a cluster of T. It is *nested* in $\mathcal{C}$ if all of the following are satisfied:

$$\inf\{k | row_k T \cap \mathcal{C}' \neq \emptyset\} > \inf\{k | row_k T \cap \mathcal{C} \neq \emptyset\}$$
$$\sup\{k | row_k T \cap \mathcal{C}' \neq \emptyset\} < \sup\{k | row_k T \cap \mathcal{C} \neq \emptyset\}$$
$$\inf\{k | col_k T \cap \mathcal{C}' \neq \emptyset\} > \inf\{k | col_k T \cap \mathcal{C} \neq \emptyset\}$$
$$\sup\{k | col_k T \cap \mathcal{C}' \neq \emptyset\} < \sup\{k | col_k T \cap \mathcal{C} \neq \emptyset\}$$

Define $\overline{\mathcal{C}}$ to be the union of $\mathcal{C}$ together with all clusters nested within it. We will write $periphery(\overline{\mathcal{C}})$ for the set of dominos in $\overline{\mathcal{C}}$ that are adjacent to some square of T that does not lie in $\overline{\mathcal{C}}$. Note that $periphery(\overline{\mathcal{C}})$ is a subset of the original cluster $\mathcal{C}$.

Example 4.7. In the above tableau, $\mathcal{C}'$ is nested in $\mathcal{C}$. Furthermore, $\mathcal{C} \cup \mathcal{C}' = \overline{\mathcal{C}} = T$, and $periphery(\overline{\mathcal{C}}) = \mathcal{Y}_{\mathcal{C}} \subset \mathcal{C}$.

The next two propositions describe properties of dominos that occur along the left and right edges of $\overline{\mathcal{C}}$. Recall that our definition of the cycle $\mathcal{Y}_{\mathcal{C}}$ endows $\mathcal{C}$ as well as $\overline{\mathcal{C}}$ with a choice of fixed and variable squares by defining the left and uppermost square of $I_{\mathcal{C}}$ as fixed.

Proposition 4.8. *Suppose that $\mathcal{C}$ is a non-zero cluster of a domino tableau T and that the intersection of the k-th row of T with $\mathcal{C}$ is not empty. Then the dominos $D(label(T_{k,inf_k \mathcal{C}}), T)$ and $D(label(T_{k,inf_k \overline{\mathcal{C}}}), T)$ are both of type I^+. In addition, if $\mathcal{C}$ is also closed, then the dominos $D(label(T_{k,sup_k \mathcal{C}}), T)$ and $D(label(T_{k,sup_k \overline{\mathcal{C}}}), T)$ are of type I^-.*

Proof. The first statement is true for all non-zero clusters by Definition 2.8. The second statement is the defining property of closed clusters. □

Proposition 4.9. *Suppose that $\mathcal{C}$ is a non-zero cluster of a domino tableau T. If the domino D consisting of the squares S_{pq} and $S_{p+1,q}$ lies in $periphery(\overline{\mathcal{C}})$, then*

(1) S_{pq} is fixed if type $D = I^+$ and
(2) $S_{p+1,q}$ is fixed if type $D = I^-$

Proof. Case (i). Assume that there is a D' in the $periphery(\overline{\mathcal{C}})$ of type I^+ whose uppermost square is not fixed. Then $periphery(\overline{\mathcal{C}})$ must contain two type I^+ dominos $E = \{S_{k,l}, S_{k+1,l}\}$ and $E' = \{S_{k+1,m}, S_{k+2,m}\}$ with the squares S_{kl} and $S_{k+2,m}$ fixed and $|m-l|$ minimal.

Assume $m < l$. The opposite case can be proved by a similar argument. Because E' is of type I^+, there is an integer t such that $m < t < l$, $S_{k+1,t} \in periphery(\overline{\mathcal{C}})$, and t is maximal with these properties. Let F be the domino containing $S_{k+1,t}$. F has to be $\{S_{k+1,t}, S_{k+2,t}\}$ and of type I^-. If its type was I^- or N, Definition 2.8 would force $S_{k+1,t+1}$ to be in $periphery(\overline{\mathcal{C}})$ as well. If F on the other hand was

$\{S_{k+1,t}, S_{k,t}\}$, this would contradict the minimality of $|m-l|$. We now consider two cases.

