NON-VANISHING OF DERIVATIVES OF L-FUNCTIONS ATTACHED TO HILBERT MODULAR FORMS

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ABSTRACT. This article is to show a non-vanishing property of the derivative of certain L-functions. For certain primitive holomorphic Hilbert modular forms, if the central critical value of the standard L-function does not vanish, then neither does its derivative. This is a generalization of a result by Gun, Murty and Rath in the case of elliptic modular forms. Some applications in transcendental number theory deduced from this result are discussed as well.

1. INTRODUCTION

Gun, Murty, and Rath proved a non-vanishing property of the derivative of the L-function of an elliptic modular cusp form at the center of symmetry. (See [1, Theorem 4.1].) The aim of this paper is to generalize their result to Hilbert modular forms. A precise statement of our theorem is as follows:

Theorem 1.1. *Let $\mathbf{f}$ be a holomorphic Hilbert modular cusp form of weight $k = (k_1, \cdots, k_n)$, level $\mathfrak{n}$, with trivial character, over a totally real number field F of degree n. Assume that $\mathbf{f}$ is primitive, and the weight satisfies the following conditions: $k_j \geq 4$ for all j and $k_1 \equiv \cdots \equiv k_n \equiv 0 \mod 2$. Let $k_0 = \max(k_1, \ldots, k_n)$. If $L_f(k_0/2, \mathbf{f}) \neq 0$, then*

$$\frac{L'_f(k_0/2, \mathbf{f})}{L_f(k_0/2, \mathbf{f})} = -\frac{\log \mathrm{N}(\mathfrak{n}\mathfrak{D}^2)}{2} + n\log(2\pi) - \sum_{j=1}^{n} \psi\left(\frac{k_j}{2}\right),$$

where $\mathfrak{D}$ is the different ideal of F, and ψ is the logarithmic derivative of the gamma function. Furthermore, $L'_f(k_0/2, \mathbf{f}) \neq 0$, i.e., if the central critical value is nonzero then so is the derivative at the center of symmetry.

The theorem is proven in Section 3. Some definitions and basic properties of Hilbert modular forms that are needed in the proof are introduced in 2.2, and of their L-functions in 2.3. These settings are adopted from Shimura [3]. The necessity of the hypotheses on the weight $k = (k_1, \cdots, k_n)$ are stated in Remark 3.4 and 3.5. The theorem leads us to some applications in transcendental number theory, as in [1]. See Corollary 3.2 and 3.3.

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2. PRELIMINARIES

2.1. **Notations.** Throughout this paper, let F be a totally real number field of degree n, $\mathcal{O}$ the ring of integers in F, and $\eta = (\eta_1, \cdots, \eta_n)$ the real embeddings of F with a fixed order of $\{\eta_j\}$. Any element α in F can be viewed as an element of $\mathbb{R}^n$ with this embedding, and we write $(\alpha_1, \cdots, \alpha_n)$ to mean $(\eta_1(\alpha), \cdots, \eta_n(\alpha))$.

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Let $F_{\mathfrak{p}}$ be a completion of F at each place $\mathfrak{p}$, and $\mathbb{A}_F$ the adèle ring of F. Note that it is the restricted direct product of $F_{\mathfrak{p}}$ with respect to $\mathcal{O}_{\mathfrak{p}}$, where $F_{\mathfrak{p}}$ is the completion of F at $\mathfrak{p}$, and $\mathcal{O}_{\mathfrak{p}}$ is the ring of integers of $F_{\mathfrak{p}}$. For each non-archimedean place $\mathfrak{p}$, define an open-compact subgroup $K_{\mathfrak{p}}(\mathfrak{n})$ of $\mathrm{GL}_2(F_{\mathfrak{p}})$ as

$$K_{\mathfrak{p}}(\mathfrak{n}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(F_{\mathfrak{p}}) : \begin{array}{ccc} a\mathcal{O}_{\mathfrak{p}} + \mathfrak{n}_{\mathfrak{p}} = \mathcal{O}_{\mathfrak{p}}, & b \in \mathfrak{D}_{\mathfrak{p}}^{-1}, & \\ c \in \mathfrak{n}_{\mathfrak{p}}\mathfrak{D}_{\mathfrak{p}}, & d \in \mathcal{O}_{\mathfrak{p}}, & ad - bc \in \mathcal{O}_{\mathfrak{p}}^{\times} \end{array} \right\}, \tag{2.1}$$

