CS 2200: Algorithms

Fall, 2019

Administrative Information

- Course Webpage:
  - http://www.bowdoin.edu/~smajerci/teaching/cs2200/2019spring/index.html
- Textbook (optional): Coren, Leiserson, Rivest, and Stein. *Introduction to Algorithms, 3rd edition,* MIT Press, 2009.
- My Office Hours:
  - Monday, 6:00-8:00 pm, Searles 224
  - Thursday, 11:30 am-1:00 pm, Searles 222
- TAs (Office Hours TBA):
  - Will deBruynKops
  - Jordan Ferraras
  - Kim Hancock

What you can expect from me

- Strategies for designing algorithms
- When to use those strategies
- Tools for analyzing algorithm efficiency
- Techniques for arguing algorithm correctness (a little)
- Specific algorithms
- Improved problem solving skills
- Improved ability to think abstractly

What I will expect from you

- Labs and Homework Problems (25%):
  - Generally after every two classes
  - In-Lab Problems
  - Homework Problems
  - More a learning tool than a testing tool
- 3 Exams (75%):
  - In class
  - Closed book, closed notes
    - except for one 8.5 x 11 sheet of notes (both sides)

Collaboration Levels

- Level 0 (In-Lab and In-Class Problems)
  - No restrictions on collaboration
- Level 1 (Homework Problems)
  - Verbal collaboration without code sharing
  - But many details about what is allowed
- Level 2 (not used in this course)
  - Discussions with TAs only
- Level 3 (Exams)
  - Professor clarifications only

Algorithms is a Difficult Class!

- Much more abstract than Data Structures:
  - emphasis is on *designing* the solution technique, not *implementing* a solution
- What to do:
  - Allow plenty of time to read the materials and do the homework
  - Solve all problems (even the optional ones)
  - Go to the study groups (TA hours)
  - Form a group to work with
  - Spaced study

Learning

- What helps you?
- What hinders you?

Algorithms and Programs

- An algorithm is a computational recipe designed to solve a particular problem
- Must be implemented as a program in a particular programming language
- Data structures are critical...
- ...but you already know that.

Making a telephone call to Jill

```
pick up the phone;
dial Jill’s number;
wait for person to answer;
talk;
```

Correctness

Waiting at a traffic light

```
if (light is red) {
        wait a while;
        accelerate;
}
```

Definiteness

Looking for an integer >= 0 with property P.

```
i = 0;
foundIt = testForP(i);
while (!foundIt) {
        i++;
        foundIt = testForP(i);
}
```

Finite number of steps

Packing for vacation

```
flip coin;
if (heads)
        pack paraglider;
else
        pack scuba gear;
```

Predictability

Desirable Characteristics

- THEORY suggests/requires:
  - Correctness
  - Definiteness
  - Finiteness
  - Predictability
- Practice suggests:
  - Efficiency
  - Clarity
  - Brevity

An algorithm is:

…a list of **precisely** defined steps that can be done by a computer in a **finite** (and, hopefully, relatively **short**) amount of **time** to **correctly** solve a particular type of **problem**.

Types of Problems

- STRUCTURING: transform input to satisfy Y (SORT)
- CONSTRUCTION: build X to satisfy property Y (ST)
- OPTIMIZATION: find best X satisfying property Y (TSP)
- DECISION: does the input satisfy property Y (SAT)
- APPROXIMATION: find X that almost satisfies property P and has bounded error (TSP)
- RANDOMIZED: make random choices (QuickSort)
- PARALLEL ALGORITHMS (ACO)
- ON-LINE ALGORITHMS (Job Scheduling)

Pseudocode

- High-level description of an algorithm
- More structured than English prose
- Less detailed than a program
- Preferred notation for describing algorithms
- Hides program design issues

Example: Find the maximum element of an array

```
arrayMax(A, n)

  currentMax = A[0]
  for i = 1 to n – 1
    if A[i] > currentMax
      currentMax = A[i]
  return currentMax
```

