SIGN CHANGES OF FOURIER COEFFICIENTS OF HILBERT MODULAR FORMS

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Abstract. Sign changes of Fourier coefficients of various modular forms have been studied. In this paper, we analyze some sign change properties of Fourier coefficients of Hilbert modular forms, under the assumption that all the coefficients are real. The quantitative results on the number of sign changes in short intervals are also discussed.

1. Introduction

The Fourier coefficients of modular forms are interesting objects because of their nice arithmetic and algebraic properties. It is easy to see that the Fourier coefficients of a cusp form for $\Gamma_0(N)$ change signs infinitely often if the coefficients are all real numbers. In fact, the signs of the Fourier coefficients determine a cusp form. The signs of the Fourier coefficients of cusp forms were first studied by M. Ram Murty in [9]. After that there have been more extensive study of the Fourier coefficients of other kinds of automorphic forms. In this article, we first prove a sign change result in the case of Hilbert modular forms. More precisely, we prove the following.

Theorem 1.1. *Let $\mathbf{f}$ be a Hilbert cusp form of weight $k = (k_1, \ldots, k_n)$ and level $\mathfrak{n}$, and let $C(\mathfrak{m})$ be a Fourier coefficient of $\mathbf{f}$ at each integral ideal $\mathfrak{m}$. If $\{C(\mathfrak{m})\}$ are all real, then there are infinitely many sign changes on $\{C(\mathfrak{m})\}$.*

Here, n is the extension degree of the base field. All the setting is precisely described in Section 2.

Next question which naturally arises in the case of cusp forms is to determine a bound for the first sign change to occur in the sequence of Fourier coefficients. Bounds have been obtained by Kohnen and Sengupta [7], Iwaniec, Kohnen and Sengupta [6], and Choie and Kohnen [4]. More generally, Qu [11] has obtained a similar kind of bound for the first sign change of the coefficients for the automorphic L-function attached to an irreducible unitary cuspidal representation for $\mathrm{GL}_m(\mathbb{A}_{\mathbb{Q}})$, under the assumption that all the coefficients are real. Thus it naturally comes to our mind to get a bound of similar kind in the case of Hilbert modular forms. In our next result which is stated below, we get an affirmative answer.

Theorem 1.2. *Let $\mathbf{f}$ be a primitive Hilbert cusp form of weight $k = (k_1, \ldots, k_n)$, level $\mathfrak{n}$ and with the trivial character. Write $\{C(\mathfrak{m})\}$ for Fourier coefficients of $\mathbf{f}$, and let $Q_{\mathbf{f}}$ be the analytic conductor of $\mathbf{f}$. Then there exists an integral ideal $\mathfrak{m}$ with*

$$\mathrm{N}(\mathfrak{m}) \ll_{n,\epsilon} Q_{\mathbf{f}}^{1+\epsilon}$$

such that $C(\mathfrak{m}) < 0$.

Finally, we consider the behavior of the signs of the coefficients in short intervals $(x, 2x)$ for sufficiently large x. Namely, we prove the following quantitative result for the number of sign changes in the interval $(x, 2x)$.

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Theorem 1.3. *Let* $\mathbf{f}$ *be a primitive Hilbert cusp form of weight* $k = (k_1, \ldots, k_n)$*, full level, and with the trivial character. Assume that the weight satisfies the following congruence property:* $k_1 \equiv \cdots \equiv k_n \equiv 0 \mod 2$*. For each integral ideal* $\mathfrak{m}$ *of* F*, let* $C(\mathfrak{m})$ *be a Fourier coefficient of* $\mathbf{f}$ *at* $\mathfrak{m}$*. Then, for any* r *with* $\frac{4n-1}{4n+1} < r < 1$*, at least one sign change for* $\{C(\mathfrak{m})\}$ *occurs with* $\mathrm{N}(\mathfrak{m}) \in (x, x + x^r]$*.*

This follows from a recent work of Meher and Murty [8], together with a result of Chandrasekharan and Narasimhan [3] and Ramanujan conjecture for Hilbert modular forms. It should be also noted that Theorem 1.3 guarantees that, if $\mathbf{f}$ satisfies all the hypotheses in the theorem, its Fourier coefficients $\{C(\mathfrak{m})\}$ cannot completely vanish in the interval $(x, x + x^r]$ for large x. This rather interesting remark is briefly explained in Section 5.

2. Notations and Preliminaries

This section is to recall all the basic definitions and setting on Hilbert modular forms as well as their associated L-functions. We adopt the setting from Shimura [13]. Throughout the paper, we let F be a totally real number field of degree n and h the narrow class number of F, that is the cardinality of the group of all fractional ideals of F modulo all principal ideals of F generated by totally positive elements. We write $\{t_\nu\}_{\nu=1}^h$ for a complete set of the representatives of the narrow class group. For each representative t_ν, a congruence subgroup of $\mathrm{GL}_2(F)$ is taken to be

$$\Gamma_\nu(\mathfrak{n}) := \left\{ \begin{pmatrix} a & t_\nu^{-1} b \\ t_\nu c & d \end{pmatrix} \in \mathrm{GL}_2(F) \; : \; \begin{matrix} a \in \mathcal{O}_F, & b \in \mathfrak{D}_F^{-1}, & \\ c \in \mathfrak{n}\mathfrak{D}_F, & d \in \mathcal{O}_F, & ad - bc \in \mathcal{O}_F^\times \end{matrix} \right\},$$

where $\mathfrak{D}_F$ is the different ideal of F.

