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HIGH-ENERGY COLLISION THEORY

R. J. Glauber
Department of Physics
Harvard University

Few trends are more striking nowadays than the increase of attention being devoted to the collisions of particles accelerated to high energies. The reasons for such studies lie basically in the information they furnish about the interactions of the colliding particles, and about the reaction products they generate. Both types of information may usually be obtained more readily at high than at low energies. The study of low-energy collisions ordinarily tells us only a certain measure of the strength of an interaction. At high energies, on the other hand, the shorter wavelength of the incident particles makes them sensitive probes of the region of interaction. When the wavelengths are sufficiently short, for example, the angular distribution of elastically scattered particles becomes, in a sense, a detailed map of the region of interaction. Inelastic collisions are capable of furnishing much the same information too, and as particle energies rise, the importance of such collisions grows and their variety multiplies.

The type of problem we should like to treat, say the collision of an incident particle with a nucleus consisting of many particles, is not an easy one to formulate at any energy. But at high energies the complication of the problem as evidenced, for example, by the huge number of final states available to the system, makes the prospect of reaching exact solutions quite dim indeed. Fortunately, however, the physical conditions which hold at high energies are in a number of ways well suited to the introduction of approximation methods. The major part of these lectures will be devoted to the development of such techniques.

The approximation methods we shall describe are quite elementary in structure. They all bear a certain family resemblance to the approximations used in the diffraction theory of physical optics. That is not to say that they are too familiar, however, since the situations encountered in collision theory are usually quite different from those of optics. For example, the target particles in a nucleus are free to move about in a bound state while the obstacles of diffraction theory are always fixed. It will be necessary, therefore, to develop mathematical methods for treating the quantum mechanical problem which are of much greater generality than those of physical optics. But the mathematics required is very simple in form and furnishes insight into situations of surprising complexity.

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A great deal of the work which has thus far been done on high-energy collisions has consisted of more or less empirical studies of the so-called "optical model." This model, which has been introduced and used more often on intuitive than on mathematical grounds, represents an attempt to deal with the problem of elastic scattering alone. The optical model removes from consideration particles which undergo inelastic scatterings by pretending that they have been absorbed within the nucleus. Of course any desire to discuss this effective absorption in a quantitative way leads us back again to the question of treating inelastic processes. Any mathematically comprehensive discussion of the optical model must therefore be based on a unified treatment of elastic and inelastic transitions. One of the favorable features of the approach to be described is that it is unitary in this sense, and thereby allows a very simple insight into the origin of the optical model. The model emerges as a rather natural way of describing the particular results obtained for elastic scattering, and these enable us to give an explicit construction of the optical potential for a finite-sized nucleus. We shall use it to illustrate the way in which the optical potential depends on the fluctuations and correlations in the positions of the individual nucleons.

It may be appropriate, before beginning, to say a word about how this work developed. The work on elastic scattering was done in 1952 in connection with a problem which was of particular interest in electron diffraction theory. It produced, in effect, an extension and generalization of a method described by Molière[1] in 1947. The remainer of the work, on the treatment of nonstatic interactions accumulated slowly in the period that followed. Unfortunately the material has been published only in fragments and abstracts. Various of the unpublished results have been quoted, however, in a number of papers[2-8] and so may be, to a degree, familiar. Others of the results are of more recent date and have not previously been published in any form. We shall try in these lectures to present the first systematic approach to this work, to its methods and results.

Since not everyone may be too conversant with the mathematical methods of scattering theory we shall begin with a general discussion of the simple problem of elastic scattering by a static potential. Here we shall review quite briefly some of the familiar methods of treating the problem, and shall pay particular attention to certain very useful theorems regarding the scattering amplitudes. We shall then begin the development of the high-energy approximation, and illustrate its application to a succession of problems of gradually increasing complexity. Collisions involving many-body systems will be dealt with in the last lectures.

Elastic Scattering (Formulation)

To begin with the simplest type of scattering problem, we shall assume that the incident particles are deflected by a static force field which is localized in range. The field may be represented by a potential $V(\vec{r})$. We shall take the energy of the incident particle to be

$$E = \frac{\hbar^2 k^2}{2m} \tag{1}$$

where the symbol $\vec{k}$ will be used to represent the propagation vector of the incident wave. Our problem is to solve the Schroedinger equation

$$(\nabla^2 + k^2)\psi(\vec{r}) = \frac{2m}{\hbar^2} V(\vec{r})\psi(\vec{r}) , \tag{2}$$

subject to the boundary condition that at large distances from the region occupied by the potential the wave function has the asymptotic form

$$\psi(\vec{r}) \sim e^{i\vec{k}\cdot\vec{r}} + f(\theta)\frac{e^{ikr}}{r} , \tag{3}$$

i. e., the sum of the incident plane wave and an outgoing spherical wave with scattering amplitude $f(\theta)$. We choose the incident wave to have unit density so that the incident flux is the incident velocity v. The flux scattered through solid angle $d\Omega$ is just

$$|f(\theta)|^2 \frac{1}{r^2} v\, r^2 d\Omega$$

so that the corresponding differential element, $d\sigma$, of the cross section is given by

$$d\sigma = \frac{\text{Flux through } d\Omega}{\text{Incident flux}} = |f(\theta)|^2 d\Omega . \tag{4}$$

The problem, as we have stated it thus far, falls into two parts. It is necessary to find functions which satisfy a partial differential equation and among these to choose the one satisfying an asymptotic boundary condition. Now for many purposes it is useful to have a more unified formulation of the problem, one which incorporates both the Schroedinger equation and its boundary condition. Such a statement may be obtained by means of an integral equation. As the first step in formulating an integral equation we define the Green's function, $G(\vec{r},\vec{r}')$, as a solution of the inhomogeneous wave equation

$$(\nabla^2 + k^2)G(\vec{r},\vec{r}') = \frac{2m}{\hbar^2}\delta(\vec{r}-\vec{r}') . \tag{5}$$

The similarity of this equation to the Poisson equation permits one to see that the solution has the singularity $1/|\vec{r}-\vec{r}'|$. The general solution may easily be seen to be

$$-\frac{2m}{4\pi\hbar^2}\,\frac{\alpha e^{ik|\vec{r}-\vec{r}'|}+\beta e^{-ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}$$

where $\alpha+\beta=1$. We shall define $G(r,r')$ to be the amplitude which corresponds to the steady radiation from a coherent source at r'. We, therefore, choose $\alpha=1$, so that our Green's function is

$$G(\vec{r},\vec{r}') = -\frac{2m}{4\pi\hbar^2}\,\frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \quad . \tag{6}$$

Now, it is easy to see that the expression for $\Psi(r)$ given by

$$\Psi(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} + \int G(\vec{r}-\vec{r}')V(\vec{r}')\Psi(\vec{r}')\,d\vec{r}' \tag{7}$$

satisfies the Schroedinger equation identically. To see if the asymptotic behavior is correct we expand for large $|\vec{r}|=r$, noting that

$$|\vec{r}-\vec{r}'| \longrightarrow r - \frac{\vec{r}'\cdot\vec{r}}{r}$$

as the ratio $|\vec{r}'|/r$ approaches zero. The latter ratio is indeed small when r is large since the region of the r' integration extends only over the region where V is different from zero. Now let us define a propagation vector pointing in the direction $\vec{r}$,

$$\vec{k}_r \equiv |\vec{k}|\frac{\vec{r}}{r} = k\frac{\vec{r}}{r} \quad . \tag{8}$$

From the integral equation

$$\Psi(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} - \frac{2m}{4\pi\hbar^2}\int\frac{e^{ik|r-r'|}}{|\vec{r}-\vec{r}'|}V(\vec{r}')\Psi(\vec{r}')\,d\vec{r}' \tag{9}$$

we obtain as $r\to\infty$

$$\Psi(\vec{r}) \longrightarrow e^{i\vec{k}\cdot\vec{r}} - \frac{2m}{4\pi\hbar^2}\frac{e^{ikr}}{r}\int e^{-i\vec{k}_r\cdot\vec{r}'}V(\vec{r}')\Psi(\vec{r}')\,d\vec{r}'. \tag{10}$$

We see that this has the required asymptotic form. Furthermore, we have obtained an exact expression for the scattering amplitude.

To make the notation more explicit it will be convenient to write

$\Psi_{\vec{k}}(r)$ for the wave function which develops from an incident plane wave $e^{i\vec{k}\cdot\vec{r}}$. We may also replace the symbol $f(\theta)$ for the scattering amplitude by the more general notation $f(\vec{k}',\vec{k})$ which designates the amplitude for scattering from the direction $\vec{k}$ to a direction $\vec{k}'$, where of course we have $|\vec{k}'|=|\vec{k}|$. The scattering amplitude is then given by

$$f(\vec{k}',\vec{k}) = -\frac{2m}{4\pi\hbar^2}\int e^{-i\vec{k}'\cdot\vec{r}}V(\vec{r})\Psi_{\vec{k}}(\vec{r})\,d\vec{r}. \tag{11}$$

From this it is clear that we only need to know the wave function in the region where $V(\vec{r})\neq 0$ in order to have an accurate evaluation of the scattering.

Identities Satisfied by the Scattering Amplitude

There are certain exact relations which the scattering amplitude obeys quite generally, and in developing approximation procedures it is wise never to let these stray far from sight. These relations have to do, first of all, with dynamical reversibility and, secondly, with conservation theorems and orthogonality relations. We shall briefly outline their derivations.

Suppose, first of all, one considers two wave functions $\Psi_{\vec{k}}(\vec{r})$ and $\Psi_{-\vec{k}'}(\vec{r})$ where $|\vec{k}|=|\vec{k}'|$, but $\vec{k}$ and $\vec{k}'$ are in different directions and the minus sign in the index $-\vec{k}'$ is taken only for convenience in expressing the result. It follows from the Schroedinger equation that we have the identity

$$\Psi_{-\vec{k}'}\nabla^2\Psi_{\vec{k}} - \Psi_{\vec{k}}\nabla^2\Psi_{-\vec{k}'} = 0 \; . \tag{12}$$

Integrating this expression over the volume of a large sphere and using Green's Theorem, we find

$$\int_{\substack{\text{Surf. of}\\ \text{Sphere}}}\left\{\Psi_{-\vec{k}'}\frac{\partial}{\partial r}\Psi_{\vec{k}} - \Psi_{\vec{k}}\frac{\partial}{\partial r}\Psi_{-\vec{k}'}\right\}dS = 0 \; . \tag{13}$$

Now this integration is over the surface of a very large spere so we may use the asymptotic form of the wave function in the integrand. By making use of the propagation vector in the direction $\vec{r}$ defined by Eq. (8) the integral relation Eq. (13) may be written as

$$\int\left\{e^{-i\vec{k}'\cdot\vec{r}} + f(\vec{k}_r,-\vec{k}')\frac{e^{ikr}}{r}\right\}\frac{\partial}{\partial r}\left\{e^{i\vec{k}\cdot\vec{r}} + f(\vec{k}_r,\vec{k})\frac{e^{ikr}}{r}\right\}dS - \tag{14}$$

$$-\int\left\{e^{i\vec{k}\cdot\vec{r}}+f(\vec{k}_r,\vec{k})\frac{e^{ikr}}{r}\right\}\frac{\partial}{\partial r}\left\{e^{-i\vec{k}'\cdot\vec{r}}+f(\vec{k}_r,-\vec{k}')\frac{e^{ikr}}{r}\right\}dS=0.$$

The terms which contain the product of two plane waves may be seen, when summed, to furnish a vanishing integral. The terms which contain the product of two scattering amplitudes obviously cancel as well. The only remaining contributions are from terms linear in the scattering amplitude. To demonstrate the method of deriving the identity, and incidentally to introduce a procedure which will later be of use, it suffices to discuss the first of these terms linear in f, which is of the form

$$\frac{1}{r}\int f(\vec{k}_r,-\vec{k}')(\vec{k}\cdot\vec{r}-kr)\,e^{i(\vec{k}\cdot\vec{r}+kr)}\,d\Omega_r. \tag{15}$$

By writing $\mu=\cos(\widehat{\vec{k},\vec{r}})$, and letting φ be an azimuthal angle of $\vec{r}$ about $\vec{k}$, this integral may be written as

$$k\int f(\vec{k}_r,-\vec{k}')(\mu-1)\,e^{ikr(\mu+1)}\,d\mu\,d\varphi.$$

This contains an integral of the general form

$$\int_{-1}^{1}F(\mu)\,e^{ikr\mu}\,d\mu. \tag{16}$$

If the function F is sufficiently smooth the asymptotic form of the latter integral as $r\to\infty$ is easily evaluated. Integration by parts yields

$$\left[\frac{F(\mu)\,e^{ikr\mu}}{ikr}\right]_{-1}^{1}-\frac{1}{ikr}\int F'(\mu)\,e^{ikr\mu}\,d\mu.$$

Further integration by parts will only give terms involving higher powers of $(kr)^{-1}$, which for a large sphere, $(kr\gg1)$ can all be dropped. Therefore, we secure for the integral Eq. (15) the value

$$\frac{k}{ikr}\int d\varphi\,f(\vec{k}_r,-\vec{k}')(\mu-1)\,e^{ikr(\mu+1)}\Big|_{\mu=-1}^{\mu=1}.$$

The contribution of the limit $\mu=1$, is seen to vanish. The limit $\mu=-1$ corresponds to $\vec{r}$ lying in the direction $-\vec{k}$. At this limit the exponential reduces to unity and the azimuthal integration degenerates to the trivial

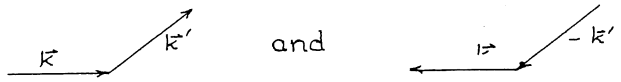
contribution of a factor 2π. The integral Eq. (15) therefore reduces asymptotically to

$$-\frac{2}{ir}\cdot2\pi\cdot f(-\vec{k},-\vec{k}').$$

By evaluating the other integral of Eq. (14) in the same way we obtain the relation

$$f(-\vec{k},-\vec{k}')=f(\vec{k}',\vec{k}). \tag{17}$$

That is, the scattering amplitudes for the sequences of propagation vectors represented by the diagrams



are identically equal. The identity evidently expresses the dynamical reversibility of the system.