Pseudocode Details

- Control flow
  - **if**…[**else**…]
  - **while**…
  - **repeat**…**until** …
  - **for**…**to** and **for**…**downto**
  - Indentation replaces braces
- Method declaration

  method (***arg*** [, ***arg***…])
- Method call (pass by value)

  method (***arg*** [, ***arg***…])
- Return value

  return ***expression***
- Java expressions
  - Also: i = j = k
  - Booleans “short circuit”
- NOTE:
  - Will use 0-based indexing, BUT
  - CLRS uses 1-based indexing!
- Usual OOP notation
  - x.f is the attribute f of object x

Sorting

- Pervasive problem
  - Data processing
  - Efficient search
  - Operations research (e.g. shortest jobs first)
  - Event-driven simulation (e.g. what happens first?)
  - Sub-routine for other algorithms (e.g. Kruskal's MST)
- Informally
  - Bunch of items
  - Each has a "key" that allows "<=" comparison
  - Put items in ascending (or descending) order according to key comparisons

Sorting

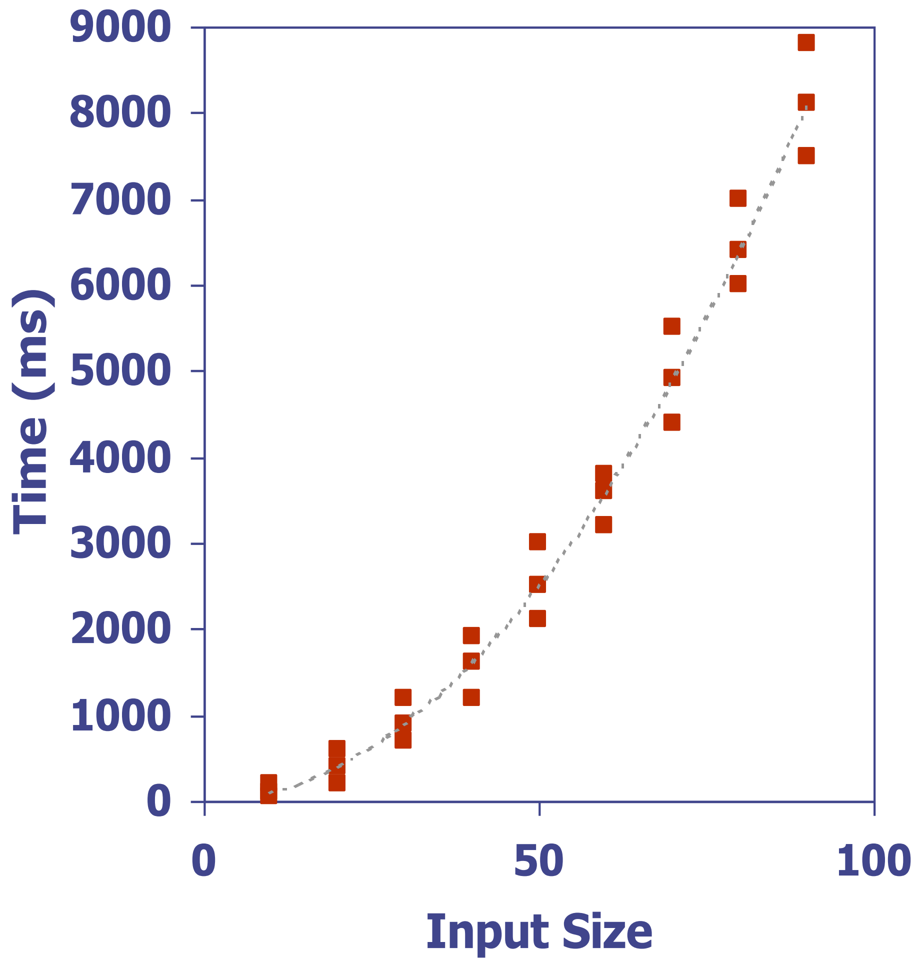
- Bubble Sort
- Selection Sort
- Insertion Sort

What About Efficiency?

- Time
- Space

Experimental Studies

- Write a program implementing the algorithm
- Run the program with inputs of varying size and composition
- Use a method like `System.currentTimeMillis()` to get a measure of the actual running time
- Plot the results
- Okay?



Not Okay

- Implementation can be difficult
- Results depend on:
  - quality of the implementation
  - language used
  - computer used
- Can only run on a limited number of inputs, which may not be representative
- Difficult to test on very large inputs
- In order to compare two algorithms, the same hardware and software environments must be used
- So what would you do?