A Hilbert modular form f_ν of weight $k := (k_1, \ldots, k_n)$ with respect to $\Gamma_\nu(\mathfrak{n})$ has a Fourier expansion, and we write it as

$$f_\nu(z) = \sum_{\substack{0 \ll \xi \in t_\nu \mathcal{O}_F, \\ \xi = 0}} a_\nu(\xi) \exp\left(2\pi i \sum_{j=1}^n \xi_j z_j\right).$$

Furthermore, we write $\mathbf{f}$ for a collection $(f_1, \ldots, f_h)$ of Hilbert modular forms f_ν $(\nu = 1, \ldots, h)$ of weight k with respect to $\Gamma_\nu(\mathfrak{n})$, respectively, and associate it with a function on $\mathrm{GL}_2(\mathbb{A}_F)$ in a usual way.

Put $k_0 := \max\{k_1, \ldots, k_n\}$. To associate the Fourier coefficients $\{a_\nu(\xi)\}$ for each f_ν with the lifted function $\mathbf{f}$, we define

$$C(\mathfrak{m}) = C(\mathfrak{m}, \mathbf{f}) = a_\nu(\xi)\xi^{-k/2}\mathrm{N}(\mathfrak{m})^{k_0/2}$$

for each integral ideal $\mathfrak{m}$ of F, where $\mathfrak{m} = \xi t_\nu^{-1}\mathcal{O}_F$ for a unique ν and some totally positive element ξ in F. This definition is well-defined because the right hand side of the above expression does not depend on the choice of ξ up to the totally positive elements in $\mathcal{O}_F^\times$.

Our goal is to analyze the sign change of such $\{C(\mathfrak{m})\}$, with an assumption of $C(\mathfrak{m})$ being real for all $\mathfrak{m}$, or equivalently the sign change of $\{a_\nu(\xi)\}$ where ν and ξ vary. We note that $C(\mathfrak{m}, \mathbf{f})$ is known to be real for all $\mathfrak{m}$ if $\mathbf{f}$ is a normalized common Hecke eigenfunction and with trivial character. This follows from Shimura [13, Proposition 2.5, and pp650].

Before concluding the section, we also recall the definition and some properties of L-function attached to a Hilbert modular form $\mathbf{f}$. The L-function $L(s, \mathbf{f})$ is defined to be

$$L(s, \mathbf{f}) := \sum_{\mathfrak{m} \subseteq \mathcal{O}_F} \frac{C(\mathfrak{m})}{\mathrm{N}(\mathfrak{m})^s}.$$

It is known that $L(s,\mathbf{f})$ converges on some half plane. Furthermore, if $\mathbf{f}$ is a cuspform, it can be analytically continued to the whole complex plane $\mathbb{C}$. Let us now define

$$\Lambda(s,\mathbf{f}) = \mathrm{N}(\mathfrak{n}\mathfrak{D}_F^2)(2\pi)^{-ns}\prod_{j=1}^{n}\Gamma\left(s-\frac{k_0-k_j}{2}\right)L(s,\mathbf{f}). \tag{2.1}$$

If $\mathbf{f}$ is a primitive form (with the trivial character), it satisfies a functional equation:

$$\Lambda(s,\mathbf{f}) = i^{\sum_j k_j}\Lambda(k_0-s,\mathbf{f}). \tag{2.2}$$

A word "primitive" is used in a usual way, that is, $\mathbf{f}$ is a new form, normalized as $C(\mathcal{O}_F,\mathbf{f})=1$, and a common eigenfunction of Hecke operators.

3. Proof for Theorem 1.1

We start this section by recalling Landau's theorem, which is a key tool to prove the first theorem.

Theorem 3.1 (Landau). *Let $\phi(s)=\sum_{n=1}^{\infty}\frac{a_n}{n^s}$ be a series that converges on some right half plane and that $a_n \geq 0$ for all but finitely many n. Then, $\phi(s)$ is either convergent everywhere or has a singularity at the abscissa of convergence of $\phi(s)$.*

For a complete proof of Landau's theorem, the reader can refer to, for instance, Murty [10, pp 266 – 267].

To prove our first theorem, we suppose that there are only finitely many sign changes. Without loss of generality, we may assume that there are only finitely many ideals $\mathfrak{m}$ such that $C(\mathfrak{m})<0$. With this assumption, Landau's theorem guarantees that $L(s,\mathbf{f})$ converges absolutely at all s since it is known that $L(s,\mathbf{f})$ can be analytically continued to the whole plane and cannot have a singularity when $\mathbf{f}$ is a cuspform. Furthermore, we claim that $L(s_{j,l},\mathbf{f})$ vanishes at $s_{j,l}=\frac{k_0-k_j}{2}-l$ for any nonnegative integer l. This follows immediately from observing a completed L-function $\Lambda(s,\mathbf{f})$ defined in (2.1), which is known to be entire when $\mathbf{f}$ is a cuspform. Since $\Gamma(s)$ has poles at negative integers, at least one of the gamma factors on the right hand side of (2.1) must have poles at $s_{j,l}$ for some (j,l). Henceforth, $L(s,\mathbf{f})=0$ at these points.

