ON THE NATURE OF e^γ AND NON-VANISHING OF DERIVATIVES OF L-SERIES AT $s = 1/2$

M. RAM MURTY AND NAOMI TANABE

Abstract. In 2011, M. R. Murty and V. K. Murty [10] proved that if $L(s, \chi_D)$ is the Dirichlet L-series attached a quadratic character χ_D, and $L'(1, \chi_D) = 0$, then e^γ is transcendental. This paper investigates such phenomena in wider collections of L-functions, with a special emphasis on Artin L-functions. Instead of $s = 1$, we consider $s = 1/2$. More precisely, we prove that

$$\exp\left(\frac{L'(1/2, \chi)}{L(1/2, \chi)} - \alpha\gamma\right)$$

is transcendental with some rational number α. In particular, if we have $L(1/2, \chi) \neq 0$ and $L'(1/2, \chi) = 0$ for some Artin L-series, we deduce the transcendence of e^γ.

1. Introduction

It is unknown if Euler's constant γ is rational or irrational. Equally unknown is the nature of the number e^γ. Thus, it is rather striking that in 2011, M. R. Murty and V. K. Murty [10] proved the following curious theorem. Let K be an imaginary quadratic field and χ_D its associated quadratic character. If $L(s, \chi_D)$ is the Dirichlet series associated to χ_D, then

$$\exp\left(\frac{L'(1, \chi_D)}{L(1, \chi_D)} - \gamma\right)$$

and π are algebraically independent. Thus, if $L'(1, \chi_D) = 0$, then e^γ and π are algebraically independent and in particular e^γ is transcendental. It is unknown if there are any quadratic characters χ_D for which $L'(1, \chi_D) = 0$. Presumably not. In [10], the authors show that such L-series are very rare, if they exist.

In this paper, we will prove a related result. Instead of considering Dirichlet L-series attached to quadratic characters, we look at Artin L-series attached to real characters. While the authors in [10] considered $s = 1$, we focus on Artin L-series at $s = 1/2$. More precisely, we prove the following:

Theorem 1.1. *Let $L(s, \chi, E/F)$ be an Artin L-series associated to a real character χ. Suppose that $L(1/2, \chi, E/F) \neq 0$. Then,*

$$\exp\left(\frac{L'(1/2, \chi, E/F)}{L(1/2, \chi, E/F)} - \frac{(d + 2r_2)}{2}\chi(1)\gamma\right)$$

is transcendental. Here $d = r_1 + 2r_2$ is the degree of F over $\mathbb{Q}$.

In particular, if there is a real Artin character χ for which $L'(1/2, \chi, E/F) = 0$ and $L(1/2, \chi, E/F) \neq 0$, then e^γ is transcendental.

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We will prove this theorem as a consequence of a more general investigation regarding Dirichlet series that satisfy functional equations. See Section 2 for the general setting and Section 3 for results on Artin L-functions.

While our main focus in this paper is the values of derivatives of Dirichlet L-functions at the central point of symmetry, the same method applies to evaluate the values at other rational points. This will be discussed in Section 4.

2. Dirichlet L-series

One of the main results in this paper is to state a non-vanishing property of derivatives of L-series at the central point of symmetry. More precisely we have the following:

Theorem 2.1. *Let $\Phi_1(s) = \sum_{n=1}^{\infty} \frac{a_n}{\mu_n^s}$ and $\Phi_2 = \sum_{n=1}^{\infty} \frac{b_n}{\lambda_n^s}$ be two Dirichlet series such that they converge in some half plane, can be meromorphically continued to the entire complex plane, and satisfy the functional equation:*

$$\Delta(s)\Phi_1(s) = \Delta(\delta - s)\Phi_2(\delta - s) \tag{2.2}$$

where $\Delta(s) = \prod_{j=1}^{l} \Gamma(\alpha_j s + \beta_j)$ with $\alpha_j \frac{\delta}{2} + \beta_j \in \mathbb{Q} \backslash \mathbb{Z}_{\leq 0}$ and $\alpha_j \neq 0$ for all j. Write $\alpha_j \frac{\delta}{2} + \beta_j = n_j + \frac{m_j}{q_j}$ with $(m_j, q_j) = 1$ and $0 \leq m_j \leq q_j - 1$. If $\Phi_1(\delta/2)$ and $\Phi_2(\delta/2)$ are both nonzero, then we have the following:

$$\frac{1}{2}\left(\frac{\Phi_1'}{\Phi_1} + \frac{\Phi_2'}{\Phi_2}\right)\left(\frac{\delta}{2}\right) = \sum_{j=1}^{l} \alpha_j \gamma - \sum_{j:\, m_j = 0} \alpha_j \sum_{t=1}^{n_j - 1} \frac{1}{t}$$
$$- \sum_{j:\, m_j \neq 0} \alpha_j \left(\sum_{t=0}^{n_j-1} \frac{1}{t + m_j/q_j} - \log(2q_j) - \frac{\pi}{2} \cot\left(\frac{\pi m_j}{q_j}\right) + \sum_{r_j=1}^{[q_j/2]} \cos\left(\frac{2\pi m_j r_j}{q_j}\right) \log \sin\left(\frac{\pi r_j}{q_j}\right)\right)$$

where γ is the Euler constant.

It is understood that the summations in the above theorem are defined to be zero where $t > n_j - 1$ for each j. The result gives interesting corollaries, stated below, regarding the transcendental nature of some values.

Corollary 2.3. *Let $\Phi_1(s)$ and $\Phi_2(s)$ be as given in Theorem 2.1. Then*

$$\exp\left(\frac{1}{2}\left(\frac{\Phi_1'}{\Phi_1} + \frac{\Phi_2'}{\Phi_2}\right)\left(\frac{\delta}{2}\right) - \sum_{j=1}^{l} \alpha_j \gamma\right) = Ce^A e^{\pi \frac{B}{2}} \prod_{j: m_j \neq 0} \prod_{r_j=1}^{[q_j/2]} \left(\sin \frac{\pi r_j}{q_j}\right)^{-\alpha_j \cos\left(\frac{2\pi m_j r_j}{q_j}\right)},$$

where

$$A := -\sum_{j:\, m_j=0} \alpha_j \sum_{t=1}^{n_j-1} \frac{1}{t} - \sum_{j:\, m_j \neq 0} \alpha_j \sum_{t=0}^{n_j-1} \frac{1}{t + m_j/q_j}, \quad B := \sum_{j: m_j \neq 0} \alpha_j \cot\left(\frac{\pi m_j}{q_j}\right), \quad \text{and} \quad C := \prod_{j: q_j \neq 1} (2q_j)^{\alpha_j}.$$

Furthermore, this value is transcendental.

Corollary 2.4. *Let ν be an algebraic number and $\mathfrak{S}_\nu$ the set of all the pairs of Dirichlet series $\phi_1(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ and $\phi_2(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}$ such that they can be meromorphically continued to a whole complex planes and satisfy the functional equation*

$$W^s \pi^{\nu s} \Delta(s) \phi_1(s) = W^{\delta - s} \pi^{\nu(\delta - s)} \Delta(\delta - s) \phi_2(\delta - s),$$

where $\Delta(s)$ is as given in Theorem 2.1, and W is an algebraic number. Further, assume that ϕ_1 and ϕ_2 do not vanish at the center of symmetry. Then, there is at most one algebraic element in the set

$$\left\{\exp\left(\frac{1}{2}\left(\frac{\phi_1'}{\phi_1}+\frac{\phi_2'}{\phi_2}\right)\left(\frac{\delta}{2}\right)-\sum_{j=1}^{l}\alpha_j\gamma\right)\ :\ (\phi_1,\phi_2)\in\mathfrak{S}_\nu\right\}.$$

We conjecture that there is no pair (ϕ_1, ϕ_2) satisfying all the hypothesis in the theorem and that has a property $\phi_1'(\delta/2) = \phi_2'(\delta/2) = 0$. Indeed, the first author, with Gun and Rath, proved that no L-series attached to a cuspform of even weight can hold such a property. The second author showed a similar result for the L-function attached to an even weight Hilbert cuspform. See [3] and [13] for details. If there is any such pair in general, then there is an immediate consequence that we obtain a specific expression of e^γ involving known transcendental numbers. This suggests that, even if there are some pairs (ϕ_1, ϕ_2) whose derivatives vanish at $s = \delta/2$, the number of such pairs must be limited as otherwise we obtain various expressions for e^γ and some of which would easily contradict with each other. We give some examples of this phenomenon in Section 4.