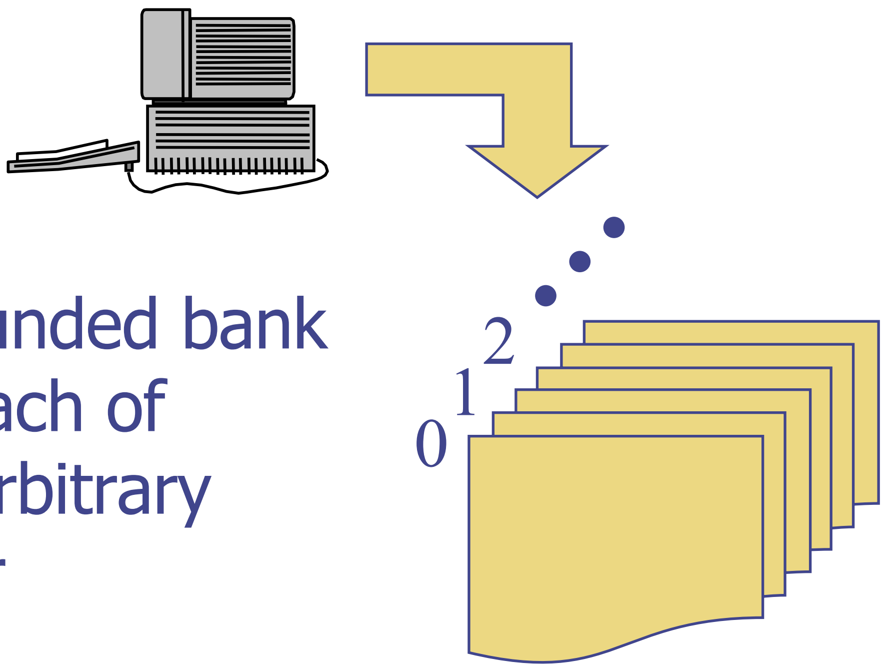
Theoretical Analysis

- Use a pseudocode description of the algorithm instead of an implementation
- Equate running time with the number of instructions executed
- Characterize this measure of running time as a function of the input size, *n*.
- Advantages:
  - Takes into account all possible inputs
  - Can analyze and compare algorithms independently of hardware and software

The Random Access Machine (RAM) Model

- A **CPU**

- An potentially unbounded bank of **memory** cells, each of which can hold an arbitrary number or character



- Memory cells are numbered and accessing any cell in memory takes unit time.

Primitive Operations

- Basic computations performed by an algorithm
- Identifiable in pseudocode
- Largely independent of any programming language
- Exact definition not important
- Each assumed to take a constant amount of time
- Each assumed to take the *same* constant amount of time

- Examples:
  - Evaluating a binary expression,

    e.g. (a + b)
  - Assigning a value to a variable
  - Indexing into an array
  - Calling a method
  - Returning from a method

Really?

- Ignores many things, e.g.
  - Memory hierarchy
  - Processor load
  - “Tricks” like:
    - Pipelining
    - Speculative execution (e.g. branch prediction)
  - Some operations really are a lot more expensive
- But, in practice, it works:
  - It accurately characterizes the rate of growth.
  - It allows us to compare different algorithms.

Counting Primitive Operations

- By inspecting the pseudocode, we can determine the maximum number of primitive operations executed by an algorithm, as a function of the input size

```
Algorithm arraySum(A, n)                          #operations
    $sum = 0$                                     $1$
    for $i = 0$ to $n - 1$                        $3n + 2$
        $sum = sum + A[i]$                        $3n$
    return $sum$                                  $1$
                                    Total         $6n + 4$
```

Growth Rate of Running Time

- Algorithm *arraySum* executes $6n + 4$ primitive operations in the worst case (and the best case).
- Changing the hardware/software environment
  - Affects this by a constant factor, but
  - Does not alter the growth ***rate***
- The fact that the running time grows at the same ***rate*** as the input size is an intrinsic property of algorithm *arraySum*