Let $\mathfrak{m}_1,\ldots,\mathfrak{m}_t$ be the complete set of ideals such that $C(\mathfrak{m}_i)<0$ with $\mathrm{N}(\mathfrak{m}_1)\leq\cdots\leq\mathrm{N}(\mathfrak{m}_t)$. It is obvious that we must have at least one such $\mathfrak{m}_1$ for the L-series $L(s,\mathbf{f})$ to converge everywhere. Indeed, if $C(\mathfrak{m})>0$ for all $\mathfrak{m}$, then $L(s_{j,l},\mathbf{f})>0$. We now rewrite the series as

$$\sum_{\mathfrak{m}\neq\mathfrak{m}_i}\frac{C(\mathfrak{m})}{\mathrm{N}(\mathfrak{m})^{s_{j,l}}} = -\frac{C(\mathfrak{m}_1)}{\mathrm{N}(\mathfrak{m}_1)^{s_{j,l}}}-\cdots-\frac{C(\mathfrak{m}_t)}{\mathrm{N}(\mathfrak{m}_t)^{s_{j,l}}}. \tag{3.2}$$

Multiplying both sides of (3.2) by $\mathrm{N}(\mathfrak{m}_t)^{s_{j,l}}$ and letting l be arbitrarily large, we observe that the right hand side of the equation (3.2) approaches $C(\mathfrak{m}_t)$, while the left hand side tends to the infinity unless $C(\mathfrak{m})=0$ for all $\mathfrak{m}$ whose norm is larger than $\mathrm{N}(\mathfrak{m}_t)$. But if it is so, then $\mathbf{f}$ has only finitely many nonzero Fourier coefficients which cannot happen. This completes the proof for Theorem 1.1.

4. Proof for Theorem 1.2

In [11], Qu proved a similar statement for an irreducible unitary cuspidal representation for $\mathrm{GL}_m(\mathbb{A}_{\mathbb{Q}})$. (See Theorem 1.3.) Our goal is essentially to expand her result to $\mathrm{GL}_2(\mathbb{A}_F)$, i.e., the base field to be any totally real number field F while m is fixed to be 2.

To prove our case, let x be a real number such that $C(\mathfrak{m})\geq 0$ for all ideals $\mathfrak{m}$ with $\mathrm{N}(\mathfrak{m})\leq x$, and set

$$S(x) := \sum_{\mathrm{N}(\mathfrak{m})\leq x}\tilde{C}(\mathfrak{m})\left(\log\frac{x}{\mathrm{N}(\mathfrak{m})}\right)^{2n}$$

where

$$\tilde{C}(\mathfrak{m}) := \frac{C(\mathfrak{m})}{\mathrm{N}(\mathfrak{m})^{(k_0-1)/2}}. \tag{4.1}$$

Clearly, our normalization $\tilde{C}(\mathfrak{m})$ does not affect the results on sign changes. We prove that $x \ll Q_{\mathbf{f}}^{1+\epsilon}$ by finding a upper bound and a lower bound of $S(x)$.

4.1. **Upper Bound for $S(x)$.** Applying the Perron's formula, $S(x)$ can be written as

$$S(x) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} L\left(s + \frac{k_0-1}{2}, \mathbf{f}\right)\frac{x^s}{s^{2n+1}}ds.$$

Since $L(s,\mathbf{f})$ can be analytically continued to the whole complex plane, the integrand in the above integral is analytic for any $\sigma = \Re(s) > 0$. Thus, the line of integration can be moved to $\sigma = \epsilon$, and therefore we obtain that

$$S(x) = \frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty} L\left(s + \frac{k_0-1}{2}, \mathbf{f}\right)\frac{x^s}{s^{2n+1}}ds.$$

We now recall a result of Harcos [5]. (See also Qu [11].)

Lemma 4.2. *Let $\epsilon > 0$ be arbitrary and $(k_0-1)/2 < \sigma < (k_0+1)/2$, where $s = \sigma + it$. Then we have that*

$$L(\sigma + it, \mathbf{f}) \ll_\epsilon Q_{\mathbf{f}}(t)^{\frac{1-\sigma}{2}+\epsilon},$$

where $Q_{\mathbf{f}}(t)$ is the analytic conductor of $\mathbf{f}$ *at t.*

Applying the above lemma, we see that

$$S(x) \ll_\epsilon \int_{-\infty}^{\infty} Q_{\mathbf{f}}(t)^{1/2+\epsilon}\frac{x^\epsilon}{(|t|+\epsilon)^{2n+1}}dt.$$

Furthermore, since $Q_{\mathbf{f}}(t) \ll_n (1+|t|)^{2n+1}Q_{\mathbf{f}}$, where $Q_{\mathbf{f}}$ is the conductor of $\mathbf{f}$, we obtain that

$$S(x) \ll_{n,\epsilon} Q_{\mathbf{f}}^{1/2+\epsilon}x^\epsilon \int_{-\infty}^{\infty}\frac{(|t|+1)^{n+1/2+\epsilon}}{(|t|+\epsilon)^{2n+1}}dt \ll_{n,\epsilon} Q_{\mathbf{f}}^{1/2+\epsilon}x^\epsilon. \tag{4.3}$$