Now suppose we have a potential $V(\vec{r})$ which is invariant under inversion through the origin; i. e.,

$$V(\vec{r})=V(-\vec{r}). \tag{18}$$

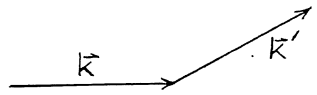
This is a symmetry relation which must also be reflected by a property of the scattering amplitude. In fact the physical situation remains unchanged if all momenta as well as coordinates are inverted, so that the scattering amplitude must have the symmetry

$$f(\vec{k}',\vec{k})=f(-\vec{k}',-\vec{k}). \tag{19}$$

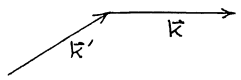
Now combining this relation with the reversibility identity, Eq. (17) we obtain

$$f(\vec{k}',\vec{k})=f(\vec{k},\vec{k}') \tag{20}$$

i. e., the scattering amplitude is the same for the diagram



as it is for



The symmetry Eq. (20) is of a type often invoked in arguments in kinetic theory. It might be referred to as detailed balancing symmetry. Note that no such relation need hold in general in the absence of the inversion symmetry Eq. (18).

Suppose we consider now two wave functions,

$$\psi_{\vec{k}}(r)$$

and a complex conjugate function

$$\psi^*_{\vec{k}'}(r),$$

and play the same game once more. As long as the potential V(r) is real we have

$$\psi^*_{\vec{k}'}\nabla^2\psi_{\vec{k}} - \psi_{\vec{k}}\nabla^2\psi^*_{\vec{k}'} = 0 . \tag{21}$$

Once again we integrate the expression over the volume of a large sphere and use Green's theorem to introduce an integration over the surface of the sphere. A number of the integrals cancel as in Eq. (14) and two terms linear in the scattering amplitude again survive. In addition there is a term quadratic in the scattering amplitudes which also survives because of the operation of complex conjugation which has been introduced in Eq. (21). When the integrals are evaluated asymptotically just as they were before we are left with the identity[9]

$$\frac{1}{2i}\{f(\vec{k}',\vec{k}) - f^*(\vec{k},\vec{k}')\} = \frac{k}{4\pi}\int f^*(\vec{k}_r,\vec{k}')f(\vec{k}_r,\vec{k})\,d\Omega_r . \tag{22}$$

For the case $\vec{k}'=\vec{k}$ this relation reduces to a particularly simple form

$$\begin{aligned} \mathrm{Im}\, f(\vec{k},\vec{k}) &= \frac{k}{4\pi}\int |f(\vec{k}_r,\vec{k})|^2 d\Omega_r \\ &= \frac{k}{4\pi}\sigma_{\text{scatt.}} , \end{aligned} \tag{23}$$

where $\sigma_{\text{scatt.}}$ is the total scattering cross section. This identity, sometimes referred to as the "optical theorem" is easily seen to represent no more than the requirement of particle conservation. Indeed the quantum mechanical particle current for a state $\psi_k(\vec{r})$ is

$$\vec{j} = \frac{\hbar}{2im}\{\psi^*_{\vec{k}}\nabla\psi_{\vec{k}} - \psi_{\vec{k}}\nabla\psi^*_{\vec{k}}\} , \tag{24}$$

so that for $\vec{k}=\vec{k}'$, Eq. (21) represents simply the conservation condition for a stationary state,

$$\nabla\cdot\vec{j} = 0. \tag{25}$$

The particular form of the identity, Eq. (23), may be understood as follows: The flux of scattered particles travelling off in spherical waves is proportional to $\sigma_{\text{scatt.}}$ Such particles can have their origin only in a decrease of the strength of the incident plane wave. This decrease is accomplished by means of a destructive interference between the plane wave and the scattered wave which takes place near the forward direction. Eq. (23) represents the balancing of this destructive interference against the outward flux.

Some further light is shed on the identity Eq. (22) particularly for the case $\vec{k}'\neq\vec{k}$ by brief reference to the time dependent formulation of scattering theory. It is well known that the operator which, applied to the initial state of a particle, yields its final state, must always be unitary. The unitarity condition, in the case of a scattering problem, may be shown to correspond precisely to the identity Eq. (22). Such unitarity conditions not only play the role of enforcing particle conservation, but enforce the preservation of orthogonality relations as well.

For reasons that we shall later discuss at length it is often convenient to deal with the mathematical abstraction of a potential $V(\vec{r})$ which takes on complex values. We shall try to show that this concept is a very natural one to use in describing interactions in systems which produce inelastic scattering in addition to elastic scattering. For the present, however, our interest is only to see how the occurrence of a complex potential as opposed to a real one influences the identities discussed earlier.

The proof of the reversibility theorem, Eq. (17), is, in fact, in no way altered by the presence of a complex potential, i.e., we still have

$$f(-\vec{k},-\vec{k}') = f(\vec{k}',\vec{k}) .$$

The derivation of the unitarity theorem, on the other hand, involved the operation of complex conjugation and, as a consequence, is appreciably altered by giving the potential a nonvanishing imaginary part. Eq. (21) must be replaced by the relation

$$\psi^*_{k'}\nabla^2\psi_k - \psi_k\nabla^2\psi^*_{k'} = \frac{4mi}{\hbar^2}\left(\mathrm{Im}V(\vec{r})\right)\psi^*_{k'}\psi_k \ , \tag{26}$$

which by the same methods we used earlier leads to the identity:

$$\frac{1}{2i}\{f(\vec{k}',\vec{k}) - f^*(\vec{k},\vec{k}')\} = \frac{k}{4\pi}\int f^*(\vec{k}_r,\vec{k}')f(\vec{k}_r,\vec{k})\,d\Omega_r - \frac{m}{2\pi\hbar^2}\int(\mathrm{Im}V(\vec{r}))\psi^*_{k'}\psi_k\,d\vec{r}. \tag{27}$$

Again to secure an interpretation of this identity for the case $\vec{k}=\vec{k}'$ it is instructive to consider the divergence of the particle current vector. In the presence of a complex potential the current is no longer divergenceless. Instead, we have

$$\nabla\cdot\vec{J} = \frac{2}{\hbar}(\mathrm{Im}V)|\psi_k|^2 \tag{28}$$

so that, for example, at a point at which $\mathrm{Im}V<0$, particles are being absorbed at a rate proportional to their local density. Since the incident beam has unit density we may introduce an absorption cross section $\sigma_{abs.}$ for the potential by saying that the total rate at which particles are disappearing is $v\sigma_{abs.}$:

$$v\sigma_{abs.} = -\int\nabla\cdot\vec{j}\,d\vec{r} = -\frac{2}{\hbar}\int(\mathrm{Im}V)|\psi_k|^2 d\vec{r}\,. \tag{29}$$

For $\vec{k}=\vec{k}'$, the identity Eq. (27) may now be written in the form of a generalized "optical theorem,"

$$\mathrm{Im}f(\vec{k},\vec{k}) = \frac{k}{4\pi}\sigma_{scatt.} + \frac{mv}{4\pi}\sigma_{abs.} = \frac{k}{4\pi}(\sigma_{scatt.}+\sigma_{abs.}) = \frac{k}{4\pi}\sigma_{tot.} \tag{30}$$

This simply states that the attenuation of the incident beam by interference is accounted for by the particles absorbed in addition to those scattered. In particular this relation shows clearly that absorption is always accompanied by scattering, at least in the forward direction, i. e., $\sigma_{abs.}>0$ implies

$$\mathrm{Im}f(\vec{k},\vec{k})>0,$$

from which it follows that

$$|f(\vec{k},\vec{k})|^2 > 0.$$

Partial Wave Expansion

The most familiar approach to scattering problems is based on an expansion of the unknown wave function in sperical harmonics. We shall only give the briefest of outlines of this method in order that it may be compared with procedures to be developed later. Further details are widely available in the literature.

When a scattering potential has axial symmetry about the propagation direction of an incident wave, the wave function may be expanded in the series

$$\psi_k(r) = \sum_l g_l(kr)P_l(\mu) \tag{31}$$

where $P_l(\mu)$ is the l-th Legendre Polynomial with argument $\mu=\cos(\hat{k},\hat{r})$. This expansion is particularly convenient for central potentials since the conservation of orbital angular momentum causes the Schroedinger equation to reduce to a succession of uncoupled ordinary differential equations for the radial functions g_l.

It is easily shown that at large distances from the scattering potential the functions $g_l(kr)$ are simple superpositions of incoming and outgoing spherical waves. The asymptotic forms of these functions may be written as

$$g_l(kr) \sim -\frac{2l+1}{2ikr}\left\{(-1)^l e^{-ikr} - C_l e^{ikr}\right\}, \tag{32}$$

where the coefficients have been chosen so that $C_l=1$ when the scattering potential vanishes, and

$$\psi_k(r) = e^{i\vec{k}\cdot\vec{r}}.$$

The deviations of the coefficients C_l from unity furnish a description of the effect of the potential on the outgoing wave, i. e., the scattering.

The asymptotic form of the entire wave function is evidently given by

$$\psi_k(r) - e^{i\vec{k}\cdot\vec{r}} \sim \frac{e^{ikr}}{2ikr}\sum_l(2l+1)(C_l-1)P_l(\mu),$$

which simply means that the scattering amplitude is

$$f(\theta) = \frac{1}{2ik}\sum_l (2l+1)(C_l - 1) P_l(\mu) . \tag{33}$$

Now it is easy to verify from the unitarity relation Eq. (22) that as long as the number of particles is conserved, i. e., $V(r)$ is real the coefficients C_l have absolute value unity. Hence it is customary to write

$$C_l = e^{2i\delta_l} . \tag{34}$$

When $V(r)$ is complex, however, the phase shifts δ_l become complex as well. In particular, when the potential is absorptive,

$$\mathrm{Im}\, V(r) < 0$$

the coefficients C_l must have absolute value less than unity, and hence

$$\mathrm{Im}\, \delta_l > 0 .$$

It is a simple matter to integrate the scattering cross section to find

$$\sigma_{scatt.} = \int |f(\theta)|^2 d\Omega = \frac{\pi}{k^2}\sum_l (2l+1)|C_l - 1|^2 . \tag{35}$$

The total cross section, on the other hand, is given still more simply by the generalized optical theorem:

$$\sigma_{tot.} = \frac{4\pi}{k}\mathrm{Im} f(0) = \frac{2\pi}{k^2}\sum_l (2l+1)(1 - \mathrm{Re}\, C_l) . \tag{36}$$

The absorption cross section is therefore

$$\sigma_{abs.} = \sigma_{tot.} - \sigma_{scatt.} = \frac{\pi}{k^2}\sum_l (2l+1)(1 - |C_l|^2) . \tag{37}$$

While these formulae appear to state the results quite explicitly, one must remember that the numbers C_l are in general not very easily found. To find each of them one must solve a radial differential equation, a task which usually can only be carried out numerically. A further complication which is particular to problems at high energies is that a large number of terms must usually be computed before the above series converge. This is to be expected since, as the energy of the incident particles rises, particles of higher angular momentum pass within the range of the potential and are deflected by it, i. e., if a is the range of the

potential, the maximum angular momentum which is affected appreciably is of order

$$a/\lambda = ka .$$

In particular the solutions of some familiar problems have recently required the computation of from 20 to 40 phase shifts.

Our goal in the following will be to develop approximation methods which allow quick and fairly accurate estimates of the solutions to certain scattering problems for $ka \gg 1$. One must emphasize, however, that where high accuracy is desired no adequate substitute has yet been found for the phase shift series, however tedious its evaluation may be.

The Born and W. K. B. Approximations

In order to set our later work in a more familiar context it may be useful to say a few words about two approximation methods which are widely used in application to high-energy problems. The first of these is the Born approximation, which is simply a perturbation expansion of the scattering amplitude in powers of the scattering potential; i. e., for potentials $V(\vec{r})$ which are not too strong, the integral equation (7) may be solved by a Liouville-Neumann expansion

$$\psi_k(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} + \int G(\vec{r}-\vec{r}')V(\vec{r}')e^{i\vec{k}\cdot\vec{r}'}d\vec{r}' + \iint G(\vec{r}-\vec{r}')V(\vec{r}')G(\vec{r}'-\vec{r}'')V(\vec{r}'')e^{i\vec{k}\cdot\vec{r}''}d\vec{r}'d\vec{r}'' + \cdots . \tag{38}$$

This series for the wave function may then be substituted in the expression Eq. (11) for the scattering amplitude to yield

$$f(\vec{k}',\vec{k}) = -\frac{2m}{4\pi\hbar^2}\left\{\int e^{i\vec{k}'\cdot\vec{r}}V(\vec{r})e^{i\vec{k}\cdot\vec{r}}d\vec{r} + \iint e^{i\vec{k}'\cdot\vec{r}}V(\vec{r})G(\vec{r}-\vec{r}')V(\vec{r}')e^{i\vec{k}\cdot\vec{r}'}d\vec{r}d\vec{r}' + \cdots\right\} . \tag{39}$$

Very wide application to scattering problems has been made of the first term of this expansion since its simplicity is often felt to compensate for all it may lack in accuracy. Unfortunately its error is usually substantial, and further terms of the series, of order higher than the first or second, are quite difficult to compute. The rate at which the Born series converges depends on the strength of the scattering and that in turn depends on the strength of the potential V, and the length of time a/v (v is the velocity) that the particle spends within the potential. The scattering will be sufficiently weak to insure rapid convergence only if the passage time a/v is much smaller than the time $\hbar/V$ required for the potential to influence the particle appreciably, i. e., rapid convergence requires

$$\frac{Va}{\hbar v} \ll 1 . \tag{40}$$

Unfortunately, this condition is virtually never fulfilled in nuclear collisions. The problems to which it may safely be applied are usually those involving the electromagnetic interactions of fast charged particles with other particles of charge, say,

$$Z \lesssim 10 .$$

Another approximation, the W. K. B. method, corresponds to the classical limit of quantum mechanics. We assume the potential to be smooth enough that the distance over which it changes its value appreciably is large compared to the wavelength, i. e., $ka \gg 1$. Now if the kinetic energy of the incident particles is large enough, the scattering will be heavily concentrated at small angles. To see this, we will suppose that θ is a typical scattering angle. Then as an order of magnitude approximation we may write

$$\theta \sim \frac{\Delta p}{p} ,$$

where p is the original momentum and Δp is the change of momentum. Hence θ is given roughly by

$$\theta \sim \frac{\int F dt}{p} ,$$

where F is the force acting on the particle. The magnitude of the force is

$$F \sim \frac{V}{a} .$$

Now the time taken by the particle in crossing the potential is approximately

$$t = \int dt \sim \frac{a}{v}$$

so that

$$\theta \sim \frac{V}{pv} \sim \frac{V}{E} . \tag{41}$$

which, shows as one might expect, that at sufficiently high energies the

scattering takes place predominantly through small angles.