where $\mathfrak{D}_{\mathfrak{p}} = \mathfrak{D}\mathcal{O}_{\mathfrak{p}}$, and similarly $\mathfrak{n}_{\mathfrak{p}} = \mathfrak{n}\mathcal{O}_{\mathfrak{p}}$. We also write

$$K_{\circ}(\mathfrak{n}) := \prod_{\mathfrak{p}<\infty} K_{\mathfrak{p}}(\mathfrak{n}).$$

Let h be the narrow class number, and $\{t_\nu\}_{\nu=1}^{h}$ a set of representatives of the narrow class group where the archimedean part $t_{\nu,\infty}$ of each t_ν is 1. Then, $\mathrm{GL}_2(\mathbb{A}_F)$ can be decomposed as

$$\mathrm{GL}_2(\mathbb{A}_F) = \cup_{\nu=1}^{h} \mathrm{GL}_2(F) x_\nu^{-\iota} \left(\mathrm{GL}_2^{+}(F_\infty) K_\circ(\mathfrak{n}) \right) \quad \text{(a disjoint union)}, \tag{2.2}$$

with $x_\nu^{-\iota} := \begin{pmatrix} t_\nu^{-1} & \\ & 1 \end{pmatrix}$.

For a fixed integral ideal $\mathfrak{n}$ of F and for each ν, we also define a congruence subgroup $\Gamma_\nu(\mathfrak{n})$ of $\mathrm{GL}_2(F)$ as

$$\Gamma_\nu(\mathfrak{n}) = \left\{ \begin{pmatrix} a & t_\nu^{-1} b \\ t_\nu c & d \end{pmatrix} : \begin{array}{ccc} a \in \mathcal{O}, & b \in \mathfrak{D}^{-1}, & \\ c \in \mathfrak{n}\mathfrak{D} & d \in \mathcal{O} & ad - bc \in \mathcal{O}^{\times} \end{array} \right\}.$$

2.2. **Hilbert modular forms.** Let $k = (k_1, \cdots, k_n) \in \mathbb{Z}^n$. For a holomorphic function f on $\mathfrak{h}^n$ and an element $\gamma = (\gamma_1, \cdots, \gamma_n)$ in $\mathrm{GL}_2(\mathbb{R})^n$, define

$$f||_k \gamma(z) = \prod_i \det \gamma_i^{k_i/2} \mathrm{j}(\gamma_i, z_i)^{-k_i} f(\gamma z),$$

with $\mathrm{j}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z \right) = cz + d$

By a holomorphic Hilbert modular form $\mathbf{f} = (f_1, \cdots, f_h)$ of weight $k = (k_1, \cdots, k_n)$, level $\mathfrak{n}$, and with trivial character, we mean that $\mathbf{f}$ is a function on $\mathrm{GL}_2(\mathbb{A}_F)$ defined as

$$\mathbf{f}(g) = \mathbf{f}(\gamma x_\nu^{-\iota} g_\infty k_\circ) = (f_\nu ||_k g_\infty)(\mathbf{i}),$$

where $g = \gamma x_\nu^{-\iota} g_\infty k_\circ$ in the decomposition given in (2.2), and each f_ν is a holomorphic function on $\mathfrak{h}^n$ and at all cusps, and satisfies the automorphy condition, $f_\nu ||_k \gamma = f$, for $\gamma \in \Gamma_\nu(\mathfrak{n})$. Such a function f_ν has a Fourier expansion given as

$$f_\nu(z) = \sum_{0 \ll \xi \in t_\nu \mathcal{O}, \text{ or } \xi = 0} a_\nu(\xi) e^{2\pi i \xi z},$$

where $e^{2\pi i \xi z} = \exp\left(2\pi i \sum_{j=1}^{n} \xi_j z_j \right)$. If the constant term of $f_\nu ||_k \gamma$ in its Fourier expansion is 0, for any γ in $\mathrm{GL}_2^{+}(F)$, then f_ν is called a cusp form. A holomorphic Hilbert modular form $\mathbf{f}$ is called a cusp form if f_ν is a cusp form for all ν.