4.2. **Lower Bound for $S(x)$.** To find a lower bound, it is easier to use the correspondence between a Hilbert modular form and an automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$. Indeed, any primitive Hilbert cusp form $\mathbf{f}$ can be assigned to an irreducible cuspidal automorphic representation $\Pi = \Pi_{\mathbf{f}}$ of $\mathrm{GL}_2(\mathbb{A}_F)$. The existence and uniqueness of such a representation Π is well-known. The details of such a correspondence can be found in Section 4 of [12]. Write $\Pi = \otimes_{\mathfrak{p}}\Pi_{\mathfrak{p}}$ where $\Pi_{\mathfrak{p}}$ is a local representation at any place $\mathfrak{p}$. It is known that, for each prime ideal $\mathfrak{p}$ that does not divide the level $\mathfrak{n}$ or the different ideal $\mathfrak{D}_F$ of the base field F, the local representation $\Pi_{\mathfrak{p}}$ is a spherical representation induced from some unramified characters $\chi_{1,\mathfrak{p}}$ and $\chi_{2,\mathfrak{p}}$, and we denote it as $\Pi_{\mathfrak{p}} = \pi(\chi_{1,\mathfrak{p}}, \chi_{2,\mathfrak{p}})$. Furthermore, the following is true for such places $\mathfrak{p}$.

Lemma 4.4. *Let Π be an automorphic representation given as above. Then, for any unramified place $\mathfrak{p}$, i.e., $\mathfrak{p} \nmid \mathfrak{n}$ and $\mathfrak{p} \nmid \mathfrak{D}_F$, we have*

$$\chi_{1,\mathfrak{p}}(\varpi_{\mathfrak{p}}) + \chi_{2,\mathfrak{p}}(\varpi_{\mathfrak{p}}) = \tilde{C}(\mathfrak{p})$$

where $\varpi_{\mathfrak{p}}$ is a uniformizer of $F_{\mathfrak{p}}$ and $\tilde{C}(\mathfrak{p})$ is as in (4.1).

A proof of the above lemma is omitted here as a detailed proof can be found, for example, in Raghuram-Tanabe [12, pp 305 – 306]. This relation between $(\chi_{1,\mathfrak{p}}, \chi_{2,\mathfrak{p}})$ and $\tilde{C}(\mathfrak{p})$ gives us the following proposition.

Proposition 4.5. *Let $\mathbf{f} \in \mathcal{S}_k(\mathfrak{n}, \mathbb{1})$ be a primitive form and $\tilde{C}(\mathfrak{m}) = C(\mathfrak{m}, \mathbf{f})\mathrm{N}(\mathfrak{m})^{-(k_0-1)/2}$ as in (4.1). Then, for any prime ideal $\mathfrak{p}$ not dividing either the level $\mathfrak{n}$ or the different $\mathfrak{D}_F$, we have*

$$|\tilde{C}(\mathfrak{p}^2)| + |\tilde{C}(\mathfrak{p})| \geq \frac{1}{2}.$$

To prove the proposition, we will need three more lemmas:

Lemma 4.6. *Let $\mathbf{f} \in \mathcal{S}_k(\mathfrak{n}, \mathbb{1})$ be a Hecke eigenform. Then $\{C(\mathfrak{m})\}$ is multiplicative, and furthermore the following equality is satisfied for any unramified prime $\mathfrak{p}$ and any integer m greater than 1:*

$$C(\mathfrak{p}^m) = C(\mathfrak{p})C(\mathfrak{p}^{m-1}) - q_{\mathfrak{p}}^{k_0-1}C(\mathfrak{p}^{m-2}),$$

with $q_{\mathfrak{p}}$ being the cardinality of the residue field $\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}\mathcal{O}_{\mathfrak{p}}$.

This follows from Shimura [13, (2.23)] and in particular by taking $\mathfrak{a} = \mathfrak{p}^{m-1}$ and $\mathfrak{b} = \mathfrak{p}$.

Lemma 4.7. (Qu, [11, Lemma 5.2]). *For a set of complex numbers $\{\beta_l\}_{l=1}^m$, define the coefficients α_i as*

$$\sum_{i=0}^{\infty} \alpha_i X^i = \prod_{l=1}^{m} (1 - \beta_l X)^{-1},$$

and also put

$$b_j = \beta_1^j + \cdots \beta_m^j$$

for any $j \geq 1$. Then, for any $t \geq 1$, we have

$$t\alpha_t = \sum_{j=1}^{t} b_j \alpha_{t-j}.$$

Lemma 4.8. (Brumley, [2, Lemma 1]). *For a set of complex numbers $\{\beta_l\}_{l=1}^m$, define the coefficients α_i as*

$$\sum_{i=0}^{\infty} \alpha_i X^i = \prod_{i,j=1}^{m} \left(1 - \beta_i \overline{\beta_j} X\right)^{-1}.$$

If $|\prod_{l=1}^m \beta_l| = 1$, then $\alpha_m \geq 1$.