2.1. **Proof of Theorem 2.1.** By taking the logarithmic derivative of (2.2) with respect to s and substituting $s = \delta/2$, we see that

$$\left(\frac{\Phi_1'}{\Phi_1}+\frac{\Phi_2'}{\Phi_2}\right)\left(\frac{\delta}{2}\right) = -2\sum_{j=1}^{l}\alpha_j\psi\left(\alpha_j\frac{\delta}{2}+\beta_j\right) \tag{2.5}$$

where $\psi(s)$ is the logarithmic derivative of the gamma function. To proceed further, let us recall some properties of the digamma function from [11]:

Proposition 2.6. *Let $\psi(s)$ be the digamma function, that is the logarithmic derivative of the gamma function. Then ψ has the following properties, with γ being the Euler constant.*

(1) $\psi(s+1) = \psi(s) + \dfrac{1}{s}$
(2) $\psi(1) = -\gamma$
(3) *Let $(m, q) = 1$ and $1 \le m < q$. Then,*

$$\psi\left(\frac{m}{q}\right) = -\gamma - \log(2q) - \frac{\pi}{2}\cot\left(\frac{\pi m}{q}\right) + \sum_{r=1}^{[q/2]}\cos\left(\frac{2\pi m r}{q}\right)\log\sin\left(\frac{\pi r}{q}\right)$$

It can be deduced from the above proposition that, at any rational point $n + m/q$ with $(m, q) = 1$ and $0 \le m < q$, we have

$$\psi\left(n+\frac{m}{q}\right) = \begin{cases}\psi\left(\frac{m}{q}\right)+\sum_{t=0}^{n-1}\frac{1}{t+m/q} & \text{if } m\neq 0,\\ -\gamma+\sum_{t=1}^{n-1}\frac{1}{t} & \text{if } m=0.\end{cases}$$

We note that the summations $\sum_{t=1}^{n-1} 1/t$ and $\sum_{t=0}^{n-1} 1/(t+(m/q))$ in the above equation are taken to be zero in case $n = 0$ and $n = 1$, respectively.

The desired result is obtained by applying this to each term in (2.5). □

2.2. **Proof of Corollary 2.3.** The first part is an immediate consequence of Theorem 2.1. To see that the expression gives a transcendental number, we need Baker's theorem. (See, for example, [1, Theorem 2.3].)

Lemma 2.7 (Baker). *If $\alpha_1, \ldots, \alpha_m$, β_0, $\beta_1, \ldots, \beta_m$ are algebraic, and α_i (for all i) and β_0 are nonzero, then*

$$e^{\beta_0}\alpha_1^{\beta_1}\cdots\alpha_m^{\beta_m}$$

is transcendental.

Note that A, B, C, $\sin(\pi r_j/q_j)$, and $\cos(2\pi m_j r_j/q_j)$ are all algebraic for all j and r_j. In particular, A, C, and $\sin(\pi r_j/q_j)$ are nonzero. Rewriting $e^{\pi B/2}$ as

$$e^{\pi \frac{B}{2}} = \left(e^{-\pi i}\right)^{\frac{B}{2} i},$$

we can apply Baker's theorem to the right hand side of the expression given in Corollary 2.3 to complete the proof. □

2.3. **Proof of Corollary 2.4.** Setting $\mu_n = \lambda_n = n(W\pi^\nu)^{-1}$ in Theorem 2.1 and Corollary 2.3, we obtain that

$$\exp\left(\frac{1}{2}\left(\frac{\phi_1'}{\phi_1} + \frac{\phi_2'}{\phi_2}\right)\left(\frac{\delta}{2}\right) + \log W^2 + 2\nu\log\pi - \sum_{j=1}^{l}\alpha_j\gamma\right) = Ce^A e^{\pi\frac{B}{2}} \prod_{j,r_j}\left(\sin\frac{\pi r_j}{q_j}\right)^{-\alpha_j\cos\left(\frac{2\pi m_j r_j}{q_j}\right)},$$

or equivalently

$$\exp\left(\frac{1}{2}\left(\frac{\phi_1'}{\phi_1} + \frac{\phi_2'}{\phi_2}\right)\left(\frac{\delta}{2}\right) - \sum_{j=1}^{l}\alpha_j\gamma\right) = CW^{-2}\pi^{-2\nu} e^A e^{\pi\frac{B}{2}} \prod_{j,r_j}\left(\sin\frac{\pi r_j}{q_j}\right)^{-\alpha_j\cos\left(\frac{2\pi m_j r_j}{q_j}\right)}. \tag{2.8}$$