Now the problem, of course, is not a classical but a quantum mechanical one. When a particle is known to pass through a region of dimension a the transverse component of its momentum is rendered uncertain to the extent

$$\delta p \sim \frac{\hbar}{a} .$$

The angle through which it is scattered is therefore rendered uncertain to the extent

$$\delta\theta \sim \frac{\hbar}{ap} .$$

The situation is a semi-classical one only in the limit in which particle trajectories become uniquely defined, in particular in the limit in which the uncertainty in the scattering angle becomes much smaller than the scattering angle itself

$$\frac{\theta}{\delta\theta} \sim \frac{V}{E} \frac{ap}{\hbar} \gg 1$$

or

$$\frac{Va}{\hbar v} \gg 1 . \tag{42}$$

Thus the region of applicability of the W. K. B. method does not overlap that of the Born approximation at all. We have in fact a considerable gap of values of

$$\frac{Va}{\hbar v}$$

to which neither method applies. (A unique exception to this statement must be made, however, for the case of one-dimensional problems. In these the scattering angle is either 0 or π and the argument used to derive Eq. (42) no longer applies. In one dimension the W. K. B. approximation may be accurate for arbitrary values of

$$\frac{Va}{\hbar v} .)$$

Since a certain ambiguity of terminology regarding the W. K. B. method is prevalent, it may be of help to spend a moment considering the approximation explicitly. We write the wave function in the form

$$\psi_k(r) = e^{\frac{i}{\hbar}S^{(0)}+S^{(1)}} \tag{43}$$

where $S^{(0)}$ and $S^{(1)}$ are functions to be determined. We define the magnitude of the momentum, as it depends on position, by

$$p^2(r) = 2m[E-V(r)]. \tag{44}$$

We then substitute these into the Schroedinger equation and secure

$$-\left(\nabla(S^{(0)}-i\hbar S^{(1)})\right)^2 + i\hbar\nabla^2(S^{(0)}-i\hbar S^{(1)}) + p^2(r) = 0.$$

Now, we consider the exponent of the expression above for $\psi_k(r)$ as an expansion in powers of $\hbar$, so that this last equation gives us the two equations

$$(\nabla S^{(0)})^2 = p^2(r) \tag{45}$$

and

$$2\nabla S^{(0)}\cdot\nabla S^{(1)} + \nabla^2 S^{(0)} = 0. \tag{46}$$

The first of these is just the classical Hamilton-Jacobi equation whose solution may be expressed as the action integral

$$S^{(0)}(r) = \int_{\substack{\text{dynamical}\\ \text{path}}}^{r} p(r')\,dr'. \tag{47}$$

Note that, written in the form

$$\nabla S^{(0)} = \vec{p}(r),$$

his equation states that the possible trajectories of the particle are nornal to the surfaces of constant $S^{(0)}$.

By using the integrating factor

$$e^{2S^{(1)}}$$

he second equation yields

$$\nabla\cdot\left(\nabla S^{(0)} e^{2S^{(1)}}\right) = 0$$

or

$$\nabla\cdot\left(\vec{p}(r)|\psi|^2\right) = 0 \tag{48}$$

which, on interpreting

$$\frac{1}{m}\vec{p}(r)$$

as the velocity field and

$$|\psi|^2$$

as the density of particles, is seen to be just the hydrodynamical equation of continuity. Now suppose a beam of particles is projected toward a scattering center. The density of particles, according to the equation, is at all points the density present if the particles simply move along their classical trajectories. Hence cross sections, for example, computed by the W. K. B. method will be precisely the same as those computed classically. (An exception arises only if the potential is sufficiently complicated that more than one classical trajectory can lead to the point of observation. In that case, Eq. (43) must be replaced by a sum of similarly constructed terms, one for each path, and non-classical interference effects arise.)

A number of authors have applied the name of the W. K. B. approximation to a rather different procedure based on the partial wave expansion. They integrate the radial equations to find the individual phase shifts by a one-dimensional W. K. B. method. The region of applicability of this procedure is altogether different from that of the one described above, although, of course, the two overlap at the classical extreme. It is questionable whether such an alternative procedure should be referred to simply as the W. K. B. approximation since its use of exact angular eigenfunctions leads to such different mathematical properties. This approximation is, in fact, related to one which we shall describe presently.

The High-Energy Approximation in One Dimension

We shall now begin the development of an approximation which is better suited to many of the purposes of high-energy studies than any of the methods mentioned earlier. While the method to be discussed is not without limitations of its own, these, as we shall see, allow one to estimate correctly the intensity of a predominant part of the scattering.

To begin the development at the simplest possible point, and one that will later prove quite useful, we shall consider a one-dimensional scattering problem. Of course it is necessary to bear in mind a very special property of scattering problems in one dimension. The scattering process can take place in only two directions, either preserving the sense of motion of the particle or sending it directly backward. There are no compromises. While this makes the problem a trifle unrealistic, it has the advantage of making it mathematically more transparent.

The Schroedinger equation in one dimension is

$$\left(\frac{d^2}{dx^2} + k^2\right)\psi(x) = \frac{2m}{\hbar^2}V(x)\psi(x) . \tag{49}$$

Now we shall assume that the energy of the incident particle greatly exceeds the magnitude of the potential $V(x)$, and is also large enough that the particle wavelength is much smaller than the potential width a

$$\frac{V}{E} \ll 1 , \quad ka \gg 1 . \tag{50}$$

(In order of magnitude relations such as this the symbol V is to be interpreted as a measure of the absolute magnitude of the potential.) Under these conditions we are justified in assuming that back-scattering will be very weak, that the wave function of the particle may to a good approximation be written in the form

$$\psi(x) = e^{ikx}\varphi(x) , \tag{51}$$

where $\varphi(x)$ is a function which varies slowly over a particle wavelength. Substituting into the Schroedinger equation, we secure

$$\left(2ik\frac{d}{dx} + \frac{d^2}{dx^2}\right)\varphi(x) = \frac{2m}{\hbar^2}V(x)\varphi(x) . \tag{52}$$

Now our approximation consists in dropping the

$$\frac{d^2}{dx^2}$$

term since we assume φ varies slowly in a wavelength. In that case the equation reduces to

$$\frac{d\varphi}{dx} = -\frac{i}{\hbar v}V(x)\varphi(x) . \tag{53}$$

Now if Eq. (51) is to reduce to the incident plane wave at $x=-\infty$ (i. e., back-scattering is neglected) we require as a boundary condition $\varphi(-\infty)=1$. Thus we secure

$$\varphi(x) = e^{-\frac{i}{\hbar v}\int_{-\infty}^{x}V(x')dx'} \tag{54}$$

and

$$\psi(x) = e^{ikx - \frac{i}{\hbar v}\int_{-\infty}^{x}V(x)dx} . \tag{55}$$

Note that if the exponential, $\varphi(x)$, were expanded in a power series, the successive terms would represent those of the Born approximation series. In this way, we may verify directly that the expansion parameter of the series is

$$Va/\hbar v .$$

The approximation we have just described may be derived in another way which is also fairly instructive. Here we begin with the one-dimensional version of the integral equation for scattering

$$\psi(x) = e^{ikx} + \int G(x-x')V(x')\psi(x')dx' . \tag{56}$$

The one-dimensional Green's function we require may be expressed as

$$G(x-x') = -\frac{m}{\pi\hbar^2}\int_{-\infty}^{\infty}\frac{e^{i\lambda(x-x')}}{\lambda^2-k^2-i\epsilon}d\lambda , \tag{57}$$

where the outgoing wave boundary condition for G requires that we take the limit of this expression as $\epsilon\to 0$ through positive values. The result is simply

$$G(x-x') = -\frac{i}{\hbar v}e^{ik|x-x'|} . \tag{58}$$

We again express $\psi(x)$ in the form

$$\psi(x) = e^{ikx}\varphi(x) ,$$

so that the integral equation for φ becomes

$$\varphi(x) = 1 - \frac{i}{\hbar v}\int_{-\infty}^{\infty} e^{ik|x-x'|-ik(x-x')}V(x')\varphi(x')dx' \qquad (59)$$

$$= 1 - \frac{i}{\hbar v}\int_{-\infty}^{x} V(x')\varphi(x')dx' - \frac{i}{\hbar v}\int_{x}^{\infty} e^{2ik(x-x')}V(x')\varphi(x')dx'. \qquad (60)$$

When the two regions of integration $x'<x$ and $x'>x$ are separated, their integrands are seen to vary in altogether different ways. Now if the functions $V(x)$ and φ both vary slowly in a particle wavelength, the rapidly oscillating exponential in the second integrand may be expected to reduce its contribution considerably in magnitude. As a first approximation, therefore, we shall neglect the integral over the region $x'>x$. It is clear from the form of this integral that we are thereby neglecting back-scattering. The integral equation which remains is simply

$$\varphi(x) = 1 - \frac{i}{\hbar v}\int_{-\infty}^{x} V(x')\varphi(x')dx', \qquad (61)$$

which may be solved trivially by differentiating, so that we are again led to the differential equation

$$\frac{d\varphi(x)}{dx} = -\frac{i}{\hbar v}V(x)\varphi(x)$$

with the boundary condition

$$\varphi(-\infty) = 1,$$

and the solution

$$\psi(x) = e^{ikx - \frac{i}{\hbar v}\int_{-\infty}^{x}V(x')dx'}. \qquad (55)$$

The restrictions underlying this result may be clearly seen from the above. We require that both $V(x)$ and $\varphi(x)$ vary slowly within a wavelength. The first of these conditions is $ka \gg 1$ where a is the width of the potential. The second condition, is indicated by the form derived for $\varphi(x)$. We evidently require

$$k \gg \frac{V}{\hbar v}$$

or

$$1 \gg V/E.$$

These are the conditions, Eq. (50), stated earlier.

It should be noted particularly that even though the assumptions

$$ka \gg 1$$

and

$$V/E \ll 1$$

are required above, no restriction has been placed on the product of these two quantities. Now their product is

$$ka \cdot \frac{V}{E} = 2\frac{Va}{\hbar v}, \qquad (62)$$

so we see that the present approximation, in contrast with those discussed earlier, remains valid for arbitrary values of the important parameter

$$Va/\hbar v.$$

Before discussing higher approximations we might point out that the form Eq. (55) for the wave function may also be reached by means of the W. K. B. method. One has only to expand the familiar W. K. B. approximation to a one-dimensional wave function in power of V. But, unfortunately this is a shortcut confined to one dimensional problems. The one-dimensional W. K. B. approximation, as we have noted earlier, is unusual in that it need not require

$$Va/\hbar v \gg 1,$$

and may in this case overlap the present approximation. In two or more dimensions, however, this overlap disappears. The direct generalization of the method we are discussing yields results which only coincide with those of the W. K. B. approximation in the limit

$$Va/\hbar v \longrightarrow \infty.$$

In order to improve the accuracy of the approximation, explicit account must be taken of the back-scattered wave. For this purpose, we write the wave function as,

$$\psi(x) = e^{ikx}\varphi_+(x) + e^{-ikx}\varphi_-(x). \qquad (63)$$

a form which possesses sufficient generality that the functions φ_+ and φ_- both need only vary slowly within a wavelength. There are various ways of writing equations for the determination of $\varphi_\pm$. One method is to construct a pair of coupled integral equations equivalent to Eq. (56). A much simpler procedure, however, is to write a pair of coupled first order differential equations equivalent to the Schroedinger equation,

$$\frac{d\varphi_+(x)}{dx} = -\frac{i}{\hbar v}V(x)\varphi_+(x) - \frac{i}{\hbar v}e^{-2ikx}V(x)\varphi_-(x) \tag{64}$$

$$\frac{d\varphi_-(x)}{dx} = +\frac{i}{\hbar v}V(x)\varphi_-(x) + \frac{i}{\hbar v}e^{2ikx}V(x)\varphi_+(x) . \tag{65}$$

The second order equation for $\psi(x)$ to which these reduce is precisely the wave Eq. (49). The boundary conditions on $\varphi_\pm$ may be seen from the form of the expression Eq. (63) for $\psi(x)$. Since a plane wave is incident from $-\infty$ we choose

$$\varphi_+(-\infty) = 1 ,$$

and since there can be no back-scattered wave present at $+\infty$, we choose

$$\varphi_-(\infty) = 0 .$$

The amplitude for back-scattering may then be defined to be

$$\varphi_-(-\infty) .$$

If back-scattering is neglected, i.e., φ_- is assumed to vanish as a first approximation, Eq. (64) reduces to the form, Eq. (53), considered earlier. Since the back-scattering is characteristically quite weak under the conditions Eq. (50), an expansion begun in this way must converge rapidly. We may therefore use an iteration procedure in the solution, writing $\varphi_\pm$ in the form

$$\begin{aligned} \varphi_+ &= \varphi_+^{(0)} + \varphi_+^{(2)} + \cdots , \\ \varphi_- &= \varphi_-^{(1)} + \varphi_-^{(3)} + \cdots , \end{aligned} \tag{66}$$

where the numerical indices indicate the number of iterations required to reach that particular term, i. e., we begin with

$$\varphi_+^{(0)}(x)$$

given by Eq. (54), substitute this in Eq. (65), solve for

$$\varphi_-^{(1)}(x)$$

then use this in Eq. (64), and so on.