Proof of Proposition 4.5. If $\mathfrak{p}$ is a unramified place, it can be easily verified by Lemma 4.4 and Lemma 4.6 that

$$\left(1 - \chi_{1,\mathfrak{p}}(\varpi_{\mathfrak{p}})q_{\mathfrak{p}}^{-s}\right)^{-1} \left(1 - \chi_{2,\mathfrak{p}}(\varpi_{\mathfrak{p}})q_{\mathfrak{p}}^{-s}\right)^{-1} = \sum_{m=0}^{\infty} \frac{\tilde{C}(\mathfrak{p}^m)}{\mathrm{N}(\mathfrak{p})^{ms}}.$$

It should be noted that $\chi_{1,\mathfrak{p}}\chi_{2,\mathfrak{p}}$ gives the central character for $\Pi_{\mathfrak{p}} = \Pi(\mathbf{f})_{\mathfrak{p}}$ which coincides with the $\mathfrak{p}$-component of the character for $\mathbf{f}$. Since we only consider a Hilbert modular form with trivial character, we have that $\chi_{1,\mathfrak{p}}\chi_{2,\mathfrak{p}} \equiv \mathbb{1}$ as well. Applying Lemma 4.7, we see that $2\tilde{C}(\mathfrak{p}^2) = B_1\tilde{C}(\mathfrak{p}) + B_2$, with $B_j = \chi_{1,\mathfrak{p}}(\varpi_{\mathfrak{p}})^j + \chi_{2,\mathfrak{p}}(\varpi_{\mathfrak{p}})^j$. In particular, $B_1 = \tilde{C}(\mathfrak{p})$ by Lemma 4.4, and so we have that

$$2\tilde{C}(\mathfrak{p}^2) = \tilde{C}(\mathfrak{p})^2 + B_2. \tag{4.9}$$

Now, define the coefficients α_i as

$$\sum_{m=0}^{\infty} \alpha_m \left(q_{\mathfrak{p}}^{-s}\right)^m = \prod_{i,j=1}^{2} \left(1 - \chi_{i,\mathfrak{p}}(\varpi_{\mathfrak{p}})\overline{\chi_{j,\mathfrak{p}}(\varpi_{\mathfrak{p}})}q_{\mathfrak{p}}^{-s}\right)^{-1},$$

and put

$$\begin{aligned} A_j &:= \chi_{1,\mathfrak{p}}\overline{\chi_{1,\mathfrak{p}}}(\varpi_{\mathfrak{p}})^j + \chi_{2,\mathfrak{p}}\overline{\chi_{1,\mathfrak{p}}}(\varpi_{\mathfrak{p}})^j + \chi_{1,\mathfrak{p}}\overline{\chi_{2,\mathfrak{p}}}(\varpi_{\mathfrak{p}})^j + \chi_{2,\mathfrak{p}}\overline{\chi_{2,\mathfrak{p}}}(\varpi_{\mathfrak{p}})^j \\ &= (\chi_{1,\mathfrak{p}}^j + \chi_{2,\mathfrak{p}}^j)(\overline{\chi_{1,\mathfrak{p}}^j + \chi_{2,\mathfrak{p}}^j})(\varpi_{\mathfrak{p}}) \\ &= \left|(\chi_{1,\mathfrak{p}}^j + \chi_{2,\mathfrak{p}}^j)(\varpi_{\mathfrak{p}})\right|^2. \end{aligned}$$

Then, Lemma 4.7 gives that $2\alpha_2 = A_1\alpha_1 + A_2$, or more precisely

$$2\alpha_2 = \left|(\chi_{1,\mathfrak{p}} + \chi_{2,\mathfrak{p}})(\varpi_{\mathfrak{p}})\right|^4 + \left|(\chi_{1,\mathfrak{p}}^2 + \chi_{2,\mathfrak{p}}^2)(\varpi_{\mathfrak{p}})\right|^2. \tag{4.10}$$

This is because $\alpha_1 = A_1 = |(\chi_{1,\mathfrak{p}} + \chi_{2,\mathfrak{p}})(\varpi_{\mathfrak{p}})|^2$ by Lemma 4.7.

We now claim that either

$$|(\chi_{1,\mathfrak{p}} + \chi_{2,\mathfrak{p}})(\varpi_{\mathfrak{p}})| \geq 1 \quad \text{or} \quad \left|(\chi_{1,\mathfrak{p}}^2 + \chi_{2,\mathfrak{p}}^2)(\varpi_{\mathfrak{p}})\right| \geq 1. \tag{4.11}$$

Suppose, on the contrary, that both values are less than 1. Then, (4.10) gives us that $2\alpha_2 < 1 + 1 = 2$, or $\alpha_2 < 1$, which cannot be true. Indeed, Lemma 4.8 is applicable here because $\chi_{1,\mathfrak{p}}\chi_{2,\mathfrak{p}} \equiv \mathbb{1}$ as mentioned earlier, and thus we must have $\alpha_2 \geq 1$. This completes the proof of the claim (4.11).

If the first inequality in (4.11) holds, then the assertion of Proposition 4.5 follows immediately from Lemma 4.4. If $|\tilde{C}(\mathfrak{p})| < 1$, i.e., the first inequality in (4.11) fails, then we must have $|B_2| = |(\chi_{1,\mathfrak{p}}^2 + \chi_{2,\mathfrak{p}}^2)(\varpi_{\mathfrak{p}})| \geq 1$ by (4.11). Together with (4.9), we see that

$$\begin{aligned} 2|\tilde{C}(\mathfrak{p}^2)| &\geq |B_2| - |\tilde{C}(\mathfrak{p})^2| \\ &\geq 1 - |\tilde{C}(\mathfrak{p})|. \end{aligned}$$