Let $\mathfrak{m}$ be an integral ideal in F. Then it can be uniquely written as $\mathfrak{m} = t_\nu^{-1} \xi \mathcal{O}$ where ξ is a totally positive element in $t_\nu \mathcal{O}$. Let $C(\mathfrak{m}, \mathbf{f}) = a_\nu(\xi) \xi^{-k/2} \mathrm{N}(\mathfrak{m})^{k_0/2}$, with $k_0 = \max(k_1, \ldots, k_n)$. We say $\mathbf{f}$ is normalized if $C(\mathcal{O}, \mathbf{f}) = 1$. If $\mathbf{f}$ is a common eigenfunction of Hecke operators $T_{\mathfrak{m}}$, its eigenvalue is $C(\mathfrak{m}, \mathbf{f})/C(\mathcal{O}, \mathbf{f})$. In particular, the eigenvalue is $C(\mathfrak{m}, \mathbf{f})$ if $\mathbf{f}$ is normalized.

Proposition 2.3 (Shimura, [3]). *Let $\mathbf{f}$ be a holomorphic Hilbert modular form of weight k, level $\mathfrak{n}$, with trivial character. If f is an eigenfunction of $T_{\mathfrak{m}}$, for an ideal $\mathfrak{m}$ prime to $\mathfrak{n}$, with its eigenvalue $\lambda(\mathfrak{m})$, then $\lambda(\mathfrak{m}) = \overline{\lambda(\mathfrak{m})}$, i.e., $\lambda(\mathfrak{m})$ is real.*

We also define $\mathbf{f}|\mathrm{J}_{\mathfrak{n}}$ as follows: For each ν, pick a totally positive element q_ν. There exists a unique index λ so that $t_\nu t_\lambda \mathcal{O}\mathfrak{n}\mathfrak{D}^2 = q_\nu \mathcal{O}$. Put $\beta_\nu = \begin{pmatrix} & 1 \\ -q_\nu & \end{pmatrix}$, and $f'_\lambda = (-1)^k f_\nu ||_k \beta_\nu$. Then $\mathbf{f}|\mathrm{J}_{\mathfrak{n}}$ is defined to be $\mathbf{f}|\mathrm{J}_{\mathfrak{n}} = (f'_1, \cdots, f'_h)$. Then $f|\mathrm{J}_{\mathfrak{n}}$ has the same weight and level as $\mathbf{f}$. Furthermore, we have the following:

Proposition 2.4 (Shimura, [3])**.** *Let* $\mathbf{f}$ *be a primitive form with conductor* $\mathfrak{n}$*. Then* $\mathbf{f}|\mathrm{J}_n$ *is a nonzero constant times the complex conjugation of* $\mathbf{f}$*.*

2.3. **L-functions of Hilbert modular forms.** Let $\mathbf{f}$ be a holomorphic Hilbert modular form of weight $k = (k_1, \cdots, k_n)$ and level $\mathfrak{n}$. The finite L-function attached to $\mathbf{f}$ is defined to be

$$L_f(s, \mathbf{f}) = \sum_{\mathfrak{m};\ \text{integral}} \frac{C(\mathfrak{m}, \mathbf{f})}{\mathrm{N}(\mathfrak{m})^s}.$$

Define the completed L-function as

$$L(s, \mathbf{f}) = \mathrm{N}(\mathfrak{n}\mathfrak{D}^2)^{s/2}(2\pi)^{-ns} \prod_{j=1}^{n} \Gamma\left(s - \frac{k_0 - k_j}{2}\right) L_f(s, \mathbf{f}).$$

It converges for $\Re(s) \gg 0$, and has an analytic continuation to $\mathbb{C}$. The completed L-function satisfies the functional equation:

$$L(s, \mathbf{f}) = i^{\sum k_j} L(k_0 - s, \mathbf{f}|\mathrm{J}_{\mathfrak{n}}). \tag{2.5}$$

3. Proof of Theorem 1.1

Gun, Murty, and Rath proved the case $n = 1$ in [1, Theorem 4.1]. So we assume that $n \geq 2$.