The first approximation to the back-scattering amplitude is found to be

$$\varphi_-^{(1)}(-\infty) = -\frac{i}{\hbar v}\int_{-\infty}^{\infty} e^{2ikx}V(x)e^{-\frac{2i}{\hbar v}\int_{-\infty}^{x}V(x')\,dx'}dx . \tag{67}$$

If the potential V is as smooth as assumed, and

$$\frac{V}{E} \ll 1 ,$$

this is indeed quite small. The higher terms in the expansions of the forward and backward amplitudes introduce corrections which are of order V/E and 1/ka relative to Eq. (54) and Eq. (67).

The High-Energy Approximation in Three Dimensions

Three-dimensional problems are, of course, the ones that interest us most. We shall try to develop a means of treating these that stays as close as possible in spirit to the one-dimensional procedure just described. The integral equation in three dimensions is

$$\psi_k(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} - \frac{2m}{4\pi\hbar^2}\int\frac{e^{ik|r-r'|}}{|r-r'|}V(r')\psi_k(r')d\vec{r}' .$$

Here again we separate the wave function into the product of the incident plane wave and a factor which modulates it

$$\psi(\vec{r}) = e^{i\vec{k}\cdot\vec{r}}\varphi(\vec{r}) \tag{68}$$

and obtain the equation for φ

$$\varphi(\vec{r}) = 1 - \frac{2m}{4\pi\hbar^2}\int\frac{e^{ik|\vec{r}-\vec{r}'|-i\vec{k}\cdot(\vec{r}-\vec{r}')}}{|\vec{r}-\vec{r}'|}V(\vec{r}')\varphi(\vec{r}')d\vec{r}' . \tag{69}$$

We now define a new position variable $\vec{r}''$ by

$$\vec{r}'' = \vec{r} - \vec{r}' ,$$

so that the above equation becomes

$$\psi(\vec{r}) = 1 - \frac{2m}{4\pi\hbar^2}\int \frac{e^{i(kr''-\vec{k}\cdot\vec{r})}}{r''} V(r-r'')\psi(r-r'')\,d\vec{r}''. \qquad (70)$$

We also assume, as before, that the product $V\psi$ varies slowly within a particle wavelength, $1/k$, so that negligibly small contributions to the integral on the right come from regions in which the exponential oscillates rapidly. If we consider points $\vec{r}$ which lie within the volume occupied by the potential, the largest contributions to the integral will come from values of $\vec{r}''$ lying close in direction to $\vec{k}$, since for these the exponential is nearly stationary. The quantitative expression of this approximation is obtained by carrying out the angular integration over $\vec{r}''$ by means of the asymptotic method discussed earlier in connection with the expression Eq. (16). The differential element $d\vec{r}''$ may be written as

$$d\vec{r}'' = r''^2\, dr''\, d\mu\, d\phi,$$

where

$$\mu = \cos(\hat{k}, \hat{r}'')$$

and ϕ is an azimuthal angle.

Now, let us suppose, to be specific, that the product $V\psi$ varies appreciably only within a distance d. We shall discuss this distance later, but for the time being we shall assume it to be much larger than $\lambda\!\!\!^{-} = 1/k$. If the integration over μ is carried out by parts and only the leading term retained, we secure

$$\psi(\vec{r}) = 1 + \frac{2m}{4\pi\hbar^2}\int \frac{r''^2 dr'' d\varphi}{r''}\left[\frac{e^{ikr''(1-\mu)}}{ikr''} V(r-r'')\psi(r-r'')\right]_{\mu=-1}^{\mu=1} + O\left(\frac{1}{kd}\right)$$

The terms neglected by the asymptotic approximation are as indicated, of relative order $1/kd$. Now the limit $\mu=-1$ corresponds to $\vec{r}''$ anti-parallel to $\vec{k}$. Since the exponential varies rapidly in this case the contributions of the $\mu=-1$ term is of order $1/kd$ and is therefore negligibly small. We are thus left simply with the term corresponding to $\vec{r}''$ parallel to $\vec{k}$,

$$\psi(\vec{r}) = 1 - \frac{i}{\hbar v}\int_0^\infty V(r-r'')\psi(r-r'')\Big|_{\vec{r}''\parallel\vec{k}}\, dr'' \qquad (71)$$

$$\uparrow \vec{r}''$$

The appearance of this equation is somewhat simpler in cartesian coordinates. We choose the positive z axis to lie in the direction of propagation $\vec{k}$, thus obtaining

$$\psi(x,y,z) = 1 - \frac{i}{\hbar v}\int_{-\infty}^{z} V(x,y,z')\,\psi(x,y,z')\,dz' . \qquad (72)$$

This equation is seen to be of precisely the form encountered in the one-dimensional problem. In fact the present approximation treats the three-dimensional problem as a bundle of parallel one-dimensional ones. The solution to Eq. (72) is immediately seen to be

$$\psi(x,y,z) = e^{-\frac{i}{\hbar v}\int_{-\infty}^{z} V(x,y,z')\,dz'} \qquad (73)$$

so that the approximate wave function is

$$\psi(x,y,z) = e^{ikz - \frac{i}{\hbar v}\int_{-\infty}^{z} V(x,y,z')\,dz'} . \qquad (74)$$

Now this expression is missing a good many of the things one looks for in a three-dimensional wave function, e. g., a spherical outgoing wave, but we must remember that the arguments from which it is derived are only intended to hold within the volume occupied by the potential. The expression Eq. (74) therefore need not represent the wave function well elsewhere. Fortunately, as we pointed out in connection with Eq. (11), it is only necessary to know the wave function within the volume of the potential in order to find the scattering amplitude.

Before evaluating the scattering amplitude, it may be of interest to indicate another way of phrasing the approximation just described.[10] For this purpose we introduce the Fourier integral representation of the three dimensional Green's function

$$G(\vec{r}-\vec{r}') = -\frac{2m}{(2\pi)^3\hbar^2}\int_{\epsilon\to+0} \frac{e^{i\vec{\lambda}\cdot(\vec{r}-\vec{r}')}}{\lambda^2 - k^2 - i\epsilon}\, d\vec{\lambda} . \qquad (75)$$

Now the situation we are attempting to describe, in which we characteristically require

$$ka \gg 1$$

and

$$V/E \ll 1$$

is one in which, as we have noted earlier, the scattering is heavily concentrated at small angles. It is, in fact, very unlikely that in traversing the potential the particle will be deflected greatly from its initial direction $\vec{K}$. If we were to treat the problem in momentum space, we could secure an approximation to the wave function by expanding its momentum dependence about the value $\vec{K}$. An equivalent procedure may be based on the expansion of the momentum space dependence of the Green's function about the point $\vec{K}$, i. e., in the problem in question only values of $\vec{\chi}$ near $\vec{K}$ in the integrand of Eq. (75) will play roles of any importance. We may therefore write

$$\vec{\chi} = \vec{K} + \vec{n}$$

in the integrand, retaining the first power of n and expanding higher powers. When this reduced form of the Green's function is substituted into the usual three-dimensional integral equation, the equation simplifies once more to the form (72).

This second form of the approximation, although a bit more cumbersome than the first, brings out a certain limitation of the approach more clearly. The approximate wave function, Eq. (74), is only adequate for the treatment of small-angle scattering. It does not contain, in general, a correct estimate of the Fourier amplitudes corresponding to large momentum transfer. In fact, quantitatively, the limitation on scattering angles may be shown to be given roughly by

$$\Theta^2 kd << 1 \tag{76}$$

where d is again the distance within which $V\varphi$ varies. Although this limitation is a strong one, we shall see that it is nevertheless consistent with an accurate representation of the total scattered intensity.

In order to evaluate the scattering amplitude, it will be convenient to define certain coordinate vectors. Let $\hat{K}$ be a unit vector,

$$|\hat{K}| = 1,$$

pointing in the incident propagation direction $\vec{K}$ which, as before, will also be taken to determine the positive z-axis. Then any position vector $\vec{r}$ may be resolved into two components.

$$\vec{r} = \vec{b} + \hat{K}z, \tag{77}$$

where $\vec{b}$ is a vector lying in a plane perpendicular to $\vec{K}$ (see Fig. 1). With this notation, $\psi(\vec{r})$ as given by Eq. (74) may be written

$$\psi(\vec{r}) = e^{i\vec{K}\cdot\vec{r} - \frac{i}{\hbar v}\int_{-\infty}^{z} V(\vec{b}+\hat{K}z')\,dz'}. \tag{78}$$

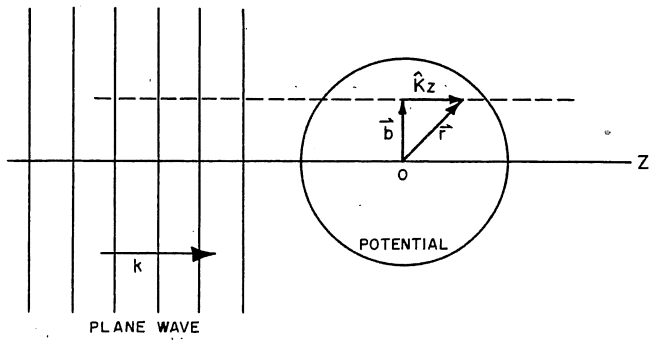


Fig. 1

Here, if the potential is centered at the origin, the distance $b = |\vec{b}|$ evidently has the interpretation of an impact parameter.

Now, substituting the above wave function into the expression Eq. (11) for the scattering amplitude, we secure

$$f(\vec{k}',\vec{k}) = -\frac{2m}{4\pi\hbar^2}\int e^{-i\vec{k}'\cdot\vec{r}} V(\vec{r}) e^{i\vec{k}\cdot\vec{r} - \frac{i}{\hbar v}\int_{-\infty}^{z} V(\vec{b}+\hat{K}z')\,dz'}\,dz\,d^{(2)}b$$

where $d^{(2)}b$ denotes integration over the plane of impact vectors. This can be written as

$$f(\vec{k}',\vec{k}) = -\frac{2m}{4\pi\hbar^2}\int e^{i(\vec{k}-\vec{k}')\cdot(\vec{b}+\hat{K}z)} V(\vec{b}+\hat{K}z) e^{-\frac{i}{\hbar v}\int_{-\infty}^{z} V(\vec{b}+\hat{K}z')\,dz'}\,dz\,d^{(2)}b. \tag{79}$$

Now, energy conservation requires $|\vec{k}'| = |\vec{k}|$ so that for small scattering angles the vector $\vec{k}-\vec{k}'$ is nearly perpendicular to $\vec{k}$. In fact, the error of approximating the exponential

$$\exp[i(\vec{k}-\vec{k}')\cdot\hat{K}z]$$

by unity is only of order

$$(1-\cos\Theta)kd \sim \Theta^2 kd,$$

where θ is the scattering angle and d the distance within which $V\varphi$ varies appreciably. As we have noted earlier, the limitation

$$\theta^2 kd \ll 1$$

already underlies the wave function used so that this approximation introduces no further restriction. With this simplification, the z-integration is simply that of an exact differential and leads to

$$f(\vec{k}',\vec{k}) = \frac{k}{2\pi i}\int e^{i(\vec{k}-\vec{k}')\cdot\vec{b}}\left\{e^{-\frac{i}{\hbar v}\int_{-\infty}^{\infty}V(\vec{b}+\hat{k}z)dz'} - 1\right\} d^{(2)}b \ . \qquad (80)$$

This is the basic result for elastic scattering. It may be thought of as corresponding to a picture in which each portion of the incident wave passes through the potential along a straight-line path and suffers a shift of phase characteristic of that path.

For potentials with azimuthal symmetry we may carry the integration one step further by noting that

$$\frac{1}{2\pi}\int_0^{2\pi} e^{i\lambda\cos\phi}\,d\phi = J_0(\lambda) \qquad (81)$$

where $J_0(\lambda)$ is the zeroth order Bessel function. So, using the small-angle expression

$$(\vec{k}-\vec{k}')\cdot\vec{b} = kb\theta\cos\phi$$

we obtain for axially symmetric potentials

$$f(\theta) = \frac{k}{i}\int_0^{\infty} J_0(kb\theta)\{e^{i\chi(b)} - 1\}\,b\,db \ , \qquad (82)$$

where

$$\chi(\vec{b}) = -\frac{1}{\hbar v}\int_{-\infty}^{\infty} V(\vec{b}+\hat{k}z)\,dz \ . \qquad (83)$$

While the methods used in carrying out the above approximation may not be too familiar, results of the form indicated have a long and venerable history in the study of physical optics. The expression Eq. (82) may be recognized as the type of formula one would use to discuss diffraction by a transparent obstacle. The traditional way, for example, of finding the diffraction pattern formed by a thin lens would proceed as follows: one would assume that the wave is given within and near the lens by a modulated plane wave such as Eq. (76). This expression would be used to evaluate the wave on an imaginary screen placed immediately behind the lens and it would then be possible to evaluate the wave in the half-space behind the screen by the use of Green's theorem.