(a) Assume $t = l-1$. Because E and F lie in $periphery(\overline{\mathcal{C}})$ and hence in $\mathcal{C}$, $\mathcal{C}$ must contain a domino of type N of the form $\{S_{u,l-1}, S_{u,l}\}$ with $u > k+2$ and u minimal with this property. The set of squares $\{S_{p,l-1} | k+2 < p < u\} \cup \{S_{pl} | k+1 < p < u\}$ must be tiled by dominos, which is impossible, as its cardinality is odd.

(b) Assume $t < l-1$. We will contradict the maximality of t. Because E and F both lie in $\mathcal{C}$, $\mathcal{C}$ must contain a sequence H_α of dominos of type N satisfying

$$H_\alpha = \{S_{k+1+f(\alpha),t+2\alpha}, S_{k+1+f(\alpha),t+2\alpha+1}\}$$

where $0 \leq \alpha \leq \frac{l-t+1}{2}$. We choose each H_α such that for all α, $f(\alpha)$ is minimal with this property. Because the sets $\{S_{k+p,l} | k+1 < p < k+1+f(\frac{l-t+1}{2})\}$ and $\{S_{k+p,t} | k+2 < p < k+1+f(0)\}$ have to be tiled by dominos of type I^+ and I^- respectively, $f(0)$ has to be even and $f(\frac{l-t+1}{2})$ has to be odd. Hence there is a β such that $f(\beta)$ is even and $f(\beta+1)$ is odd.

Assume $f(\beta) < f(\beta+1)$, but the argument in the other case is symmetric. Let G be the domino containing the square $S_{k+1+f(\beta),t+2\beta+2}$. G must belong to $\mathcal{C}$, as H_β and G is either of type I^- or N. The type of G cannot be N, however, as this would contradict the condition on f. Hence G must be of type I^-. If G equals $\{S_{k+1+f(\beta),t+2\beta+2}, S_{k+f(\beta),t+2\beta+2}\}$. Then by successive applications of Definition 2.8, the set of dominos

$$\{\{S_{k+f(\beta)-\gamma\epsilon,t+2\beta+\epsilon}, S_{k+1+f(\beta)-\gamma-\epsilon,t+2\beta+\epsilon}\}\}$$

with $\epsilon = 1$ or 2 and $0 \leq \gamma \leq f(\beta)-2$ is contained in $\mathcal{C}$ as well. But this means that $t+2\beta+\epsilon$ for $\epsilon = 1$ or 2 satisfies the defining property of t, contradicting its maximality.

Case (ii). We would like to show that the bottom square is fixed for every I^- domino in $periphery(\overline{\mathcal{C}})$. It is enough to show that this is true for one such domino, as an argument similar to that in case (i) can be repeated for the others. Let $l = \inf\{k | row_k T \cap \overline{\mathcal{C}} = \emptyset\}$. Then by 4.8 and the definition of fixed, we know that $S_{l,\inf_l \overline{\mathcal{C}}}$ is fixed. As

$\{S_{l,\sup_l \overline{\mathcal{C}}}, S_{l+1,\sup_l \overline{\mathcal{C}}}\}$ is a domino of type I^- in *periphery*$(\overline{\mathcal{C}})$, we have found the desired domino. □

Lemma 4.10. *The following inclusions hold when $\mathcal{C}$ is an open or closed cluster: periphery $(\overline{\mathcal{C}}) \subset \mathcal{Y}_{\mathcal{C}} \subset \overline{\mathcal{C}}$.*

Proof. Recall that our choice of a fixed square in $I_{\mathcal{C}}$ defines the fixed squares in all of $\overline{\mathcal{C}}$. Define $\widetilde{\mathcal{C}}$ as $\overline{\mathcal{C}}$ when $\mathcal{C}$ is closed and $\overline{\mathcal{C}}$ union with all empty holes and corners of $|T|$ adjacent to $\mathcal{C}$ when $\mathcal{C}$ is open [2](1.5.5). We show that the image $MT(D,T)$ of D in *periphery*$(\overline{\mathcal{C}})$ lies in $\widetilde{\mathcal{C}}$. This shows the second inclusion, as if any domino in *periphery*$(\overline{\mathcal{C}})$ stays in $\overline{\mathcal{C}}$ under moving through, then so must the cycle $\mathcal{Y}_{\mathcal{C}}$. The first inclusion is a consequence of the argument and the definitions of moving through and clusters. We differentiate cases accounting for different domino positions along *periphery*$(\overline{\mathcal{C}})$.