If there are two algebraic numbers of this form, their quotient must be also algebraic. This gives a contradiction. Indeed, for two such algebraic numbers, if the values corresponding to A in the above equation are different, their quotient is of the form $e^{\beta_0}\alpha_1^{\beta_1}\cdots\alpha_r^{\beta_r}$, up to algebraic constants, which is transcendental by Baker's theorem (Lemma 2.7). In case the values corresponding to A are the same for both pairs of Dirichlet series, i.e., the quotient of those values is of the form $\alpha_1^{\beta_1}\cdots\alpha_r^{\beta_r}$ up to algebraic constants, we may apply a different version of Baker's theorem shown below. (See [1, Theorem 2.4] for details).

Lemma 2.9 (Baker)**.** *Suppose that $\alpha_1, \ldots, \alpha_r$ are algebraic numbers not equal to 0 or 1 and that $\beta_1, \ldots, \beta_r$ are algebraic such that $1, \beta_1, \ldots, \beta_r$ are linearly independent over $\mathbb{Q}$. Then, the product*

$$\alpha_1^{\beta_1}\cdots\alpha_r^{\beta_r}$$

is transcendental.

We note that, if the β_j's are not all linearly independent over $\mathbb{Q}$ in our setting, the lemma above still applies by writing such β_j as a linear combination of the others and rearranging the form. This completes the proof of Corollary 2.4. □

Remark 2.10. *Unlike Corollary 2.3, an existence of a pair (ϕ_1, ϕ_2), in Corollary 2.4, with vanishing derivative at the central point does not imply the transcendence of e^γ immediately. Instead, applying the same idea as in the proof of Corollary 2.3 to the equation (2.8), we deduce that $e^\gamma\pi^{-2\nu/\alpha}$ is transcendental, where $\alpha = \sum_j \alpha_j$.*

3. Artin L-functions

We now direct our attention to Artin L-functions. First let us briefly recall the construction of an Artin L-function $L(s, \rho, E/F)$ attached to ρ. The details can be found in, for example, Cogdell-Kim-Murty [2] or Murty [9].

Let E/F be a Galois extension of number fields, and $G := \mathrm{Gal}(E/F)$ its Galois group. Let (ρ, V) be a finite dimensional representation of G, and say $\dim V = n$.

Let $\mathfrak{p}$ be any prime ideal of F and $\mathfrak{P}$ for a prime ideal of E lying above $\mathfrak{p}$.

We write $\sigma_{\mathfrak{P}}$ for the Frobenius automorphism for $\mathfrak{P}$ so that

$$\sigma_{\mathfrak{P}}(x) \equiv x^{\mathrm{N}(\mathfrak{p})} \mod \mathfrak{P}$$

for all x in $\mathcal{O}_E$. Then, the Artin L-function $L(s, \rho, E/F)$ attached to ρ is defined as

$$L(s, \rho, E/F) = \prod_{\mathfrak{p}<\infty} \det(I - \mathrm{N}(\mathfrak{p})^{-s}\rho(\sigma_{\mathfrak{P}})|_{V^{I_{\mathfrak{P}}}})^{-1}$$

where $I_{\mathfrak{P}}$ is the inertia group for $\mathfrak{P}$, i.e.,

$$I_{\mathfrak{P}} = \{\tau \in G \,:\, \tau(x) \equiv x \mod \mathfrak{P} \quad \forall x \in \mathcal{O}_E\},$$

and $\mathfrak{p}$ runs through all the prime ideals of F. Note that the right hand side of the equation defining $L(s, \rho, E/F)$ does not depend on the choice of $\mathfrak{P}$ because all $\sigma_{\mathfrak{P}}$'s are conjugate in G as long as $\mathfrak{P}$ lies above $\mathfrak{p}$. Therefore, we may replace ρ with any class function of G, or in particular, with a character $\chi = \chi_\rho$ associated to ρ. We also denote this L-function as $L(s, \chi, E/F)$.