Precisely this method of deriving Eq. (82) as a quantum mechanical result was introduced by Molière[1] in 1947. The classic derivation, however, using the imaginary screen, becomes rather awkward in the presence of long range or slowly decreasing potentials. It is furthermore a bit less easy to generalize in other directions than the procedure we have outlined.

We shall now take a closer look at the accuracy and the limitations of the approximations we have made. We have assumed that if $V\varphi$ varies in a distance d, we may consistently neglect terms of relative order $1/kd$. Now what is the distance d? The potential varies in a distance a, and according to Eq. (73), $\varphi(\vec{r})$ varies appreciably in a distance $\hbar v/V$. Evidently the distance d is, in order of magnitude, the smaller of these, i.e., for

$$\frac{Va}{\hbar v} < 1$$

we have

$$d \sim a$$

and for

$$\frac{Va}{\hbar v} > 1$$

we have

$$d \sim \hbar v/V \ .$$

In either of these cases we evidently require, as before, both the conditions

$$ka \gg 1 \qquad \text{and} \qquad E/V \ll 1 \qquad (84)$$

To find the angular range of the approximation we remember the limitation

$$\theta^2 kd \ll 1 \ .$$

For the case

$$Va/\hbar v < 1$$

this means the approximation is only consistent for angles smaller than an angle of order of magnitude

$$1/\sqrt{ka}\ ;$$

hence

$$\theta < O\left(\frac{1}{\sqrt{ka}}\right) \quad \text{for} \quad \left(\frac{Va}{\hbar v} < 1\right). \tag{85}$$

For

$$Va/\hbar v > 1$$

on the other hand, the limitation is

$$\theta < O\left(\sqrt{\frac{V}{E}}\right) \quad , \quad \left(\frac{Va}{\hbar v} > 1\right). \tag{86}$$

Both of these, according to our assumptions Eq. (84), are indeed small angles. However, it is quite important to note that nearly all of the scattered intensity is concentrated, in both cases, at angles which are much smaller still, i. e., for

$$Va/\hbar v < 1$$

it is clear from the Born approximation that an average angle of scattering is

$$\langle\theta\rangle \sim \frac{1}{ka} \quad , \quad \left(\frac{Va}{\hbar v} < 1\right), \tag{87}$$

while for

$$Va/\hbar v > 1$$

the argument given in connection with the W. K. B. method (see relation (41)) shows

$$\langle\theta\rangle \sim \frac{V}{E} \quad , \quad \left(\frac{Va}{\hbar v} > 1\right). \tag{88}$$

In both extremes the typical scattering angles are well within the angular range of the approximation. These inequalities have a most important consequence. They allow the approximation to furnish satisfactory estimates of total cross sections in spite of its limited angular range. We shall presently demonstrate the self-consistency of this feature of the high-energy approximation by showing that its expression for the small-angle scattering amplitude obeys the optical theorem.

While the approximation we have been discussing is limited in application to small-angle scattering, there is a simple way in which its angular range may be increased somewhat. This corresponds to removing an artificial asymmetry which we have allowed to creep into the derivation of the approximate expression for the scattering amplitude. In the foregoing derivation the incident propagation vector $\vec{k}$ played a unique role in the determination of the approximate wave function while the final propagation vector $\vec{k}'$ appeared only in a plane wave factor. Now the scattering amplitude always has a certain symmetry under the interchange of these two vectors. The most general such symmetry is that under velocity reversal which we noted earlier, in Eq. (17). It is a simple matter to find a formulation of the scattering amplitude, which has the advantage over Eq. (11) of being more manifestly symmetric under velocity reversal. We shall not go into the details of the altered derivation since, to a large degree, they resemble those we have just discussed. Suffice it to say, that the only change required in the result Eq. (80) for the scattering amplitude is a slight shift in the direction of the unit vector $\hat{R}$. Instead of choosing $\hat{R}$ to lie in the incident direction, we choose it to lie half-way between the incident and final directions, i. e., we choose

$$\hat{K} = \frac{\vec{k} + \vec{k}'}{|\vec{k} + \vec{k}'|} \ . \tag{89}$$

The revised approximation thus represents a wave scattered from a direction $\vec{k}$ to a direction $\vec{k}'$ as undergoing, in travelling through a potential, phase shifts which are appropriate to a path deflected through half the final scattering angle, (see Fig. 2). This is evidently a crude correction for some of the bending of particle paths that takes place within the potential. The determination of $\hat{R}$ by means of Eq. (89) actually simplifies the evaluation of the integral Eq. (80) for the scattering amplitude since, as long as energy is conserved, $\hat{R}$ is then perpendicular to $\vec{k}-\vec{k}'$ i. e.,

$$(\vec{k}-\vec{k}')\cdot(\vec{k}+\vec{k}') = k^2 - k^2 = 0 \ . \tag{90}$$

With this alteration, the scattering amplitude for an axially symmetric potential becomes

$$f(\theta) = \frac{k}{i}\int_0^\infty J_0\left(2kb\sin\frac{\theta}{2}\right)\left\{e^{i\chi(b)} - 1\right\} b\,db, \tag{91}$$

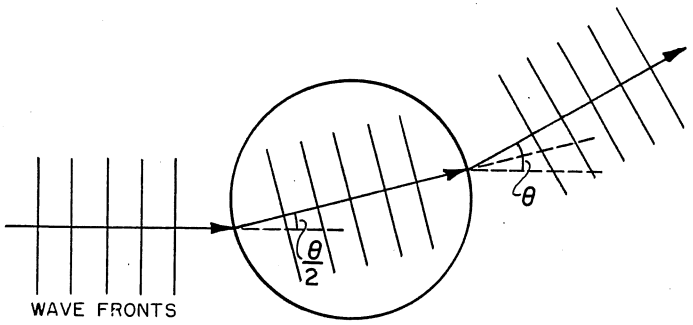


Fig. 2

i. e., the only change is to replace Θ in the argument of the Bessel function by

$$2\sin\frac{\Theta}{2}\ .$$

To the extent that we restrict ourselves to the consideration of very small angles the change described is hardly a very great one. But it must be remembered that the angular distributions of scattering we are dealing with are peaked near the forward direction and as a consequence vary rapidly with angle. Even so slight a shift of the angular scale at small angles as we have just described may be of significance in improving the angular range over which the approximation holds, although, of course, the order-of-magnitude inequalities Eq. (87) and Eq. (88) still apply.

Further arguments in favor of the accuracy of Eq. (91) may be adduced. In particular, if the exponential, $\exp[i\chi(b)]$, is expanded in a power series, the sucessive terms correspond to the various orders of the Born approximation. In the form Eq. (91) the first Born amplitude is reproduced precisely for all Θ so that the approximations characteristic of the high-energy procedure are confined to the second and higher order amplitudes. Still another indication of the accuracy of Eq. (91) will be encountered in showing its correspondence with the phase shift series.

An important test of the self-consistency of the high-energy approximation is furnished by the unitarity theorem (22). To illustrate this we shall present a rather simplified check of the optical, or conservation, theorem. According to this theorem, the total scattering

cross section should be given by

$$\sigma_{tot.} = \frac{4\pi}{k}\,\mathrm{Im}\,f(\vec{k},\vec{k})\ .$$

Now the expression for the scattering amplitude is

$$f(\vec{k}',\vec{k}) = \frac{k}{2\pi i}\int e^{i(\vec{k}-\vec{k}')\cdot\vec{b}}\left\{e^{i\chi(\vec{b})}-1\right\}d^{(2)}b\ ,$$

so that the total cross section should be

$$\sigma_{tot.} = 2\int\left(1-\mathrm{Re}\,e^{i\chi(\vec{b})}\right)d^{(2)}b. \tag{92}$$

To verify this expression we must integrate the differential cross section over the sphere $|\vec{k}'| = k$

$$\int|f(\vec{k}',\vec{k})|^2 d\Omega_{k'} = \left(\frac{k}{2\pi}\right)^2\int e^{i(\vec{k}-\vec{k}')\cdot(\vec{b}-\vec{b}')}\left\{e^{i\chi(\vec{b})}-1\right\}\left\{e^{-i\chi(\vec{b}')}-1\right\}d^{(2)}b\,d^{(2)}b'\,d\Omega_{k'}\ . \tag{93}$$

Now to make a quick estimate of this integral we shall take advantage of the fact that the scattering is concentrated near the forward direction. We shall replace the integration over the sphere $|\vec{k}'| = k$ by an integration over the plane in $\vec{k}'$-space which is tangent to the sphere at $\vec{k}' = \vec{k}$, i. e., in the forward direction. In this case we may write

$$d\Omega_{k'} \approx \frac{d^{(2)}k'}{k^2}\ , \tag{94}$$

where the differential element $d^{(2)}k'$ lies in a plane perpendicular to $\vec{k}$. Furthermore, we note that

$$\int e^{i(\vec{k}-\vec{k}')\cdot(\vec{b}-\vec{b}')}d^{(2)}k' = (2\pi)^2\delta^{(2)}(\vec{b}-\vec{b}')\ , \tag{95}$$

where $\delta^{(2)}(\vec{b}-\vec{b}')$ is a two dimensional delta-function.

By carrying out the angular integration in this approximate way, we find

$$\sigma_{scatt.} = \int|e^{i\chi(\vec{b})}-1|^2 d^{(2)}b\ . \tag{96}$$

Hence for χ real, i. e., in the absence of absorption, we have

$$\sigma_{scatt.} = 2\int(1 - \mathrm{Re}\, e^{i\chi(\vec{b})})\, d^{(2)}b\,, \tag{97}$$

which indeed agrees with the expression Eq. (92) obtained from the conservation theorem. It is a simple matter to extend the foregoing proof to include the cases $\vec{K}' \neq \vec{K}$ of the unitarity theorem Eq. (22), as long as $\vec{K}$ and $\vec{K}'$ lie close together in direction. Furthermore the proof may be improved considerably in accuracy by the introduction of a more precise means of carrying out the angular integration. That can be accomplished by taking the angular dependence of amplitude (91) quite literally and making use of the orthogonality relations for Bessel functions in evaluating the integral, Eq. (93).

As a by-product of the foregoing analysis we may derive an expression for the absorption cross section which is present if the phase shifts are complex. We have only to subtract expression Eq. (97) for the scattering cross section from Eq. (92) for the total cross section;

$$\begin{aligned}\sigma_{abs.} &= \sigma_{tot} - \sigma_{scatt.} \\ &= \int(1 - |e^{i\chi(\vec{b})}|^2)\, d^{(2)}b\,.\end{aligned} \tag{98}$$

Relation of the High-Energy Approximation to Other Methods

It may improve our perspective a bit to show how the approximation we have been discussing is related to the various approaches mentioned earlier. The correspondence of the method with the Born approximation approach, has already been indicated. The power series expansion of

$$\exp[i\chi(\vec{b})]$$

on which the correspondence is based is only useful for

$$Va/\hbar v \ll 1\,.$$

The W. K. B. limit,

$$Va/\hbar v \gg 1\,,$$

is also implicitly contained in the approximation. In this limit the phase shift function is large in absolute magnitude compared with unity, so that the function

$$\exp[i\chi(\vec{b})]$$

oscillates rapidly. Now for any particular direction of scattering we must integrate the expression

$$\exp[i(\vec{K} - \vec{K}')\cdot\vec{b} + i\chi(\vec{b})]$$

over the $\vec{b}$-plane. Now in the classical limit this expression will oscillate so rapidly that the only appreciable contributions to the integral will come from the points of stationary phase. These, by symmetry, must lie in the plane containing $\vec{K}$ and $\vec{K}'$. It is easy to verify that these stationary points represent just the classical impact vectors of particles scattered in the direction $\vec{K}'$. For the simpler choices of scattering potential there is usually only one stationary point for each $\vec{K}'$; but there are, of course, potentials for which more than one classical path is possible and others for which there is none for scattering in certain directions. It is an elementary matter to integrate the expression for the scattering amplitude using the method of stationary phase. The result, at small angles, agrees with the W. K. B. approximation.

The correspondence of the high-energy approximation with the partial wave expansion becomes clear on comparing the expressions, Eq. (96) and Eq. (35), for the scattering cross sections. In the high-energy approximation the summation over angular momenta is replaced by an integration over impact vectors. If we adopt the asymptotic correspondence

$$kb \longleftrightarrow l + \tfrac{1}{2}\,, \tag{99}$$

the summation and integration become identical in form. The coefficients

$$C_l = e^{2i\delta_l}$$

must then be identified with the factors

$$e^{i\chi(b)}\,,$$

so that we have the correspondence

$$\chi\left(\frac{l+\frac{1}{2}}{k}\right) \longleftrightarrow 2\delta_l\,. \tag{100}$$

These correspondences may be illustrated, for example, with the absorption cross section Eq. (37)

$$\frac{\pi}{k^2}\sum_l (2l+1)(1 - |e^{2i\delta_l}|^2) \longleftrightarrow \frac{\pi}{k^2}\int 2kb\,kdb\,(1 - |e^{i\chi(\vec{b})}|^2)$$

$$\longleftrightarrow \int (1 - |e^{i\chi(\vec{b})}|^2)\, d^{(2)}b .$$

To show the correspondence of the partial wave expansion for the scattering amplitude,

$$f(\theta) = \frac{1}{2ik} \sum_l (2l+1)\{e^{2i\delta_l} - 1\} P_l(\cos\theta) , \tag{101}$$

with the high-energy approximation, we require an approximate form for the Legendre polynomials at small angles.