Case (i). Take $D = \{(k, S_{ij}), (k, S_{i+1,j})\}$ and suppose $\mathit{type}\, D = I^+$. Because D lies on *periphery*$(\overline{\mathcal{C}})$, Proposition 4.9 implies that S_{ij} is fixed. Due to Definition 2.8 (1) and Definition 2.9, $S_{i,j+1} \in \widetilde{\mathcal{C}}$.

(a) Suppose $S_{i-1,j+1}$ in not in $\overline{\mathcal{C}}$. Then $r = \mathit{label}(S_{i-1,j+1}) < k$. Otherwise $S_{i-1,j}$ and S_{ij} would both belong to the same cluster by Definition 2.8 (1). Since $S_{i-1,j}$ and $S_{i-1,j+1}$ are in the same cluster by Definition 2.8 (2) or (3), this contradicts our assumption. Now [2](1.5.26) forces $MT(D,T)$ to equal $\{(k, S_{ij}), (k, S_{i,j+1})\}$, and since S_{ij} and $S_{i,j+1}$ both belong to $\widetilde{\mathcal{C}}$, so must $MT(D,T)$.

(b) Suppose now that $S_{i-1,j+1} \in \widetilde{\mathcal{C}}$. Then the square $S_{i-1,j} \in \overline{\mathcal{C}}$ as well since by Definition 2.8 (2) or (3), they both belong to the same cluster. Now [2](1.5.26) implies $MT(D,T) \subset \{S_{ij}, S_{i-1,j}, S_{i,j+1}\}$. As all of these squares lie in $\widetilde{\mathcal{C}}$, we must also have $MT(D,T) \subset \widetilde{\mathcal{C}}$.

Case (ii). Suppose $D = \{(k, S_{ij}), (k, S_{i,j+1})\}$ and that the square $S_{i,j+1}$ is fixed. By Definition 2.8 (1) and Definition 2.9, $S_{i,j+2} \in \widetilde{\mathcal{C}}$.

(a) Suppose $S_{i-1,j+1}$ is not in $\overline{\mathcal{C}}$. Then $S_{i-1,j+2}$ lies in $|T|$ but not in $\overline{\mathcal{C}}$, as by Definition 2.8 (2) or (3), they both belong to the same cluster. The definition of a cluster forces $r = \mathit{label}(S_{i-1,j+2}) < k$ and [2](1.5.26(ii)) implies $MT(D,T) = \{S_{i,j+1}, S_{i,j+2}\}$. Since the squares $S_{i,j+1}$ as well as $S_{i,j+2}$ are both contained in $\widetilde{\mathcal{C}}$, so is $MT(D,T)$.

(b) Suppose $S_{i-1,j+1}$ lies in $\overline{\mathcal{C}}$. Then because the domino $MT(D,T)$ must be a subset of $\{S_{i,j+1}, S_{i,j+2}, S_{i-1,j+1}\}$, it must also be a subset of $\overline{\mathcal{C}}$.

Case (*iii*). Suppose $D = \{(k, S_{ij}), (k, S_{i,j+1})\}$ and that the square S_{ij} is fixed. Then $S_{i,j-1} \in \overline{\mathcal{C}}$ by Definition 2.8 (3).

(a) Suppose first that $S_{i+1,j-1}$ is not in $\overline{\mathcal{C}}$. Then $r = label(S_{i+1,j-1}) > k$ by either Definition 2.8 (1) or (3). But [2](1.5.26(iii)) forces $MT(D,T)$ to be precisely $\{S_{ij}, S_{i,j-1}\}$ which is a subset of $\overline{\mathcal{C}}$.

(b) If $S_{i+1,j-1} \in \overline{\mathcal{C}}$, then $S_{i+1,j} \in \widetilde{\mathcal{C}}$ as well, since by Definition 2.8 (2) or (3), they either must belong to the same cluster or $S_{i+1,j}$ is an empty hole or corner. But by [2](1.5.26(iii)(iv)), $MT(D,T)$ is a subset of $\{S_{ij}, S_{i+1,j}, S_{i,j-1}\}$, all of whose squares lie in $\widetilde{\mathcal{C}}$.

Case (*iv*). Suppose $D = \{(k, S_{ij}), (k, S_{i+1,j})\}$ and that the domino D is of type I^-. The square $S_{i+1,j}$ is then fixed and $S_{i+1,j-1} \in \overline{\mathcal{C}}$.