We now define the local factors at archimedean places, and complete the L-function $L(s, \chi, E/F)$. The decomposition group $D_{\mathfrak{P}}$ at an archimedean place $\mathfrak{P}$ is given as

$$D_{\mathfrak{P}} = \begin{cases} \{1\} & \text{if } E_{\mathfrak{P}} = F_{\mathfrak{p}} \\ \{1, \omega_{\mathfrak{P}}\} & \text{if } E_{\mathfrak{P}} = \mathbb{C} \text{ and } F_{\mathfrak{p}} = \mathbb{R}. \end{cases}$$

For $\mathfrak{P}$ such that $F_{\mathfrak{p}} = \mathbb{R}$, put $n_{\mathfrak{p}}^+ = \dim V^{\rho(\omega_{\mathfrak{P}})}$ and $n_{\mathfrak{p}}^- = n - n_{\mathfrak{p}}^+$. (Recall that $n = \dim V$.) Then, the local L-factor at each archimedean place $\mathfrak{p}$ is defined to be

$$L_{\mathfrak{p}}(s, \chi_{\mathfrak{p}}) = \begin{cases} \pi^{-n(s+1/2)} \left(\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)\right)^n & \text{if } F_{\mathfrak{p}} = \mathbb{C}, \\ \left(\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\right)^{n_{\mathfrak{p}}^+}\left(\pi^{-(s+1)/2}\Gamma\left(\frac{s+1}{2}\right)\right)^{n_{\mathfrak{p}}^-} & \text{if } F_{\mathfrak{p}} = \mathbb{R}, \end{cases} \tag{3.1}$$

and the completed Artin L-function is

$$\Lambda(s, \chi) := A(\chi)^{s/2} L(s, \chi, E/F) \prod_{\mathfrak{p}|\infty} L_{\mathfrak{p}}(s, \chi_{\mathfrak{p}}) \tag{3.2}$$

with the constant $A(\chi)$ given by

$$A(\chi) = |D_F|^n \mathrm{N}(\mathfrak{f}(\chi)).$$

Here, D_F is the discriminant of F and $\mathfrak{f}(\chi)$ is the Artin conductor. The completed L-function satisfies the following functional equation:

$$\Lambda(s, \chi) = W(\chi)\Lambda(1 - s, \overline{\chi}) \tag{3.3}$$

where $W(\chi)$ is the Artin root number, which is a complex number with absolute value 1.

Let us put $\deg F/\mathbb{Q} = d = r_1 + 2r_2$, where r_1 and $2r_2$ are the numbers of real and complex embeddings of F, respectively. Then the equation (3.2) can be written as

$$\Lambda(s, \chi) = A(\chi)^{s/2} L(s, \chi, E/F) \pi^{-\frac{s}{2}(a+b) - \frac{b}{2}} \Gamma\left(\frac{s}{2}\right)^a \Gamma\left(\frac{s+1}{2}\right)^b, \tag{3.4}$$

where $a = 2nr_2 + \sum_{\mathfrak{p}\colon \mathrm{real}} n_{\mathfrak{p}}^+$ and $b = 2nr_2 + \sum_{\mathfrak{p}\colon \mathrm{real}} n_{\mathfrak{p}}^-$. An Artin L-function of this form is said to be of Hodge type (a, b). Now we restate Theorem 1.1:

Theorem 3.5. *Let E/F be a Galois extension of number fields, and (ρ, V) a finite dimensional representation of the Galois group $G := \mathrm{Gal}(E/F)$. If the Artin L-function $L(s, \chi, E/F)$ associated to the character $\chi = \chi_\rho$ is of the Hodge type (a, b) and if both $L(1/2, \chi, E/F)$ and $L(1/2, \overline{\chi}, E/F)$ are nonzero, then we have the following property:*

$$\exp\left(\frac{L'\left(\frac{1}{2}, \chi, E/F\right)}{L\left(\frac{1}{2}, \chi, E/F\right)} + \frac{L'\left(\frac{1}{2}, \overline{\chi}, E/F\right)}{L\left(\frac{1}{2}, \overline{\chi}, E/F\right)} - (a + b)\gamma\right) = A(\chi)^{-1}(8\pi)^{a+b} e^{\frac{\pi}{2}(a-b)},$$

where γ is the Euler constant. In particular, this value is transcendental.

Furthermore, if $L'(1/2,\chi,E/F)$ and $L'(1/2,\overline{\chi},E/F)$ both vanish for some character χ, then e^γ is transcendental.