Now, in spherical coordinates, regions far from the origin and close to the polar axis are described in a way which differs only very slightly from that of cylindrical coordinates. Consequently, there is an approximate correspondence between the solution of the wave equation in the two systems. This correspondence may be expressed by the asymptotic relation

$$P_l(\cos\theta) \sim J_0\left(2(l+\tfrac{1}{2})\sin\tfrac{\theta}{2}\right) + \tfrac{1}{4}\sin^2\tfrac{\theta}{2} + \cdots . \tag{102}$$

Although this relation is most accurate for large l and small angles, the remainder term adds only a 6 per cent correction for $l=0$ and $\theta=60^\circ$. By discarding the remainder term and using the previously noted correspondences, we may reduce the partial wave expression Eq. (101) to precisely the form Eq. (92) of the high-energy expansion. The expansion Eq. (102) is seen to furnish some additional weight for the use of the argument

$$2\sin\tfrac{\theta}{2}$$

rather than θ in the Bessel function.

Some Illustrative Examples

Before we proceed to generalize the high-energy approximation in various ways, it may be helpful to illustrate its use in some concrete examples. The simplest of these is one familiar from diffraction theory. We consider the case of an absorptive (i. e., negative imaginary) potential which is confined to a sphere of radius a, one which absorbs effectively enough that the sphere may be considered opaque. Note that the absolute value of the imaginary potential need not be very large to produce this effect. Even if the potential is only weakly absorptive in any volume element, it will be opaque if its radius is large enough.

Assuming the sphere to be opaque, we write

$$e^{i\chi(\vec{b})} = \begin{cases} 0 & , b < a \\ 1 & , b > a \end{cases} , \tag{103}$$

so that the scattering amplitude becomes

$$f(\theta) = -\frac{k}{i}\int_0^a J_0(2kb\sin\tfrac{\theta}{2})\, b\, db \tag{104}$$

$$= ia\,\frac{J_1(2ka\sin\theta/2)}{2\sin\theta/2} .$$

The differential scattering cross section is thus

$$|f(\theta)|^2 = (ka^2)^2\,\frac{J_1^2(2ka\sin\theta/2)}{[2ka\sin\theta/2]^2} , \tag{105}$$

which has the appearance typical of a short-wavelength diffraction pattern, shown in Fig. 3.

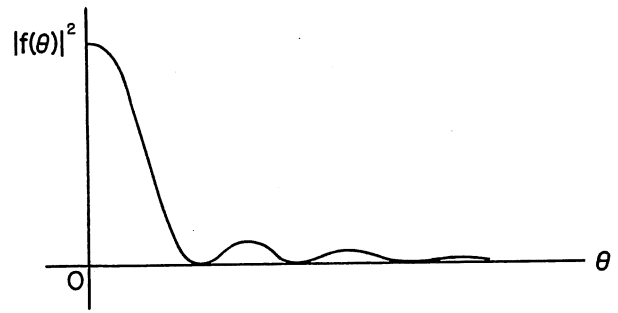


Fig. 3

The total scattering cross section is given simply by

$$\sigma_{scatt.} = \int |e^{i\chi(\vec{b})} - 1|^2\, d^{(2)}b = \pi a^2 , \tag{106}$$

since the only nonzero contribution to the integral comes from the shadow region, $b < a$. For the same reason we have

$$\sigma_{abs.} = \int (1 - |e^{i\chi}|^2) d^{(2)}b = \pi a^2 , \tag{107}$$

and

$$\sigma_{tot.} = 2\int (1 - \mathrm{Re}\, e^{i\chi}) d^{(2)}b = 2\pi a^2 . \tag{108}$$

The latter result is just twice the classical absorption cross section. The small-angle scattering, i. e., the intensity diffracted about the edges of the sphere, is of course, the non-classical effect responsible for the factor of two. Note that nothing essential in the discussion above rests on the choice of spherical shape of the potential. The scattering cross section will equal the absorption cross section for an opaque potential of any shape whose dimensions are large compared with the wavelength.

As a somewhat more general example let us consider the case of a square potential well of radius a;

$$V(r) = \begin{cases} V_0 , & r < a \\ 0 , & r > a . \end{cases} \tag{109}$$

The phase shift function is then

$$\chi(b) = \begin{cases} -\frac{2V_0}{\hbar v}\sqrt{a^2 - b^2} , & b < a \\ 0 , & b > a . \end{cases} \tag{110}$$

In this case there is no longer any elementary way of carrying out the impact parameter integration to find the angular dependence of the scattering amplitude. The integral is best treated by means of a series expansion, due to I. Shapiro,[8] which is a bit too lengthy to present here. The integration required to find the amplitude for scattering in the forward direction, however, may easily be carried out in closed form, and from the imaginary part of this amplitude we secure an expression for the total cross section.

In terms of the parameter

$$\alpha = \frac{V_0 a}{\hbar v} \qquad (V_0 \text{ real}) \tag{111}$$

the ratio of the total scattering cross section to the geometrical cross section may be written as

$$\frac{\sigma_{tot.}}{\pi a^2} = 2 + \frac{1}{\alpha^2} - \frac{2}{\alpha}\left(\frac{\cos 2\alpha}{2\alpha} + \sin 2\alpha\right) . \tag{112}$$

A plot of

$$\sigma_{tot}/\pi a^2$$

versus α is shown in Fig. 4. It is of particular interest to note the limiting values

$$\frac{\sigma_{tot.}}{\pi a^2} = \begin{cases} 2\alpha^2 & \text{for} \quad \alpha \ll 1 \\ 2 & \text{as} \quad \alpha \to \infty . \end{cases} \tag{113}$$

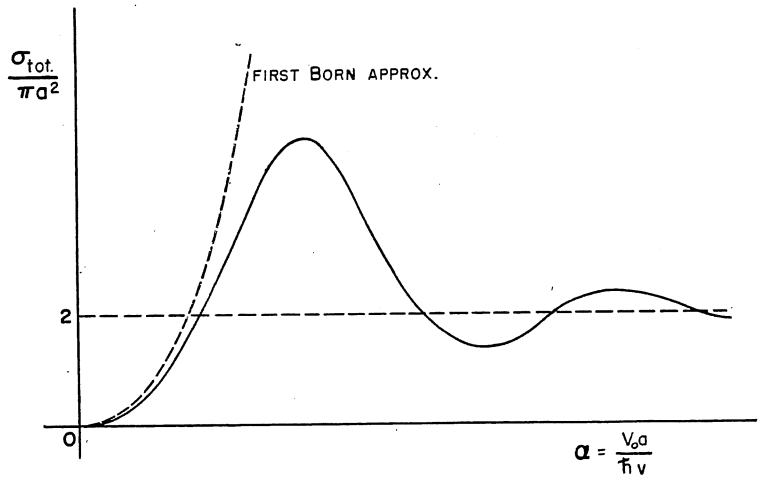


Fig. 4

The first of these is the Born approximation. The ratio 2 which appears in the limit

$$V_0 a/\hbar v \to \infty$$

is due to the contributions of two different types of scattering. Particles may be scattered either by penetrating the sphere and suffering relatively large deflections

$$\Theta \sim |V_0|/E$$

or by being diffracted about it and suffering relatively small ones,

$$\Theta \sim \frac{1}{ka} .$$

For the case of a Gaussian potential

$$V(r) = V_0 e^{-r^2/a^2} \qquad (V_0 \text{ real}) \tag{114}$$

it is also easy to obtain the expression for the integrated scattering cross section. If again we write

$$\alpha = V_0 a/\hbar v$$

the result may be expressed as

$$\frac{\sigma_{\text{scatt.}}}{\pi a^2} = 2\int_0^{\sqrt{\pi}\alpha} \frac{1-\cos t}{t}\, dt \tag{115}$$

which is shown plotted as a function of α in Fig. 5. In this case, the

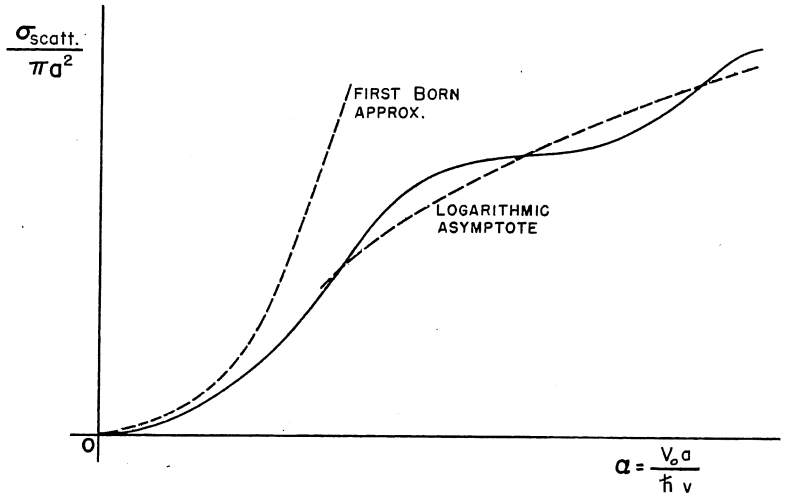


Fig. 5

values for small and large α are

$$\frac{\sigma_{\text{scatt.}}}{\pi a^2} = \begin{cases} \frac{\pi}{2}\alpha^2 , & \text{for } \alpha \ll 1 \\ 2\{\log(\sqrt{\pi}\alpha) + .577 + \cdots\} & \text{for } \alpha \gg 1 . \end{cases} \tag{116}$$

The interesting point here is that in the limit

$$\alpha = V_0 a/\hbar v \longrightarrow \infty$$

the ratio

$$\frac{\sigma_{\text{scatt.}}}{\pi a^2}$$

increases without bound. This may be a bit surprising since the decrease of the Gaussian potential with distance is notoriously rapid. The explanation lies in the fact that the limit

$$\frac{V_0 a}{\hbar v} \longrightarrow \infty$$

is, in effect, the classical limit $\hbar \to 0$. The infinite value approached by the cross section is common to all potentials which possess gradients, however small, at unbounded distances from the origin. In the classical limit all particles passing through such potentials must be considered as scattered no matter how small the scattering angle may be. Hence, a potential with an infinitely extended tail will always lead to an infinite scattering cross section in the classical limit.

An example which is of considerable practical value is furnished by the Coulomb potential,

$$V(r) = Ze^2/r .$$

In this case, however, we are immediately confronted by the difficulty that the integral which represents the phase shift,

$$\chi(b) = -\frac{Ze^2}{\hbar v}\int_{-\infty}^{\infty} \frac{dz}{\sqrt{b^2 + z^2}} ,$$

diverges logarithmically at both extremes of the range of integration. The difficulty here is essentially one common to all methods of approximation which assume that the incident wave is simply a plane wave. The slow decrease of the Coulomb field with distance has the effect that an

incoming wave is appreciably distorted in phase at large distances from the center of force. The only consistent way to treat scattering by a field which decreases so slowly would be to take this distortion into account.

Now in actual practice, scattering experiments virtually never involve Coulomb fields which decrease so very slowly. It is almost always possible to say that the charge in question is neutralized within a certain distance by surrounding charges of opposite sign. Usually this distance is quite small compared with laboratory dimensions, and it is often large compared with the particle wavelengths. This is typically the case, for example, in nuclear scattering experiments, where the distance in question is simply the atomic radius. Now in such cases, the difficulty noted above is no longer present. The field falls off rapidly enough with distance to permit an elementary application of the approximation method.

The screened Coulomb potential, we may assume, can be written in the form

$$V(r) = \frac{Ze^2}{r} F(r) \tag{117}$$

where

$$F(r) \longrightarrow 0$$

as

$$r \longrightarrow \infty$$

a property which insures the convergence of the integral for $\chi(b)$. In order to secure a simple illustration we may consider the particular choice

$$F(r) = \begin{cases} 1, & r < a \\ 0, & r > a \end{cases} \tag{118}$$

which leads to the phase shift function

$$\chi(b) = \begin{cases} -2\frac{Ze^2}{\hbar v} \log\left(\frac{a + \sqrt{a^2 - b^2}}{b}\right), & b < a \\ 0, & b > a \end{cases} \tag{119}$$

The scattering amplitude which we wish to calculate is given by the expression

$$f(\theta) = \frac{k}{i} \int_0^\infty J_0(2kb \sin\theta/2)\left\{e^{i\chi(b)} - 1\right\} b\,db. \tag{91}$$

The integral required is not an elementary one, but it may be simplified appreciably by considering the limit of large screening radii,

$$ka \longrightarrow \infty.$$

This limit, in fact, represents well the situation which prevails in most nuclear scattering problems, where ka usually exceeds 10^5. Now it is easy to see that in such cases the effect of the screening on the scattered intensity is virtually imperceptible. The screening affects particles whose impact parameters exceed the radius a, and so alters the scattered intensity only at the unobservably small angles,

$$\theta \sim 1/ka.$$

Particles which are deflected to angles appreciably larger than this are ones which have suffered close encounters with the Coulomb field. We should therefore expect the Rutherford formula to hold for the intensity at angles

$$\theta \gg 1/ka,$$

and we shall try to show that this is so.