(a) Assume that $S_{i+2,j-1} \in \overline{\mathcal{C}}$. Then $S_{i+2,j} \in \widetilde{\mathcal{C}}$. Since $MT(D,T)$ is the domino $\{S_{i+1,j}, S_{i+1,j-1}\}$ or $\{S_{i+1,j}, S_{i+2,j}\}$. Hence $MT(D,T)$ lies in $\overline{\mathcal{C}}$ as both possibilities are contained in $\overline{\mathcal{C}}$.

(b) Assume $S_{i+2,j-1}$ is not in $\overline{\mathcal{C}}$. We have $r = label(S_{i+2,j-1}) > k$, for otherwise $D(r,T)$ and hence $S_{i+2,j-1}$ would lie in $\overline{\mathcal{C}}$. But then $MT(D,T) = \{S_{i+1,j}, S_{i+1,j-1}\}$, so it is contained in $\overline{\mathcal{C}}$.

These cases describe all possibilities by Proposition 4.9. □

What remains is to see that the initial cycle $\mathcal{Y}_{\mathcal{C}}$ is contained within the cluster $\mathcal{C}$ itself. It is enough to show that its intersection with any closed cluster nested in $\mathcal{C}$ is empty, as open clusters cannot be nested. Our proof relies on the notion of X-boxing [2](1.5.2). We restate the relevant result.

Proposition 4.11 ([2](1.5.9) and (1.5.22))**.** *Suppose that the dominos $D(k,T)$ and $D(k',T)$ both belong to the same X-cycle. Then*

(1) $D(k,T)$ is X-boxed iff $MT(D(k,T),T)$ is not X-boxed.

(2) $D(k,T)$ and $D(k',T)$ are both simultaneously X-boxed or not X-boxed.

Lemma 4.12. *If $\mathcal{C}' \subset \overline{\mathcal{C}}$ is a closed cluster nested in $\mathcal{C}$, then $\mathcal{Y}_{\mathcal{C}} \cap \mathcal{C}' = \emptyset$.*

Proof. It is enough to show that $\mathit{periphery}(\mathcal{C}') \cap \mathcal{Y}_{\mathcal{C}} = \emptyset$, as this forces $\mathcal{C}' \cap \mathcal{Y}_{\mathcal{C}} = \emptyset$. We divide the problem into a few cases.

Case (i). Suppose $\{\mathit{type}\ \mathcal{Y}_{\mathcal{C}}, \mathit{type}\ \mathcal{Y}_{\mathcal{C}'}\} = \{C, D'\}$. We investigate the intersection of $\mathit{periphery}(\mathcal{C}')$ with $\mathcal{Y}_{\mathcal{C}}$. It cannot contain dominos of types I^+ and I^-; because the boxing property is constant on cycles according to Proposition 4.11(ii), such dominos would have to be simultaneously C and D-boxed, which is impossible. If $D(k,T) \in \mathit{periphery}(\mathcal{C}') \cap \mathcal{Y}_{\mathcal{C}'}$ is of type (N), $D(k,T)$ and $MT(D(k,T),T)$ are both C and D'-boxed. This contradicts Proposition 4.11(i), forcing $\mathit{periphery}(\mathcal{C}') \cap \mathcal{Y}_{\mathcal{C}} = \emptyset$. The proof is virtually identical when the set $\{\mathit{type}\ \mathcal{Y}_{\mathcal{C}}, \mathit{type}\ \mathcal{Y}_{\mathcal{C}'}\}$ equals $\{B, D\}$ instead.

Case (ii). Suppose $\{\mathit{type}\ \mathcal{Y}_{\mathcal{C}}, \mathit{type}\ \mathcal{Y}_{\mathcal{C}'}\} = \{C, D\}$. The proof is similar to the first case, except this time, dominos of type N cannot be simultaneously C and D-boxed. Again, the proof is identical when the set $\{\mathit{type}\ \mathcal{Y}_{\mathcal{C}}, \mathit{type}\ \mathcal{Y}_{\mathcal{C}'}\}$ equals $\{B, D'\}$ instead.