We note that $a+b=n(d+2r_2)$, and thus the statement of Theorem 3.5 is consistent with that of Theorem 1.1. This theorem has interesting corollaries:

Corollary 3.6. *Let (ρ,V) be a finite dimensional representation of $G:=Gal(E/F)$, and $L(s,\chi_\rho,E/F)$ its associated Artin L-function. For any $L(s,\chi,E/F)$ satisfying the properties*

$$L(1/2,\chi_{\rho,i},E/F)\neq 0 \quad \text{and} \quad L'(1/2,\chi_{\rho,i},E/F)=0, \tag{3.7}$$

with $\chi_{\rho,1}=\chi_\rho$ and $\chi_{\rho,2}=\overline{\chi}_\rho$, the value $A(\chi)^{1/(a+b)}$ coincide where (a,b) is the Hodge type of the L-function.

In particular, we find some remarkable relations between the non-vanishing of the derivative and the root discriminants if we restrict the representation ρ to be trivial, and take a totally real number field F as the base field:

Corollary 3.8. *Suppose F is a totally real number field such that its associated Dedekind zeta function $\zeta_F(s)$ has the properties that $\zeta_F(1/2)\neq 0$ and $\zeta_F'(1/2)=0$. The root discriminants rd_F for any such field F coincide.*

This follows immediately from Corollary 3.6, as $a+b$ simply represents the extension degree of F and $A(\chi)=|D_F|$. A further observation can be made as follows:

Corollary 3.9. *There are at most finitely many Dedekind zeta functions ζ_F satisfying $\zeta_F(1/2)\neq 0$ and $\zeta_F'(1/2)=0$ if the base field F is totally real and an abelian extension over $\mathbb{Q}$.*

Corollary 3.10. *There are at most finitely many zeta functions ζ_F such that $\zeta_F(1/2)\neq 0$ and $\zeta_F'(1/2)=0$ if $F/\mathbb{Q}$ is totally real and solvable with a fixed length.*

The rest of this section is devoted to proving all the statements claimed above.

3.1. **Proof of Theorem 3.5.** Equations (3.3) and (3.4) give a functional equation;

$$L(s,\chi,E/F)\Gamma\left(\frac{s}{2}\right)^a\Gamma\left(\frac{s+1}{2}\right)^b=W(\chi)A(\chi)^{1/2-s}\pi^{s(a+b)-\frac{a+b}{2}}L(1-s,\overline{\chi},E/F)\Gamma\left(\frac{1-s}{2}\right)^a\Gamma\left(\frac{2-s}{2}\right)^b.$$

Taking the logarithmic derivatives of this equation with respect to s and evaluating it at $s=1/2$, we see that

$$\frac{L'(1/2,\chi,E/F)}{L(1/2,\chi,E/F)}+\frac{L'(1/2,\overline{\chi},E/F)}{L(1/2,\overline{\chi},E/F)}=-\log A(\chi)+(a+b)\log\pi-a\psi\left(\frac{1}{4}\right)-b\psi\left(\frac{3}{4}\right). \tag{3.11}$$

It follows from the third statement in Proposition 2.6 that

$$\psi\left(\frac{1}{4}\right)=-\gamma-3\log 2-\frac{\pi}{2}, \quad \text{and} \quad \psi\left(\frac{3}{4}\right)=-\gamma-3\log 2+\frac{\pi}{2},$$

and thus the equation (3.11) can be written as

$$\frac{L'(1/2,\chi,E/F)}{L(1/2,\chi,E/F)}+\frac{L'(1/2,\overline{\chi},E/F)}{L(1/2,\overline{\chi},E/F)}=-\log A(\chi)+(a+b)\log(8\pi)+(a+b)\gamma+\frac{\pi}{2}(a-b). \tag{3.12}$$

The desired result is obtained by exponentiating the equation (3.12). Furthermore, the value

$$A(\chi)^{-1}(8\pi)^{a+b}e^{\frac{\pi}{2}(a-b)}$$

is transcendental because π and e^π are algebraically independent over $\mathbb{Q}$. That is due to a result of Nesterenko [12]. □

3.2. **Proof of Corollary 3.6.** Suppose $L(s, \chi_1, E_1/F_1)$ and $L(s, \chi_2, E_2/F_2)$ both satisfy the properties (3.7) and we write their Hodge types as (a_i, b_i) for $i = 1, 2$, respectively. Then Theorem 3.5 says that, for each χ_i,

$$e^{(a_i+b_i)\gamma} = A(\chi_i)(8\pi)^{-(a_i+b_i)}e^{-\frac{\pi}{2}(a_i-b_i)},$$

and so the value

$$A(\chi_i)^{1/(a_i+b_i)}(8\pi)^{-1}\exp\left(-\frac{\pi}{2}\frac{a_i-b_i}{a_i+b_i}\right)$$

coincides, and the value equals e^γ. Equivalently, we have that

$$A(\chi_1)^{1/(a_1+b_1)}A(\chi_2)^{-1/(a_2+b_2)} = \exp\left(\frac{\pi}{2}\left(\frac{a_1-b_1}{a_1+b_1} - \frac{a_2-b_2}{a_2+b_2}\right)\right).$$