To carry out the integration for the scattering amplitude, it helps to note that the integral

$$\int_0^\infty J_0(2kb \sin\theta/2) b\,db = \frac{1}{2\pi} \int e^{i(\vec{k} - \vec{k}')\cdot\vec{b}} d^{(2)}b$$

is simply a two dimensional delta-function which vanishes for $\vec{k}' \neq \vec{k}$. Hence for angles $\theta \neq 0$ we need only calculate the integral

$$f(\theta) = \frac{k}{i} \int_0^\infty J_0(2kb \sin\theta/2) e^{i\chi(b)} b\,db, \quad (\theta \neq 0). \tag{120}$$

Now, because of the way in which the Bessel function oscillates, the most significant contributions to this integral will come from the region

$$b \sim 1/k\theta.$$

In this region we have

$$b/a \sim 1/ka\theta.$$

If we restrict the calculation to angles

$$\theta \gg 1/ka ,$$

we evidently have

$$b/a \ll 1$$

in the region which contributes most. We therefore evaluate the integral by expanding $\chi(b)$ in powers of b/a. The expansion of χ is

$$\chi(b) = 2\frac{Ze^2}{\hbar v}\log\frac{b}{2a} + \frac{Ze^2}{\hbar v}\frac{b^2}{2a^2} + O\left(\frac{b^4}{a^4}\right). \qquad (121)$$

The approximation of $\chi(b)$ by the first term of this series leads to an integral, Eq. (120), of a form which is well known. The scattering amplitude which results is

$$f(\theta) = -\frac{2k\frac{Ze^2}{\hbar v}}{(2k\sin\theta/2)^2}\, e^{-i\left\{2\frac{Ze^2}{\hbar v}\log(2ka\sin\theta/2) - 2\eta\right\}} \qquad (122)$$

where η is the phase angle

$$\eta = \arg\Gamma\left(1 + i\frac{Ze^2}{\hbar v}\right) \qquad (123)$$

We have thus verified that for angles

$$\theta \gg 1/ka$$

the scattered intensity

$$|f(\theta)|^2$$

indeed follows the Rutherford formula. The only effect of the screening on the scattering amplitude at these angles is the addition of a constant to its complex phase angle, which otherwise also agrees with the exact result for a Coulomb field. This phase constant contains the term

$$2\frac{Ze^2}{\hbar v}\log ka$$

which must occur whatever the form of the screening function $F(r)$, and to it there is added in general a constant term, independent of the screening radius, but which does depend on the shape of the screening function.

For example, if we had chosen the screening function

$$F(r) = e^{-r/a} \qquad (124)$$

the resulting scattering amplitude for

$$\theta \gg 1/ka$$

would be precisely the result Eq. (122) multiplied by the phase factor

$$e^{2i\frac{Ze^2}{\hbar v}C}$$

where C is Euler's constant (C=.577).

In the foregoing discussion we have shown that the introduction of screening to a Coulomb field makes it possible for the high-energy approximation to solve the scattering problem accurately. If the screening radius a is large, the effect of screening on the intensities scattered to angles

$$\theta \gg 1/ka$$

is negligible. The specific manner in which the field is screened affects only a multiplicative phase factor in the scattering amplitude. Now in applications to nuclear physics we are virtually never interested in the overall phase of the scattering amplitude when dealing with charged particles. This phase could, in principle, be measured through interference effects, but our discussion shows that it is more sensitive to atomic parameters than to nuclear ones, and hence is not of great interest. Since few nuclear measurements are sensitive to the effects of screening, it ordinarily makes little difference what form the screening is assumed to take.

In nuclear applications, the incident particle is usually subject to nuclear forces as well as to the Coulomb field. The use of the method for such cases requires only that the phase shift functions $\chi(b)$ for the nuclear and Coulomb interactions be superposed, i. e., the nuclear phase shift function is added to Eq. (121) and the sum substituted for $\chi(b)$ in Eq. (120). The amplitude calculated in this way takes proper account of the two kinds of scattering plus all of the interferences that occur between them.

Treatment of More General Static Interactions

Before passing on to the treatment of time-varying potentials and the many-body problem, it will be helpful to indicate how the high-energy approximation may be applied to spin-dependent and velocity-dependent

interactions. We shall suppose that the incident particle has spin 1/2, and shall represent its angular momentum vector (in units of $\hbar$) by

$$\vec{S} = \frac{1}{2}\vec{\sigma}$$

where $\vec{\sigma}$ is the familiar Pauli spin vector

$$\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij} .$$

The orbital angular momentum of the particle (in units of $\hbar$) is represented by the operator

$$\vec{L} = \frac{1}{\hbar}(\vec{r}\times\vec{p}).$$

A particular form of nuclear interaction which possesses all of the required invariances and is both spin and velocity dependent may be obtained by multiplying the scalar product $\vec{\sigma}\cdot\vec{L}$ by any spherically symmetric function of position. More generally we may consider an interaction made up of a central potential $V_c(r)$ and a spin-orbit potential $(\vec{\sigma}\cdot\vec{L})V_s(r)$,

$$V(r) = V_c(r) + V_s(r)\vec{\sigma}\cdot\vec{L} . \tag{125}$$

The treatment of this potential[11] will be of use to us both to illustrate the method and to serve, at a later point, as a means of identifying the spin-orbit potential which is effective in the many-body problem.

In order to generalize the approximation to include the spin variable, we shall write the wave function in the form

$$\psi_k(\vec{r}) = e^{ik\cdot r}\mathcal{P}(\vec{r})u_i , \tag{126}$$

where u_i is a two-component spinor representing the initial state of the particle, and $\mathcal{P}(\vec{r})$ is an operator which induces the appropriate change of spin state. By making the approximations described earlier, and further assuming

$$|V_s|/E << 1 ,$$

the equation determining $\mathcal{P}(\vec{r})$ may be reduced once again to the form Eq. (72). Its solution is

$$\mathcal{P}(\vec{r}) = e^{-\frac{i}{\hbar v}\int_{-\infty}^{z}\{V_c(\vec{b}+\hat{k}z')+V_s(\vec{b}+\hat{k}z')\vec{\sigma}\cdot(\vec{b}\times\hat{k})k\}dz'} \tag{127}$$

Again by a procedure of the type described earlier we may use the expression for $\mathcal{P}(\vec{r})$ to derive the scattering amplitude,

$$f(k',k) = \frac{k}{2\pi i}\int e^{i(\vec{k}-\vec{k}')\cdot\vec{b}}\left\{e^{i\chi_c(\vec{b})+i\chi_s(\vec{b})\vec{\sigma}\cdot(\vec{b}\times\hat{k})k}-1\right\}d^{(2)}b. \tag{128}$$

Here χ_c and χ_s are the phase shifts
1

$$\chi_c(\vec{b}) = -\frac{1}{\hbar v}\int_{-\infty}^{\infty}V_c(\vec{b}+\hat{k}z')dz', \quad \chi_s(\vec{b}) = -\frac{1}{\hbar v}\int_{-\infty}^{\infty}V_s(\vec{b}+\hat{k}z')dz' . \tag{129}$$

Now the exponential containing the spin operator in the integrand of Eq. (128) is considerably simpler in structure than it may appear. In fact, any function whatever of $\vec{\sigma}$ must reduce to a linear function. In the above case the reduction is easily verified by using the anticommutation rule to simplify the power series expansion. The scattering amplitude may then be written in the form

$$f(k',k) = \frac{ik}{2\pi}\int e^{i(\vec{k}-\vec{k}')\cdot\vec{b}}\left\{\Gamma_c(\vec{b})+i\vec{\sigma}\cdot\left(\frac{\vec{b}}{b}\times\hat{k}\right)\Gamma_s(\vec{b})\right\}d^{(2)}b, \tag{130}$$

where Γ_c and Γ_s are the axially symmetric functions

$$\Gamma_c(\vec{b}) = 1 - e^{i\chi_c(\vec{b})}\cos[kb\chi_s(\vec{b})] \tag{131}$$

$$\Gamma_s(\vec{b}) = -e^{i\chi_c(\vec{b})}\sin[kb\chi_s(\vec{b})] . \tag{132}$$

The scattering amplitude we are constructing is to be regarded as an operator which transforms the initial spin state of the particle into its final state. As such, the most general form it can take is

$$f(k',k) = f(k',k) + (\vec{\sigma}\cdot\vec{n})g(k',k) , \tag{133}$$

where f and g are spin-independent amplitudes and $\hat{n}$ is a unit vector perpendicular to the plane of scattering,

$$\hat{n} = \frac{\bar{k}\times\bar{k}'}{|\bar{k}\times\bar{k}'|} \quad . \tag{134}$$

By carrying out the integration of Eq. (130) over the azimuthal angle we find

$$f(k',k) = ik\int_0^\infty J_0(|\bar{k}-\bar{k}'|b)\Gamma_c(\bar{b})\,b\,db \tag{135}$$

$$g(k',k) = -ik\int_0^\infty J_1(|\bar{k}-\bar{k}'|b)\Gamma_s(\bar{b})\,b\,db \quad . \tag{136}$$

The appearance of the Bessel function J_1 in the latter integral is due to dependence of the spin-orbit potential on the azimuthal angle. The amplitude g may be seen to vanish for forward scattering, as it must, since in that case there is no longer any axial vector to represent a possible direction of spin polarization. The particles scattered at nonvanishing angles are in general polarized. The effect which gives rise to the polarization may be thought of as a species of double refraction caused by the spin dependence of the interaction.

It is also of interest to consider more general forms of operator-dependent interactions. We shall suppose, for simplicity, that the problem is one-dimensional, and that we are given an interaction operator V(x). The equation for the modulating function $\varphi(x)$ is, as before,

$$\frac{d\varphi(x)}{dx} = -\frac{i}{\hbar v}V(x)\varphi(x) \quad , \tag{53}$$

but where questions of commutation are concerned we must exercise care in integrating it. In particular if the values of the operator-function V(x) do not commute with one another for all x, it is no longer correct to write the solution to Eq. (53) as a simple exponential. It is possible, however, to write the solution to Eq. (53) as a power series

$$\varphi(x) = 1 - \frac{i}{\hbar v}\int_{-\infty}^{x}V(x')dx' + \left(\frac{i}{\hbar v}\right)^2\int_{-\infty}^{x}dx'\int_{-\infty}^{x'}dx''\,V(x')V(x'') + \cdots \quad . \tag{137}$$

When the operators commute, the expansion is simply that of the exponential

$$\exp\left[-\frac{i}{\hbar v}\int_{-\infty}^{x}V(x')dx'\right] .$$

When the operators fail to commute, we shall introduce for the series (137) the notation

$$\varphi(x) = \left\{e^{-\frac{i}{\hbar v}\int_{-\infty}^{x}V(x')dx'}\right\}_+ \quad . \tag{138}$$

The bracket $\{\ \}_+$ is to be taken to mean that in the sucessive terms of the power series expansion the operators V (x) are written in order of increasing argument from right to left.

A corresponding notation may be introduced in the three-dimensional problem. The formal expression for the scattering amplitude which one finds requires that the expression $\exp[iX(b)]$ in the integrand be replaced by the more general form

$$\left\{e^{-\frac{i}{\hbar v}\int_{-\infty}^{\infty}V(\bar{b}+\hat{k}z)dz}\right\}_+ \quad . \tag{139}$$

This expression reduces to the simple exponential whenever the different values of the potential along a straight-line path commute with one another (as they do for example to first order in V/E for a spin-orbit potential). An example for which the more general formulation is required is furnished by the tensor force between two nucleons.

Time-dependent Interactions

When the systems with which the incident particle interacts are no longer inert their influence on the particle becomes time-dependent. The simplest case to treat is the interaction of the particle with a time-dependent external field, V(x, t). Once again we consider a problem with only one spatial dimension so that the time-dependent Schroedinger equation is

$$\left(\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + i\hbar\frac{\partial}{\partial t}\right)\psi(x,t) = V(x,t)\psi(x,t) \quad , \tag{140}$$

and the wave function may be written in the form

$$\psi(x,t) = e^{i(kx-\omega t)}\varphi(x,t) \quad , \tag{141}$$

where $\omega = E/\hbar = \hbar k^2/2m$. The approximation we make is once again the assumption that $\varphi(x,t)$ varies slowly as a function of x. For this reason we again drop the term $\partial^2\varphi/\partial x^2$ from the Schroedinger equation, and find the equation for φ,

$$\left(\frac{\partial}{\partial x} + \frac{1}{v}\frac{\partial}{\partial t}\right)\varphi(x,t) = -\frac{i}{\hbar v}V(x,t)\varphi(x,t) \quad . \tag{142}$$

This equation is easily reduced to a form similar to Eq. (53) by introducing as an independent variable the quantity $s = t - x/v$, and using it to replace t. We shall assume that a steady plane wave is incident, which originates at $x = -\infty$. Since the approximation neglects reflection, the boundary condition on φ is $\varphi(-\infty, t) = 1$ for all t. The solution to Eq. (142) which applies in this case is

$$\varphi(x,t) = e^{-\frac{i}{\hbar v}\int_{-\infty}^{x} V(x', t - \frac{x-x'}{v})\,dx'} , \tag{143}$$

i.e., in integrating the potential to find the shift of phase of the wave function, the potential must be evaluated for each point at the time corresponding to the passage of the particle.

Since the function φ is explicitly time-dependent the wave function $\Psi(x,t)$ will no longer depend on time simply as $\exp[-i\omega t]$. The introduction of other Fourier components into the time dependence means that particles are scattered inelastically. It also implies certain restrictions on the validity of the approximation (143) which must be added to those discussed earlier.

The approximation is consistently one which assumes that the momentum change of the incident particle is small. When inelastic scattering is present it must be assumed that the change in the absolute value of the momentum is small (as well as any change in direction that may take place in three dimensions). Hence the approximation is only valid for scattering which is not too strongly inelastic, $\Delta E/E \ll 1$. For this reason we must restrict consideration to fields $V(x,t)$ whose dominant frequencies are much smaller than ω.

It is also of interest to generalize the approximation to the case in which the values of the interaction operator fail to commute at the different points along a trajectory. This would be the case, for example, for a particle traversing a region containing a quantized field. The solution for $\varphi(x,t)$ in such a case is

$$\varphi(x,t) = \left\{ e^{-\frac{i}{\hbar v}\int_{-\infty}^{x} V(x', t - \frac{x-x'}{v})\,dx'} \right\}_{+} , \tag{144}$$

where the ordering bracket refers, as before, to the variable x'. This expression is of use more generally; e.g. in considering collisions in which the target possesses internal degrees of freedom.