Let $\mathbf{f}$ be a normalized common eigenform for $T_{\mathfrak{m}}$. Then, as given in 2.2, the eigenvalue for $T_{\mathfrak{m}}$ is $C(\mathfrak{m}, \mathbf{f})$. Moreover, it is real by Proposition 2.3. It follows by Proposition 2.4, that $\mathbf{f}|\mathrm{J}_{\mathfrak{n}} = c \cdot \mathbf{f}$ with some constant c. Therefore, the finite L-function attached to $\mathbf{f}|\mathrm{J}_{\mathfrak{n}}$, i.e., attached to $c \cdot \mathbf{f}$ is:

$$L_f(s, \mathbf{f}|\mathrm{J}_{\mathfrak{n}}) = L_f(s, c\mathbf{f}) = \sum \frac{c \cdot C(\mathfrak{m}, \mathbf{f})}{\mathrm{N}(\mathfrak{m})^s} = cL_f(s, \mathbf{f}).$$

The functional equation given in (2.5) can be written as

$$\mathrm{N}(\mathfrak{n}\mathfrak{D}^2)^{s/2}(2\pi)^{-ns} \prod_{j=1}^{n} \Gamma\left(s - \frac{k_0 - k_j}{2}\right) L_f(s, \mathbf{f}) = c \cdot i^{\sum k_j} \mathrm{N}(\mathfrak{n}\mathfrak{D}^2)^{(k_0 - s)/2}(2\pi)^{-n(k_0 - s)} \prod_{j=1}^{n} \Gamma\left(\frac{k_0 + k_j}{2} - s\right) L_f(k_0 - s, \mathbf{f}).$$

Taking the logarithmic derivative on both sides with respect to s, one has

$$\frac{\log \mathrm{N}(\mathfrak{n}\mathfrak{D}^2)}{2} - n\log(2\pi) + \sum_{j=1}^{n} \psi\left(s - \frac{k_0 - k_j}{2}\right) + \frac{L'_f(s, \mathbf{f})}{L_f(s, \mathbf{f})} = -\frac{\log \mathrm{N}(\mathfrak{n}\mathfrak{D}^2)}{2} + n\log(2\pi) - \sum_{j=1}^{n} \psi\left(\frac{k_0 + k_j}{2} - s\right) - \frac{L'_f(k_0 - s, \mathbf{f})}{L_f(k_0 - s, \mathbf{f})}.$$

The first part of the theorem is obtained by letting $s = k_0/2$.

To show the second part, let us first note that $\psi(k) = \mathrm{H}_{k-1} - \gamma$ where $\mathrm{H}_{k-1} := \sum_{m=1}^{k-1} 1/m$ is the $(k-1)$-th harmonic number, and γ is the Euler's constant. So if $L'_f(k_0/2, \mathbf{f}) = 0$, one has

$$n\left(\gamma + \log(2\pi)\right) = \frac{1}{2}\log \mathrm{N}(\mathfrak{n}) + \log(d_F) + \sum_{j=1}^{n} \mathrm{H}_{k_j/2-1}, \tag{3.1}$$

where d_F is the discriminant of F. By our assumption on k_j's, $\min\{\sum_j H_{k_j/2-1}\} = n$ that is attained when all the k_j's are 4. Using this and the Minkowski bound:

$$|d_F| \geq \frac{n^{2n}}{(n!)^2},$$

we see that $2n\log(n) - 2\log(n!) + n$ is a lower bound of the right hand side of the equation (3.1). But, for $n \geq 7$, this value is larger than $n(\gamma + \log(2\pi))$ which is bounded above by $2.4151n$. Hence (3.1) cannot be attained.

Now, we only need to check when $n \leq 6$. The table below shows the minimal discriminant of each degree extension; see Voight [4, Table 3].

n	2	3	4	5	6
minimal d_F	5	49	725	14641	300125

Applying each minimal d_F in (3.1) for $n \geq 4$, one can check that the right hand side exceeds the left hand side for any weight and level, as long as all the k_j's are at least 4.

If $n = 2$ or 3, one needs to examine several cases. Without loss of generality, let us assume that $k_i \leq k_{i+1}$. If $n = 3$ and the weight is at least $k = (4,4,6)$, the right hand side of (3.1) exceeds the left hand side for any level and any discriminant. So the only remaining case is $k = (4,4,4)$. But it can be easily verified that the equality in (3.1) never be satisfied. Checking the case $n = 2$ similarly completes the proof for the second part of the theorem.