It follows that

$$2\left(|\tilde{C}(\mathfrak{p}^2)| + |\tilde{C}(\mathfrak{p})|\right) \geq 1$$

which complete the proof of Proposition 4.5. □

We now obtain a lower bound for $S(x)$ as follows by applying Proposition 4.5:

$$\begin{aligned} S(x) &\geq \sum_{\mathrm{N}(\mathfrak{m}) \leq x/2} \tilde{C}(\mathfrak{m}) \left(\log \frac{x}{\mathrm{N}(\mathfrak{m})}\right)^{2n} \\ &\geq (\log 2)^{2n} \sum_{\substack{\mathrm{N}(\mathfrak{p}) \leq (x/2)^{1/2} \\ \mathfrak{p} \nmid \mathfrak{n}, \mathfrak{D}_F}} \left(\tilde{C}(\mathfrak{p}^2) + \tilde{C}(\mathfrak{p})\right) \\ &\geq \frac{(\log 2)^{2n}}{2} \sum_{\substack{\mathrm{N}(\mathfrak{p}) \leq (x/2)^{1/2} \\ \mathfrak{p} \nmid \mathfrak{n}, \mathfrak{D}_F}} 1 \gg \frac{x^{1/2}}{\log x}. \end{aligned} \tag{4.12}$$

4.3. **Completing the proof for Theorem 1.2.** We now complete the proof of Theorem 1.2 by comparing the upper and lower bounds we found in Section 4.1 and 4.2. It is now clear that it gives a contradiction if $x \gg Q_{\mathbf{f}}^{1+\epsilon}$. More precisely, equations (4.3) and (4.12) give

$$\frac{x^{1/2}}{\log x} \ll S(x) \ll_{n,\epsilon} Q_{\mathbf{f}}^{1/2+\epsilon} x^{\epsilon}.$$

5. Proof for Theorem 1.3

The theorem follows from a result of the first author and Murty in [8]:

Theorem 5.1. (Meher and Murty, [8, Theorem 1.1]). *Let $\{a_n\}$ be a real sequence such that, for some real numbers α, β, γ, and c, it satifies that:*

(1) $a_n = O(n^{\alpha})$ *for all* n,

(2) $\displaystyle\sum_{n \leq x} a_n = O\left(x^{\beta}\right)$,

(3) $\displaystyle\sum_{n \leq x} a_n^2 = cx + O(x^{\gamma})$.

If $\alpha + \beta$ and γ are both less than one, then for any r with $\max\{\alpha+\beta, \gamma\} < r < 1$, *there is at least one sign change for $\{a_n\}$ with $n \in (x, x + x^r]$.*

To apply the above theorem to our case, we will prove the proposition below.

Proposition 5.2. *Let* $\mathbf{f}$ *be a Hilbert cusp form satisfying all the hypothesis given in Theorem 1.3. Let*

$$\tilde{C}(\mathfrak{m}) := \frac{C(\mathfrak{m})}{\mathrm{N}(\mathfrak{m})^{(k_0-1)/2}}.$$

Then, for all $\mathfrak{m}$ *and for any* $\epsilon > 0$*, the following conditions are satisfied.*

(1) $\sum_{\mathrm{N}(\mathfrak{m})=n} \tilde{C}(\mathfrak{m}) = O\left(n^{\epsilon}\right)$ *for all* n,

(2) $\sum_{\mathrm{N}(\mathfrak{m})\le x} \tilde{C}(\mathfrak{m}) = O\left(x^{\frac{2n-1}{2n+1}+\epsilon}\right)$,

(3) $\sum_{\mathrm{N}(\mathfrak{m})\le x} \tilde{C}(\mathfrak{m})^2 = cx + O\left(x^{\frac{4n-1}{4n+1}+\epsilon}\right)$ *with some* c.

The proof of Theorem 1.3 is completed by applying Proposition 5.2 to Theorem 5.1. We remarked a non-vanishing property of the Fourier coefficients at the end of Section 1. Indeed, if all the coefficients $C(\mathfrak{m})$ are zero where the norms $\mathrm{N}(\mathfrak{m})$ of $\mathfrak{m}$ are in the interval $(x, x + x^r]$, then the third condition in the above proposition must fail. It can be observed in the proof of Proposition 5.1. See [8] for details.

The rest of this section is devoted to proving the proposition.

Proof of Proposition 5.2. The first statement is nothing but Ramanujan conjecture for Hilbert modular forms, which is known to be satisfied. See Blasius [1]. To prove the second and third statements, we now recall a theorem due to Chandrasekharan and Narasimhan:

Theorem 5.3. (Chandrasekharan and Narasimhan, [3, Theorem 4.1]) *Let*

$$\phi(s) = \sum_{n\ge 1} \frac{a(n)}{n^s} \qquad \text{and} \qquad \psi(s) = \sum_{n\ge 1} \frac{b(n)}{n^s}$$

be two Dirichlet series. Suppose that the functional equation

$$\Delta(s)\phi(s) = \Delta(\delta - s)\psi(\delta - s)$$

is satisfied with some $\delta > 0$ *where*

$$\Delta(s) = \prod_{i=1}^{l} \Gamma(\alpha_i s + \beta_i).$$

Furthermore, suppose that the only singularities of ϕ *are poles. Put* $\alpha := \sum_{i=1}^{l} \alpha_i$, $A(x) := \sum_{n\le x} a(n)$, *and*

$$Q(x) = \frac{1}{2\pi i} \int_C \frac{\phi(s)}{s} x^s ds,$$

where C *encloses all the singularities of the integrand. Then we have*

$$A(x) - Q(x) = O\left(x^{\frac{\delta}{2}-\frac{1}{4\alpha}+2\alpha\eta u}\right) + O\left(x^{q-\frac{1}{2\alpha}-\eta}\left(\log x\right)^{r-1}\right) + O\left(\sum_{x<n\le x'} |a(n)|\right),$$