The left hand side being an algebraic value, it forces the exponent on the right hand side to be zero, which gives that

$$A(\chi_1)^{1/(a_1+b_1)} = A(\chi_2)^{1/(a_2+b_2)}$$

as claimed. □

3.3. **Proof of Corollaries 3.9 and 3.10.** For an abelian extension $F/\mathbb{Q}$, the lower bound of the root discriminant rd_F tends to infinity as the extension degree increases. More precisely, we quote the following lemma from Murty [8]:

Lemma 3.13. [8, Corollary 2] *For any abelian extension $F/\mathbb{Q}$ of degree d and discriminant D_F,*

$$\frac{1}{d}\log|D_F| \geq \frac{1}{2}\log d.$$

Hence there is an upper bound for the extension degree where fields share the same root discriminant. Together with the Hermite Theorem stated below, the proof of Corollary 3.9 is completed.

Lemma 3.14 (Hermite)**.** *Let S be a finite set of primes. The set of algebraic number fields of degree n that are unramified outside S (that is, any prime dividing the discriminant d_F is in S) is finite.*

We note that reader can refe to [5, pp 273 - 278] for a complete proof of the Hermite Theorem.

Corollary 3.10 follows immediately from the following lemma by taking $K = \mathbb{Q}$:

Lemma 3.15. [6, Theorem 1] *Fix a number field K. For any positive integer k and positive real number N, the following set $Y_{k,N,K}$ is finite:*

$$Y_{k,N,K} := \{L : L/\mathbb{Q}\text{ is finite, } L/K \text{ is solvable with length } k,\ rd_L \leq N\}.$$

□

We note that it is known that there are infinitely many number fields with bounded root discriminants if the extension $F/\mathbb{Q}$ is either unramified or tamely ramified. See Martinet [7] for the case of unramified extensions and Hajir and Maire [4] for tamely ramified extensions.

□

4. Concluding Remarks

It was suggested in Section 2 that not too many pairs (ϕ_1, ϕ_2) have their derivatives vanishing at the central point of symmetry, under the condition that ϕ_1 and ϕ_2 themselves do not vanish at the point. For example, we compare Artin L-series studied in Section 3 with L-functions attached to a Hilbert cusp form. The second author proved a non-vanishing result for the derivatives of L-functions attached a primitive Hilbert cusp form. More precisely, she proved:

Theorem 4.1. ([13, Theorem 1.1]). *Let $2k = (2k_1, \dots, 2k_n)$ be an n-tuple of even integers with $k_j \geq 2$, and put $k_0 = \max_j\{k_j\}$. For a primitive Hilbert cusp form $\mathbf{f}$ of weight $2k$ with trivial character, if $L(k_0, \mathbf{f}) \neq 0$, then $L'(k_0, \mathbf{f}) \neq 0$.*

We are doubtful that the above theorem fails when $k_j = 1$ is allowed, but it is not yet proven. If $L'(k_0, \mathbf{f}) = 0$ for some $\mathbf{f}$ under this condition, then it can be seen from [13, Equation (3.1)] that

$$e^{\gamma} = W_1 \cdot \pi^{-1} e^{A} \tag{4.2}$$

with an algebraic number W_1 and $A = \sum_{j=1}^{n} \sum_{m=1}^{k_j-1} 1/m$. On the other hand, if $L'(1/2, \chi, E/F) = L'(1/2, \overline{\chi}, E/F) = 0$ for some χ, then Theorem 3.5 suggests that

$$e^{\gamma} = W_2 \cdot \pi^{-1} e^{B} \tag{4.3}$$

where W_2 is algebraic and $B = \frac{\pi}{2} \cdot \frac{a-b}{a+b}$. They cannot hold simultaneously unless $W_1 = W_2$ and $A = B = 0$. In particular, it claims that, if they both vanish simultaneously, then $e^{\gamma}\pi$ is algebraic. The nature of the number $e^{\gamma}\pi$ is still mostly unknown, but its algebraicity is unlikely.