Focus on the ***Rate*** of Growth: Big-O Notation

- Given functions $f(n)$ and $g(n)$, we say that $f(n)$ is $O(g(n))$ if there is a constant $c > 0$ and an integer constant $n_0 > 0$ such that:

  $$f(n) \leq cg(n) \text{ for } n \geq n_0$$

- To show this, we need to find a c and n_0 that make the inequality true.
- Example 1: $6n + 4$ is $O(n)$
  - $6n + 4 \leq cn$
  - $6 + 4/n \leq c$
  - Pick $c = 7$ and $n_0 = 4$
- Example 2: n^2 is *not* $O(n)$
  - $n^2 \leq cn$
  - $n \leq c$ *IMPOSSIBLE!*

More Big-O Examples

- $7n - 2$ is $O(n)$

  need $c > 0$ and $n_0 > 0$ such that $7n - 2 \leq cn$ for all $n \geq n_0$

  this is true for $c = 7$ and $n_0 = 2$

- $3n^3 + 20n^2 + 5$ is $O(n^3)$

  need $c > 0$ and $n_0 > 0$ such that $3n^3 + 20n^2 + 5 \leq cn^3$ for all $n \geq n_0$

  this is true for $c = 5$ and $n_0 = 20$

Big-O Rules

- If $f(n)$ is a polynomial of degree d, then $f(n)$ is $O(n^d)$. In other words:
  - Drop lower-order terms
  - Drop constant factors
- Use the "smallest" possible class of functions
  - Say "$2n$ is $O(n)$" instead of "$2n$ is $O(n^2)$"

Two Relative Growth Rate Rules

- Any positive polynomial function with degree greater than 0 grows faster than any poly-log function:

  $\lg^a n = O(n^b)$, $a > 0$, $b > 0$

- Any exponential with base greater than 1 grows faster than any polynomial function with degree greater than 0:

  $n^b = O(c^n)$, $b > 0$ and $c > 1$

Relative Growth Rates

- lg lg n
- lg n
- $\lg^2 n$ also written as $(\lg n)^2$
- $\sqrt{n}$
- n
- n lg n
- n^2
- n^3
- 2^n

Array Sum

- Ignoring constant factors makes things easier!

```
arraySum(A, n)                                   #operations
    sum = 0                                          1
    for i = 0 to n − 1                               n
        sum = sum + A[i]                             n
    return sum                                       1
                                     Total         2n + 2
```

- Algorithm *arraySum* runs in $O(n)$ time
- Just counting loop iterations!

Big-Omega

- **big-Omega**
  - $f(n)$ is $\Omega(g(n))$ if there is:
    - a constant $c > 0$, and
    - an integer constant $n_0 > 0$

    such that:

    $f(n) \geq c\, g(n)$ for all $n \geq n_0$

- $7n - 2$ is $\Omega(n)$

  need $c > 0$ and $n_0 > 0$ such that $7n - 2 \geq cn$ for $n \geq n_0$

  this is true for $c = 6$ and $n_0 = 2$

- $3n^3 + 20n^2 + 5$ is $\Omega(n^3)$

  need $c > 0$ and $n_0 > 0$ such that $3n^3 + 20n^2 + 5 \geq cn^3$ for $n \geq n_0$

  this is true for $c = 3$ and $n_0 = 20$

Big-Theta

- **big-Theta (big-O *and* big-Omega)**
  - f(n) is $\Theta(g(n))$ if there are:
    - constants $c' > 0$ and $c'' > 0$, and
    - an integer constant $n_0 > 0$

    such that:

    $c' g(n) \leq f(n) \leq c'' g(n)$ for all $n \geq n_0$
  - Notice that the two constants, c′ and c″, can be different, but n_0 must be the same for both. Just use the max!
- $7n - 2$ is $\Omega(n)$

  Already did it! $c' = 6$, $c'' = 7$, $n_0 = 2$
- $3n^3 + 20n^2 + 5$ is $\Omega(n^3)$

  Already did it! $c' = 3$, $c'' = 5$, $n_0 = 20$

Intuition for Asymptotic Notation

big-O

- f(n) is O(g(n)) if f(n) is asymptotically **less than or equal** to g(n)
- usually used to describe worst case

big-Omega

- f(n) is Ω(g(n)) if f(n) is asymptotically **greater than or equal** to g(n)
- can be used to describe best case

big-Theta

- f(n) is Θ(g(n)) if f(n) is asymptotically **equal** to g(n)
- if best and worst case are the same