for any $\eta \ge 0$*, where* $x' = x + O\left(x^{1-1/2\alpha-\eta}\right)$*,* q *is the maximum of the real parts of the singularities for* ϕ*,* r *the maximum order of a pole with real part* q*, and* $u = \gamma - \delta/2 - 1/4\alpha$ *with* γ *being the smallest real number such that* $\sum_{n=1}^{\infty} |b(n)|n^{-\gamma}$ *is finite. If in addition* $a(n) \ge 0$ *for all* n*, then we have*

$$A(x) - Q(x) = O\left(x^{\frac{\delta}{2}-\frac{1}{4\alpha}+2\alpha\eta u}\right) + O\left(x^{q-\frac{1}{2\alpha}-\eta}\left(\log x\right)^{r-1}\right).$$

We set $\phi(s) = \psi(s) = L(s, \mathbf{f})$ in Theorem 5.3 in order to estimate $\sum_{\mathrm{N}(\mathfrak{m})\le x} \tilde{C}(\mathfrak{m})$. It follows from the functional equation given in (2.2) that we have $\delta = k_0$ with $k_0 = \max_j\{k_j\}$, $\alpha = n$, and

$$A(x) = \sum_{\mathrm{N}(\mathfrak{m})\le x} C(\mathfrak{m}).$$

Ramanujan conjecture shows that $\sum_{\mathfrak{m}} \tilde{C}(\mathfrak{m})\mathrm{N}(\mathfrak{m})^{-s}$ converges absolutely for $\Re(s) \ge 1+\epsilon$ for any positive ϵ, and thus $\sum_{\mathfrak{m}} C(\mathfrak{m})\mathrm{N}(\mathfrak{m})^{-s}$ converges absolutely where the real part of s is at least $1+\epsilon+(k_0-1)/2 = (k_0+1)/2+\epsilon$. This value is taken to be γ in Theorem 5.3. Furthermore, since there is no singularity for $L(s,\mathbf{f})$, we have $Q(x) = 0$, and it follows that q and r are 0 as well. Thus, we now see that

$$\sum_{\mathrm{N}(\mathfrak{m})\le x} C(\mathfrak{m}) = O\left(x^{\frac{k_0}{2}-\frac{1}{4n}+2n\eta(\frac{1}{2}-\frac{1}{4n}+\epsilon)}\right) + O\left(x^{-\frac{1}{2n}-\eta}(\log x)^{-1}\right) + O\left(\sum_{x<m\le x'}\left|\sum_{\mathrm{N}(\mathfrak{m})=m} C(\mathfrak{m})\right|\right),$$

where $x' = x + O\left(x^{1-\frac{1}{2n}-\eta}\right)$. We see that in this case the middle term of the estimate does not contribute anything since the exponent is negative. Hence

$$\sum_{\mathrm{N}(\mathfrak{m})\le x} C(\mathfrak{m}) = O\left(x^{\frac{k_0}{2}-\frac{1}{4n}+\eta(n-\frac{1}{2}+2n\epsilon)}\right) + O\left(\sum_{x<m\le x'}\left|\sum_{\mathrm{N}(\mathfrak{m})=m} C(\mathfrak{m})\right|\right). \tag{5.4}$$

We also observe that the second term on the right hand side of equation (5.4) satisfies:

$$\sum_{x<m\le x'}\left|\sum_{\mathrm{N}(\mathfrak{m})=m} C(\mathfrak{m})\right| \ll \sum_{x<m\le x'} m^{\frac{k_0-1}{2}+\epsilon} \ll x^{1-\frac{1}{2n}-\eta+\frac{k_0-1}{2}+\epsilon}$$

by Ramanujan conjecture. Equating the exponents to optimize the value of η, i.e., setting

$$1-\frac{1}{2n}-\eta+\frac{k_0-1}{2}+\epsilon = \frac{k_0}{2}-\frac{1}{4n}+\eta\left(n-\frac{1}{2}+2n\epsilon\right),$$

we obtain that

$$\eta = \frac{2n-1+4n\epsilon}{2n(2n+1+4n\epsilon)}.$$

Using this η-value, the exponent in (5.4) is approximately equal to $(2n-1)/(2n+1)+(k_0-1)/2+\epsilon$. Therefore,

$$A(x) = O\left(x^{\frac{2n-1}{2n+1}+\frac{k_0-1}{2}+\epsilon}\right).$$

We now estimate $\sum_{\mathrm{N}(\mathfrak{m})\le x} \tilde{C}(\mathfrak{m})$. It follows that, by partial summation,

$$\sum_{\mathrm{N}(\mathfrak{m})\le x} \tilde{C}(\mathfrak{m}) = x^{-\frac{k_0-1}{2}}A(x) - \left(-\frac{k_0-1}{2}\right)\int_1^x A(t)t^{-\frac{k_0-1}{2}-1}dt,$$

and so it can be evaluated as

$$\sum_{\mathrm{N}(\mathfrak{m})\le x} \tilde{C}(\mathfrak{m}) = O\left(x^{\frac{2n-1}{2n+1}+\epsilon}\right).$$