Also, it is worthwhile to mention that, if there exists an even weight primitive Hilbert cusp form $\mathbf{f}$ such that $L(k_0, \mathbf{f}) \neq 0$ and $L'(k_0, \mathbf{f}) = 0$, then e^{γ} is transcendental under the assumption that the Schanuel's conjecture is true. (The conjecture need not be assumed in case $k_j = 1$ for all j.) We now recall the conjecture.

Conjecture 4.4 (Schanuel)**.** *For any set of complex numbers, $z_1, \dots, z_n$, that are linearly independent over $\mathbb{Q}$, the transcendental degree of the field $\mathbb{Q}(z_1, \dots, z_n, e^{z_1}, \dots, e^{z_n})$ over $\mathbb{Q}$ is at least n.*

Indeed, if Schanuel's conjecture is true, then e and π are algebraically independent because

$$\text{tr.deg}_{\mathbb{Q}}(e, \pi) = \text{tr.deg}_{\mathbb{Q}}(1, \pi i, e, e^{\pi i}) \geq 2.$$

Thus, the transcendence of e^{γ} follows from (4.2) (modulo Schanuel's conjecture).

At the end, we remark that the same method is applicable to evaluate the logarithmic derivative of L-functions not only at a central point of symmetry but also at any rational points a/q except where $\alpha_j a/q + \beta_j$ and $\alpha_j(\delta - a/q) + \beta_j$ are non-positive integers. Let us see this in the case of Artin L-functions. Using the functional equation of the Artin L-function given in equations (3.3) and (3.4), its logarithmic derivative can be written as follows:

$$\begin{aligned} &\frac{L'(s, \chi, E/F)}{L(s, \chi, E/F)} + \frac{L'(1-s, \overline{\chi}, E/F)}{L(1-s, \overline{\chi}, E/F)} \\ &= -\log A(\chi) + (a+b)\log \pi - \frac{a}{2}\left(\psi\left(\frac{s}{2}\right) + \psi\left(\frac{1-s}{2}\right)\right) - \frac{b}{2}\left(\psi\left(\frac{s+1}{2}\right) + \psi\left(\frac{2-s}{2}\right)\right). \end{aligned} \tag{4.5}$$

For any rational point in the interval $(0, 1)$, the right hand side of the above equation is easily evaluated by applying the properties of digamma functions described in Proposition 2.6. If a point is taken outside of the interval $(0, 1)$ which ought to be non-integer, then we apply a functional equation of the digamma function:

$$\psi(1-x) - \psi(x) = \pi \cot(\pi x)$$

accordingly. For instance, let us put $s = m/q$ and suppose $m/q > 0$. Then, Proposition 2.6 applies to $\psi(s/2)$ and $\psi((s+1)/2)$ directly. The other terms are written as:

$$\psi\left(1 - \frac{m}{2q}\right) = \psi\left(\frac{m}{2q}\right) + \pi \cot\left(\frac{\pi m}{2q}\right),$$

and

$$\psi\left(\frac{1}{2}-\frac{m}{2q}\right)=\psi\left(1-\left(\frac{1}{2}+\frac{m}{2q}\right)\right)=\psi\left(\frac{1}{2}+\frac{m}{2q}\right)+\pi\cot\left(\frac{\pi}{2}+\frac{\pi m}{2q}\right).$$

Therefore, inserting these in to equation (4.5), we obtain that

$$\frac{L'(m/q,\chi,E/F)}{L(m/q,\chi,E/F)}+\frac{L'(1-m/q,\overline{\chi},E/F)}{L(1-m/q,\overline{\chi},E/F)}$$
$$=-\log A(\chi)+(a+b)\log\pi-\frac{a+b}{2}\left(\psi\left(\frac{m}{2q}\right)+\psi\left(\frac{1}{2}+\frac{m}{2q}\right)\right)+\pi\left(\cot\left(\frac{\pi m}{2q}\right)-\tan\left(\frac{\pi m}{2q}\right)\right).$$

Applying Proposition 2.6 to the terms of ψ in the above, it will be of interest to investigate the possible transcendence of special values of these L-functions.

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Department of Mathematics and Statistics,
Queen's University
Jeffery Hall, University Ave.
Kingston, Ontario, K7L 3N6, Canada

murty@mast.queensu.ca
naomi@mast.queensu.ca