Example Uses of the Relatives of Big-O

- **$5n^2$ is $\Omega(n)$**

  f(n) is $\Omega(g(n))$ if there is a constant $c > 0$ and an integer constant $n_0 > 0$ such that $f(n) \geq c\, g(n)$ for $n \geq n_0$

  let $c = 1$ and $n_0 = 1$

- **$5n^2$ is $\Omega(n^2)$**

  f(n) is $\Omega(g(n))$ if there is a constant $c > 0$ and an integer constant $n_0 > 0$ such that $f(n) \geq c\, g(n)$ for $n \geq n_0$

  let $c = 5$ and $n_0 = 1$

- **$5n^2$ is $O(n^2)$**

  f(n) is **O**(g(n)) if, there is a constant $c > 0$ and an integer constant $n_0 > 0$ such that $f(n) \leq c\, g(n)$ for $n \geq n_0$

  let $c = 5$ and $n_0 = 1$

- So **$5n^2$ is $\Theta(n^2)$**

Asymptotic Analysis is Powerful

- An $O(n^{4/3} \log n)$ algorithm to test a conjecture about pyramid numbers ran about 30,000 times faster than an $O(n^2)$ algorithm at $n = 10^9$, finishing in 20 minutes instead of just over a year.

Asymptotic Analysis is Powerful

- In a race between two algorithms to solve the maximum-sum subarray problem:
  - A $\Theta(n^3)$ algorithm was implemented in tuned C code on a 533MHz Alpha 21164 (this was 2000…)
  - A $\Theta(n)$ algorithm was implemented in interpreted Basic on a 2.03 Radio Shack TRS-80 Model II
- The winner?
  - The horribly implemented, but asymptotically faster, algorithm started beating the beautifully implemented algorithm at $n = 5{,}800$.
  - At $n = 10{,}000$, the $\Theta(n^3)$ algorithm took 7 days compared to 32 minutes for the $\Theta(n)$ algorithm.

But wait a minute….

- Is the focus on large problems misguided?
- Do we really not care about large constants?
- Isn't focusing on the worst-case too pessimistic?
- What about NP-complete problems?
- What about heuristics and hacks?
- What about practical issues (e.g. cache effects)?

Let' s face the real world…

- You're a working programmer and you' ve been given a week to implement a data structure that supports client transactions so that it runs efficiently when scaled up to a much larger client base. Where do you start?
- You're an algorithm engineer building a code repository to hold fast implementations of dynamic multigraphs. You read papers describing asymptotic bounds for several approaches. Which ones do you implement?
- You're an operations research consultant hired to solve a highly constrained facility location problem. Should you build the solver from scratch or buy optimization software and tune it for the application?

"Run experiments to gain insight!"

From "A Guide to Experimental Algorithmics," Catherine C. McGeoch, 2012

Experimental Algorithmics

- The theoretical approach guarantees generality but lacks specificity.
- The empirical approach provides specificity, but the results are hard to generalize.
- "Experimental algorithmics represents a third approach that treats algorithms as laboratory subjects, emphasizing control of parameters, isolation of key components, model building, and statistical analysis."
- "[It] combines the tools of the empiricist – code and measurement – with the abstraction based approach of the theoretician. Insights from laboratory experiments can be more precise and realistic than pure theory provides, but also more general than field experiments can produce."

From "A Guide to Experimental Algorithmics," Catherine C. McGeoch, 2012

How Efficient Are Our Sorting Algorithms?

- Bubble Sort
  - worst case?
  - best case?
- Selection Sort
  - worst case?
  - best case?
- Insertion Sort
  - worst case?
  - best case?

Algorithm Design Principle

Sometimes we can devise a new
(possibly better) algorithm
by reallocating our computational efforts.

Reallocate Computational Effort: Example 1: Sorting

- Selection Sort
  - Picking next element to place is harder (always)
  - Placing it is easier
- Insertion Sort
  - Picking next element to place is easier
  - Placing it is harder (but only sometimes!)

Reallocate Computational Effort: Example 2: Searching

- Unsorted list
  - Easy to add items
  - Much harder to find an item
- Sorted list
  - Extra effort to add items (need to keep sorted)
  - Much easier to find an item