Let us next direct our attention to $\tilde{C}(\mathfrak{m})^2$. To further discuss about properties of $\tilde{C}(\mathfrak{m})^2$, we define

$$L(s, \tilde{\mathbf{f}}\times\tilde{\mathbf{f}}) := \sum_{\mathfrak{m}} \frac{\tilde{C}(\mathfrak{m})^2}{\mathrm{N}(\mathfrak{m})^s}\cdot\zeta_F(2s),$$

and write the Dedekind zeta function ζ_F as follows:

$$\zeta_F(s) = \sum_{m \geq 1} \frac{a(m)}{m^s}$$

with $a(m)$ being the number of ideals whose norm is m. Then it is easy to see that $L(s, \tilde{\mathbf{f}} \times \tilde{\mathbf{f}})$ can be written as a series $\displaystyle\sum_{m \geq 1} \frac{b(m)}{m^s}$ where

$$b(m) = \sum_{d^2 | m} \left(a(d) \sum_{\mathrm{N}(\mathfrak{m}) = m/d^2} \tilde{C}(\mathfrak{m})^2 \right). \tag{5.5}$$

Now, we would like to apply Theorem 5.3 to this series. To further proceed, let us set

$$\Lambda(s, \tilde{\mathbf{f}} \times \tilde{\mathbf{f}}) := L(s, \tilde{\mathbf{f}} \times \tilde{\mathbf{f}}) \prod_{j=1}^{n} \Gamma(s)\Gamma(s + k_j - 1).$$

Then, it is proven by Shimura, [13, Proposition 4.13], that $\Lambda(s, \tilde{\mathbf{f}} \times \tilde{\mathbf{f}})$ can be meromorphically continued to the complex plane with simple poles at $s = 0$ and 1, and it satisfies a functional equation of expected kind. Henceforth, we may put, in Theorem 5.3, as

$$\delta = 1, \quad \alpha = 2n, \quad q = 1, \quad r = 1, \quad \textit{and} \quad \gamma = 1 + \epsilon,$$

which gives us that

$$B(x) := \sum_{m \leq x} b(m) = Q(x) + O\left(x^{\frac{1}{2} - \frac{1}{8n} + 4n\eta u}\right) + O\left(x^{1 - \frac{1}{4n} - \eta}\right)$$

with $u = \dfrac{1}{2} - \dfrac{1}{8n} + \epsilon$. We also note that $Q(x)$ must be of the form $c_1 x + c_2$ for some c_1 and c_2. Thus,

$$B(x) = c_1 x + O\left(x^{\frac{1}{2} - \frac{1}{8n} + 4n\eta(\frac{1}{2} - \frac{1}{8n} + \epsilon)}\right) + O\left(x^{1 - \frac{1}{4n} - \eta}\right)$$

As we did for $A(x) = \sum_{\mathrm{N}(\mathfrak{m}) \leq x} \tilde{C}(\mathfrak{m})$, we equate the exponents of x in the above equation and obtain an optimized η, which shows that

$$B(x) = c_1 x + O\left(x^{\frac{4n-1}{4n+1} + \epsilon}\right). \tag{5.6}$$

Now, applying the Möbius inversion formula to (5.5), we see that

$$\sum_{\mathrm{N}(\mathfrak{m}) = m} \tilde{C}(\mathfrak{m})^2 = \sum_{d^2 | m} \mu(d) b\left(\frac{m}{d^2}\right),$$

and thus

$$\begin{aligned} \sum_{\mathrm{N}(\mathfrak{m}) \leq x} \tilde{C}(\mathfrak{m})^2 &= \sum_{m \leq x} \sum_{d^2 | m} \mu(d) b\left(\frac{m}{d^2}\right) \\ &= \sum_{d^2 \leq x} \mu(d) \sum_{e \leq x/d^2} b(e). \end{aligned}$$

Using (5.6), it can be written as

$$\sum_{\mathrm{N}(\mathfrak{m}) \leq x} \tilde{C}(\mathfrak{m})^2 = \sum_{d^2 \leq x} \mu(d) \left\{ c_1 \frac{x}{d^2} + O\left(\left(\frac{x}{d^2}\right)^{\frac{4n-1}{4n+1} + \epsilon}\right) \right\}.$$

It is well-known that

$$\sum_{d^2 \le x} \frac{\mu(d)}{d^2} = \frac{6}{\pi^2} + O\left(x^{-\frac{1}{2}}\right),$$

and henceforth applying this to the above expression, we obtain that

$$\sum_{\mathrm{N}(\mathfrak{m}) \le x} \tilde{C}(\mathfrak{m})^2 = \frac{6c_1}{\pi^2} x + O\left(x^{\frac{1}{2}}\right) + O\left(x^{\frac{4n-1}{4n+1}+\epsilon} \sum_{d^2 \le x} |\mu(d)| d^{-\frac{2(4n-1)}{4n+1}-\epsilon}\right).$$

Since

$$\sum_{d^2 \le x} |\mu(d)| d^{-\frac{2(4n-1)}{4n+1}-\epsilon} \le \sum_{d=1}^{\infty} d^{-\frac{2(4n-1)}{4n+1}-\epsilon} < \infty$$

and $\frac{4n-1}{4n+1} > \frac{1}{2}$, we finally get

$$\sum_{\mathrm{N}(\mathfrak{m}) \le x} \tilde{C}(\mathfrak{m})^2 = \frac{6c_1}{\pi^2} x + O\left(x^{\frac{4n-1}{4n+1}+\epsilon}\right).$$

This completes the proof of the proposition.

□

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