Corollary 3.2. *Suppose that* $\mathbf{f}$ *satisfies all the conditions given in Theorem 1.1. Then*

$$\exp\left(\frac{L_f'(k_0/2,\mathbf{f})}{L_f(k_0/2,\mathbf{f})} + \sum_{j=1}^{n}\psi\left(\frac{k_j}{2}\right)\right)$$

is transcendental.

Proof. This follows from the first part of Theorem 1.1:

$$\exp\left(\frac{L_f'(k_0/2,\mathbf{f})}{L_f(k_0/2,\mathbf{f})} + \sum_{j=1}^{n}\psi\left(\frac{k_j}{2}\right)\right) = \exp\left(n\log(2\pi) - \frac{\log \mathrm{N}(\mathfrak{n}\mathfrak{D}^2)}{2}\right) = \frac{(2\pi)^n}{\mathrm{N}(\mathfrak{n}\mathfrak{D}^2)^{1/2}},$$

which is transcendental. □

Corollary 3.3. *Fix* $k = (k_1, \cdots, k_n)$ *with* $k_j \equiv 0 \mod 2$ *for all* j*, and let* $\mathfrak{S}_k$ *be the set of all primitive Hilbert cusp forms* $\mathbf{f}$ *of weight* k *that satisfy* $L_f(k_0/2,\mathbf{f}) \neq 0$*. Then there is at most one algebraic element in the set*

$$\left\{\frac{L_f'(k_0/2,\mathbf{f})}{L_f(k_0/2,\mathbf{f})} \,:\, \mathbf{f} \in \mathfrak{S}_k\right\}.$$

Proof. The first part of Theorem 1.1 shows that the logarithmic derivatives of the finite L-functions at $k_0/2$ give the same value if two cusp forms have the same level. Suppose that there are two cusp forms $\mathbf{f}$ and $\mathbf{g}$, with different levels $\mathfrak{n}$ and $\mathfrak{m}$ respectively, and that $L_f'(k_0/2,\mathbf{f})/L_f(k_0/2,\mathbf{f})$ and $L_f'(k_0/2,\mathbf{g})/L_f(k_0/2,\mathbf{g})$ are both algebraic. But then

$$\frac{L_f'(k_0/2,\mathbf{f})}{L_f(k_0/2,\mathbf{f})} - \frac{L_f'(k_0/2,\mathbf{g})}{L_f(k_0/2,\mathbf{g})} = \frac{1}{2}\log\left(\frac{\mathrm{N}(\mathfrak{m}\mathfrak{D}^2)}{\mathrm{N}(\mathfrak{n}\mathfrak{D}^2)}\right)$$

must be also algebraic, which is a contradiction. □

Remark 3.4. The parity condition of the weight, $k_1 \equiv \cdots \equiv k_n \mod 2$, makes $\mathbf{f}$ a Hilbert modular form of algebraic type. Under this condition, any integer m with $(k_0 - k^0)/2 < m < (k_0 + k^0)/2$ is a critical point of the (finite) L-function attached to $\mathbf{f}$, where $k^0 = \min(k_1, \ldots, k_n)$. In particular, if $k_1 \equiv \cdots \equiv k_n \equiv 0 \mod 2$, then $k_0/2$ is a critical point for $L_f(s,\mathbf{f})$. (See [2, Theorem 1.4].)

Remark 3.5. When the condition $k_j \geq 4$ for all j is not satisfied, the first part of the theorem still holds. However, a difficulty arises to prove the second part, as the right hand side of (3.1) does not give a good bound. For example, if $k_1 = \cdots = k_n = 2$, one needs to show that $n(\gamma + \log(2\pi)) = 1/2\log \mathrm{N}(\mathfrak{n}) + \log(d_F)$

cannot hold. One way to show this is to prove that $e^{\gamma}\pi$ is transcendental, which to the best of our knowledge seems to be unknown.

It should be also noted that in case the degree n of F is large enough, and $k_j \geq 4$ for enough j's (but not necessarily all of them), the non-vanishing of $L'_f(k_0/2, \mathbf{f})$ can be shown in the same way.

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