Case (iii). Suppose $\{\mathit{type}\ \mathcal{Y}_{\mathcal{C}}, \mathit{type}\ \mathcal{Y}_{\mathcal{C}'}\} \subset \{B, C\}$ or $\{D, D'\}$. Then by by the definition of cycles, $\mathcal{Y}_{\mathcal{C}} \cap \mathcal{Y}_{\mathcal{C}'} = \emptyset$. We know $\mathit{periphery}(\mathcal{C}') \subset \mathcal{Y}_{\mathcal{C}'} \subset \overline{\mathcal{C}}'$ by Lemma 4.10, implying again that $\mathit{periphery}(\mathcal{C}') \cap \mathcal{Y}_{\mathcal{C}} = \emptyset$. □

5. The τ-Invariant for Orbital Varieties

A natural question is whether our method of describing orbital varieties by standard tableaux gives the same parametrization as [8]. More precisely, if $\pi : \mathrm{Irr}(\mathcal{F}_u)/A_u \to \mathrm{Irr}(\mathcal{O}_u \cap \mathfrak{n})$ is the bijection of [11], does the same tableau parameterize both $\mathcal{C} \in \mathrm{Irr}(\mathcal{F}_u)/A_u$ and its image $\mathcal{V} = \pi(\mathcal{C})$? Write $\mathcal{T}(\mathcal{C})$ for the domino tableau corresponding to the A_u-orbit $\mathcal{C} \in \mathrm{Irr}(\mathcal{F}_u)/A_u$ via the map of the previous section and $\mathcal{T}(\mathcal{V})$ for the domino tableau used to parameterize $\mathcal{V}$ in [8].

Let $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ be the set of simple roots in $\mathfrak{g}$. Write $\{e_1, \ldots, e_n\}$ for the basis of the dual of the Cartan subalgebra, and choose the indices so that $\alpha_1 = 2e_1$ in type C_n, $\alpha_1 = e_1$ in type B_n, and $\alpha_1 = e_1 + e_2$ in type D_n. The remaining simple roots are then $\alpha_i = e_i - e_{i-1}$ for $2 \leq i \leq n$. The τ-invariant, a subset of Π, is defined for orbital varieties in [7] and for components of the Springer fiber in [12]. It is constant on each A_u-orbit. For a standard domino tableau T, it can be defined in terms of the relative positions of the dominos. We say that a domino D lies *higher* than D' in a tableau T iff the rows containing squares of D have indices strictly smaller than the indices of the rows containing squares of D'. Then $\tau(T)$ consists of precisely the simple roots α_i whose indices satisfy:

(1) $i = 1$ and the domino $D(1,T)$ is vertical and, if G is of type D, $shape(T(2)) \neq [3,1]$,
(2) $i > 1$ and $D(i-1,T)$ lies higher than $D(i,T)$ in T.

The notion of the τ-invariant can be generalized using wall-crossing operators to define equivalence classes of domino tableaux, see for instance [3] and [4]. Defined on tableaux, the generalized τ-invariant is used to classify primitive ideals in groups of type B_n and C_n. In type D_n, a further generalization, the *generalized* generalized τ-invariant is necessary. According to [4], there is in fact a unique tableau of a given shape within each equivalence class of tableaux generated by the generalized τ-invariant. We show

Theorem 5.1. *Suppose that $\mathcal{C} \in Irr(\mathcal{F}_u)/A_u$ and that $\mathcal{V} = \pi(\mathcal{C})$. Then*
$$\tau(\mathcal{T}(\mathcal{C})) = \tau(\mathcal{T}(\mathcal{V})).$$

Proof. In fact, we show that all of following sets are equal.
$$\tau(\mathcal{T}(\mathcal{V})) = \tau(\mathcal{V}) = \tau(\mathcal{C}) = \tau(\mathcal{T}(\mathcal{C})).$$

The first equality follows from [8] and [7]. The second from the definition of π. We verify the third.

Recall the map $\Phi : SDT_{op,cl} \to SDT$ defined in the previous section. We prove that if $\widetilde{T} \in SDT_{op,cl}$ parameterizes the irreducible component $\mathcal{C} \in \mathrm{Irr}\mathcal{F}_u$ in [15], then its τ-invariant $\tau(\mathcal{C})$ is precisely the τ-invariant of the standard domino tableau $\Phi(\widetilde{T}) = \mathcal{T}(\mathcal{C})$ as defined above. The content of the proof is a description of the effect of Φ on the characterization of the τ-invariant of the components of the Springer fiber given in [12]:

Proposition 5.2. [12]*(II.6.29 and II.6.30) Let $X = B$, C, or D. Consider $\mathcal{C} \in$ Irr $\mathcal{F}_{u,|T|}$, that is, an irreducible component whose classifying tableau T in $SDT_{op,cl}$ has underlying domino tableau $|T|$. Then $\alpha_i \in \tau(\mathcal{C})$ iff one of the following is satisfied:*

(i) $i = 1$, $D(1,T)$ is vertical, and $X \neq D$,
(ii) $i > 1$ and $D(i-1,|T|)$ lies higher that $D(i,|T|)$ in $|T|$,

(iii) $i > 1$ *and* $\{D(i-1,T), D(i,T)\} \in CC^+(T)$,

(iv) *If* $X = D$, *then* $\alpha_1 \in \tau(\mathcal{C})$ *iff* $\{1,2\} \in CC^-(T)$ *and* $shape(T(2)) \neq [3,1]$, *while* $\alpha_2 \in \tau(\mathcal{C})$ *iff* $\{1,2\} \in CC^+(T)$.

That $\alpha_1 \in \tau(\mathcal{C})$ iff $\alpha_1 \in \tau(\Phi(\widetilde{T})$ is clear in types B_n and C_n since $D(1,T)$ never lies within a closed cluster and hence remains unaltered by Φ. In type D_n, the conditions for α_i, when $i \leq 2$, to lie in $\tau(\mathcal{C})$ described by Spaltenstein translate exactly to our conditions for α_i to lie in $\tau(\Phi(\widetilde{T})$.

For $i > 1$, suppose that either $D(i,T)$ or $D(i-1,T)$ lies in some $\mathcal{K} \in CC^+(T)$. If $\mathcal{K}$ contains more than two dominos, then [4](III.1.4) implies that $\alpha_i \in \tau(\mathcal{C})$ iff $\alpha_i \in \tau(\Phi(\widetilde{T}))$.

So suppose that $\mathcal{K}$ contains exactly two dominos. If, in fact, $\mathcal{K} = \{D(i), D(i-1)\}$, the simple root α_i must lie in $\tau(\mathcal{C})$. But $D(i-1)$ is higher than $D(i)$ in $MT(\mathcal{C}, T)$, implying by the definition of Φ that $\alpha_i \in \tau(\Phi(\widetilde{T}))$ as well. The remaining possibility is that only one of the dominos $D(i)$ and $D(i-1)$ lies in the two-domino cluster $\mathcal{K}$. Then the fact that $\alpha_i \in \tau(\mathcal{C})$ iff $\alpha_i \in (\widetilde{T})$ follows by inspection. □

References

[1] R. Carter. *Finite Groups of Lie Type*. Wiley Interscience, New York, 1985.

[2] D. Garfinkle. On the classification of primitive ideals for complex classical Lie algebras (i). Compositio Math., 75(2) (1990) 135–169.

[3] D. Garfinkle. On the classification of primitive ideals for complex classical Lie algebras (ii). Compositio Math., 81(3) (1992) 307–336.

[4] D. Garfinkle. On the classification of primitive ideals for complex classical Lie algebras (iii). Compositio Math., 88 (1993) 187–234.

[5] M. Gerstenhaber. Dominance over the classical groups. Ann. of Math., 74(2) (1961) 532–569.

[6] W. Graham and D. Vogan. Geometric quantization for nilpotent coadjoint orbits. In D. Vogan J. Tirao and J. Wolf, editors, *Geometry and representation theory of real and p-adic groups*. Birkhäuser, Boston-Basel-Berlin, 1998.

[7] A. Joseph. On the variety of a highest weight module. Journal of Algebra, 88 (1984) 238–278.

[8] W. M. McGovern. On the Spaltenstein-Steinberg map for classical lie algebras. Comm. Algebra, 27 (1999) 2979–2993.

[9] D. Mihov. Quantization of Nilpotent Coadjoint Orbits. PhD thesis, M.I.T., 1996.

[10] T. Pietraho. Orbital Varieties and Unipotent Representations of Semisimple Lie Groups. PhD thesis, M.I.T., 2001.

[11] N. Spaltenstein. On the fixed point set of a unipotent element of the variety of Borel subgroups. Topology, 16 (1997) 203–204.

[12] N. Spaltenstein. *Classes Unipotentes et Sous-Groupes de Borel.* Number 946 in Lecture Notes in Mathematics. Springer-Verlag, New York, 1982.
[13] T. A. Springer. A construction of representations of Weyl groups. Inv. Math., 44 (1978) 279–293.
[14] P. Trapa. Symplectic and orthogonal Robinson-Schensted algorithms. preprint.
[15] M.A. van Leeuwen. A Robinson-Schensted Algorithm in the Geometry of Flags for Classical Groups. PhD thesis, Utrecht, 1989.