Scattering by a Bound Particle

The target particles in actual scattering experiments are always free to move about, to a certain degree, and to absorb recoil momentum. In nuclear and atomic experiments they are usually in a bound state of some sort initially, i.e., confined to the interior of a nucleus or molecule,

and may make transitions under the impact of the incident particle to states which are either free or bound. The approximation procedure we have been describing can be extended in a natural way to treat such problems. To illustrate this generalization we shall consider a one-dimensional problem which may be described as follows:

The target particle has coordinate q. Its Hamiltonian, H(q), has eigenvalues ϵ_j which correspond to eigenstates u_j. It will be convenient to assume that at least one of these states, which we take to be the initial state u_i, is bound, and confines the particle to the neighborhood of the origin. The incident particle is described by the coordinate x, and is assumed to interact with the target particle via the potential V(x-q). We shall assume that this potential satisfies the conditions for the high-energy approximation noted earlier.

In treating this problem, there is a certain formal convenience in being able to discuss separately the behavior of the incident and target particles. A procedure which allows this, in a measure, may be based on the time-dependent form of the Schroedinger equation. If we let Ω represent the time-dependent state of the system, this equation is

$$\left\{ -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x-q) + H(q) \right\}\Omega = i\hbar\frac{\partial}{\partial t}\Omega \quad . \tag{145}$$

Now let us assume that the dynamics of the unperturbed target particle are sufficiently well understood that its motion may be described in terms of time-dependent operators, i.e., via the Heisenberg picture. To secure such a description we remove from the state vector Ω the time dependence induced by the Hamiltonian H(q), i.e., we introduce the state Ψ by means of the relation

$$\Omega = e^{-H(q)t/\hbar}\,\Psi \quad . \tag{146}$$

Substituting this into the Schroedinger equation, we secure

$$\left\{ -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + e^{iH(q)t/\hbar}V(x-q)e^{-iH(q)t/\hbar} \right\}\Psi = i\hbar\frac{\partial}{\partial t}\Psi \quad . \tag{147}$$

We now define the time-dependent coordinate operator,

$$q(t) = e^{iH(q)t/\hbar}\,q\,e^{-iH(q)t/\hbar} \quad . \tag{148}$$

Since the variables x and q commute, the Schroedinger equation reduces

to

$$\left\{\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + i\hbar\frac{\partial}{\partial t}\right\}\Psi = V(x-q(t))\Psi \quad . \tag{149}$$

This form of the equation is very similar in structure to Eq. (14 0) which we encountered in treating a particle moving through a time-dependent external field. The only difference arises from the fact that q(t) is an operator whose values at different times do not in general commute. But we have already dealt with this complication at the end of the last section, and so the approximation procedure is now straightforward.

We write the wave function, by analogy with previous work, in the form

$$\Psi(x,t) = e^{i(kx-\omega t)}\varphi(x,t)u_i \quad , \tag{150}$$

where u_i represents the initial state of the target, and φ is an operator implicitly dependent on q, which induces appropriate changes of the target state. With the approximations noted earlier, the equation determining φ is

$$\left(\frac{\partial}{\partial x} + \frac{1}{v}\frac{\partial}{\partial t}\right)\varphi(x,t) = -\frac{i}{\hbar v}V(x-q(t))\varphi(x,t) \quad . \tag{151}$$

The solution to this equation may be written as

$$\varphi(x,t) = \left\{e^{-\frac{i}{\hbar v}\int_{-\infty}^{x}V\left(x'-q\left(t-\frac{x-x'}{v}\right)\right)dx'}\right\}_+ \quad . \tag{152}$$

This expression permits us to calculate, for example, the amplitude for the incident particle to be at the point x at time t when the target particle is at the same time in the state u_f. The amplitude is given by the scalar product

$$(u_f, \Psi(x,t)) = e^{i(kx-\omega t)}\left(u_f, \left\{e^{-\frac{i}{\hbar v}\int_{-\infty}^{x}V\left(x'-q\left(t-\frac{x-x'}{v}\right)\right)dx'}\right\}_+ u_i\right) . \tag{153}$$

Now, since we know the energies of the target states i and f, it is possible to simplify the x- and t-dependence of this expression. We make use of the definition (148) of the time-dependent coordinate operator to write

$$q\left(t-\frac{x-x'}{v}\right) = e^{i\frac{H(q)}{\hbar}\left(t-\frac{x}{v}\right)}\, q\left(\frac{x'}{v}\right)\, e^{-i\frac{H(q)}{\hbar}\left(t-\frac{x}{v}\right)} \quad , \tag{154}$$

so that Eq. (153) may be reduced to the form

$$e^{i(kx-\omega t)}\left(u_f, e^{i\frac{H(q)}{\hbar}\left(t-\frac{x}{v}\right)}\left\{e^{-\frac{i}{\hbar v}\int_{-\infty}^{x}V\left(x'-q\left(\frac{x'}{v}\right)\right)dx'}\right\}_+ e^{-i\frac{H(q)}{\hbar}\left(t-\frac{x}{v}\right)}\right) \tag{155}$$

$$= e^{i\left[\left(k+\frac{\epsilon_i-\epsilon_f}{\hbar v}\right)x-\left(\omega+\frac{\epsilon_i-\epsilon_f}{\hbar}\right)t\right]}\left(u_f, \left\{e^{-\frac{i}{\hbar v}\int_{-\infty}^{x}V\left(x'-q\left(\frac{x'}{v}\right)\right)dx'}\right\}_+ u_i\right) .$$

This expression shows explicitly that the time dependence of the target coordinate is such as to preserve the conservation of energy and momentum, i.e., (155) is a wave function for the incident particle which accounts for the proper exchange of energy and momentum. Note however that the representation of the momentum change is only correct for small changes of energy, $|\epsilon_i - \epsilon_f|/E \ll 1$. This is precisely the limitation discussed in the preceding section.

In practice it would generally be extremely difficult to attempt an exact valuation of a time-ordered operator such as occurs in Eq. (155). In fact we rarely know enough about the operator q(t), which describes the motion of the target particle to construct it explicitly. Since the expression (155) is already based upon certain approximations, there is no need to evaluate it to a high order of accuracy. Instead we may evaluate it by introducing a further approximation which is consistent with those made earlier and of comparable accuracy.

The point of departure of the approximation is the neglect of the time dependence of the operator q(t). A sufficient condition for this neglect to be accurate may be seen from the Schroedinger equation (145). It is that the Hamiltonian H (q) take on values which are small compared to the incident kinetic energy E. This must be true both for the initial bound state of the target and its final state whether bound or freely recoiling. In most applications the high-energy approximation is valid only for energies which are considerably larger than those characteristic of bound states of nuclei or atoms, so that the latter energies may be justifiably neglected. The Hamiltonian H(q) must be restricted not only in its initial and final values, but must remain much smaller than E during the collision as well. For this reason the approximation only treats correctly collisions in which the momentum transfer is small. The condition that the momentum transferred be a small fraction of the incident momentum, i.e., that the

angle of scattering be small, is one of the assumptions already underlying the high-energy approximation. The calculations of the earlier sections assure us that at sufficiently high energies nearly all of the scattering will take place through small angles. The recoil momenta of the target particles will therefore be small and their energies quite small compared to E.

The effects of neglecting the time dependence of q(t) may be seen in somewhat more detail in the expression (155). The exponential time dependence of Eq. (155) indicates the energy of the incident particle. One effect of neglecting the time dependence of q(t) is the neglect of the energy increment, $\epsilon_i - \epsilon_f$, of the incident particle. This assumption, which is exact for elastic scattering, remains an accurate approximation for final states u_f which correspond to fairly inelastic collisions. The second consequence of dropping the time dependence of q(t) lies in the neglect of what we may describe as a retardation effect, i.e., in Eq. (152) the operator q is evaluated at a time at which the incident particle is at x', assuming that it reaches x at time t. The argument of q which occurs in Eq. (152) is simplified to the form x'/v by the unitary transformation (154). The physical significance of this argument may be seen by considering the function $V(x'-q_t(\frac{x'}{v}))$ in its dependence upon the integration variable x'. If, for example, the target particle is moving in the same direction as the incident particle the effect of the x'-dependence of $q_t(x'/v)$ is to increase the interval of the variable x' over which the interaction V extends. This dependence corrects for the fact that the velocity of the incident particle with respect to the target particle is different from its velocity in the laboratory system. Neglect of the retardation effect therefore requires that the velocity of the target particle be small compared to the velocity of the incident particle.

The neglect of retardation, of course, greatly simplifies the expression (155). Approximation of $q_t(x'/v)$ by $q_t(0) = q$ means that the time ordering brackets may be removed and the expression between them regarded as a simple exponential. By neglecting the energy differences $\epsilon_i - \epsilon_f$ as a part of the same approximation we find

$$(u_f, \Psi(x,t)) = e^{i(kx-\omega t)}\left(u_f, e^{-\frac{i}{\hbar v}\int_{-\infty}^{x} V(x'-q)dx'} u_i\right) \tag{156}$$

This result could, of course, have been derived in fewer steps by neglecting H(q) at an earlier stage of the argument. It is of some interest, however, to see the separation, as in Eq. (155), of the two conditions that the approximation hold, namely that the energy ratio $|\epsilon_i - \epsilon_f|/E$ and the ratio of the target and incident particle velocities both be much smaller than unity. The general expression (155) may be used to construct an expansion in powers of these parameters whose initial term is (156). We shall not derive these corrections here.

The foregoing calculations may be generalized directly to treat

three dimensional scattering problems. The approximation analogous to Eq. (156) for the time independent wave function in three dimensions is clearly

$$\Psi_{k,i}(\vec{r},\vec{q}) = e^{i\vec{k}\cdot\vec{r} - \frac{i}{\hbar v}\int_{-\infty}^{z} V(\vec{b}+\hat{K}z'-\vec{q})dz'} u_i(q) \ . \tag{157}$$

To describe a collision we must specify the initial and final states, i and f, of the target particle as well as the initial and final propagation vectors, $\vec{k}$ and $\vec{k}'$, of the incident particle. The corresponding scattering amplitude will be written as $F_{fi}(\vec{k}',\vec{k})$. An exact expression for this amplitude may be constructed in terms of the values of the two-particle wave function within the range of the potential $V(\vec{r}-\vec{q})$. The derivation is very similar to that of the expression Eq. (11) for the single particle amplitude, and the result forms an obvious generalization of it,

$$F_{fi}(\vec{k}',\vec{k}) = \tag{158}$$

$$-\frac{2m}{4\pi\hbar^2}\int e^{-i\vec{k}'\cdot\vec{r}} u_f^*(\vec{q}) V(\vec{r}-\vec{q})\, \Psi_{k,i}(\vec{r},\vec{q})\, d\vec{r}\, d\vec{q} \ .$$

When the approximate wave function Eq. (157) is substituted in this expression, the integration over the coordinate z may be carried out precisely as in Eqs. (79) and (80). The result of this integration may be written in the form

$$F_{fi}(\vec{k}',\vec{k}) = \tag{159}$$

$$\frac{k}{2\pi i}\int e^{i(\vec{k}-\vec{k}')\cdot\vec{b}} \int u_f^*(\vec{q})\left[e^{i\chi(\vec{b}-\vec{s})}-1\right] u_i(\vec{q})\, d\vec{q}\, d^{(2)}b \ .$$

Here the coordinate $\vec{s}$ represents the component of $\vec{q}$ perpendicular to the propagation direction $\hat{K}$,

$$\vec{s} = \vec{q} - \hat{K}(\hat{K}\cdot\vec{q}) \ , \tag{160}$$

and the phase shift function χ is given by

$$\chi(\vec{b}-\vec{s}) = -\frac{1}{\hbar v}\int_{-\infty}^{\infty} V(\vec{b}-\vec{s}+\hat{K}z)\, dz \ . \tag{161}$$

This is simply the phase shift function discussed earlier, but with the impact parameter relative to a fixed center replaced by the impact parameter relative to the transverse coordinate $\vec{s}$ of the target particle. Because of the infinite range of the z integration in Eq. (161) the phase shift is independent of the longitudinal coordinate $(\hat{K} \cdot \vec{q})$ of the target.

The expression (159) for the scattering amplitude will furnish the basic approximation for the considerations to follow. It may be shown by means of the expansion procedure discussed earlier that the corrections to cross sections calculated by means of Eq. (159) are of the same order as or smaller than those implicit in the high-energy approximation. In particular, for azimuthally symmetric bound states, the first corrections are of order θ^2, (where the scattering angle θ is small) and of order v_t^2/v^2 where v_t is the velocity of the target particle in a bound state. If the bound states are not azimuthally symmetric, as may be the case, e.g., for oriented target nuclei, angular asymmetries of order θ may appear, but these are likewise weak corrections for the small-angle collisions under consideration.

Optical Theorem for Scattering by Bound Particles

We have maintained that when the incident particle energy is sufficiently high the simple approximation (159) represents accurately the amplitude for the overwhelming proportion of collision processes that actually occur, whether elastic or inelastic. A check on the self-consistency of this statement may be made by showing that Eq. (159) obeys the unitarity theorem. Here we shall limit ourselves to verifying the particular case of the optical theorem, since it is of use in establishing total cross sections.

The total cross section for collisions in which the target particle, originally in the state i, makes a transition to the state f is

$$\sigma_{fi} = \int \frac{k'}{k} \left| F_{fi}(\vec{k}', \vec{k}) \right|^2 d\Omega_{k'} . \tag{162}$$

The factor k'/k in the integrand of this expression reduces to unity since we are neglecting the energy change of the incident particle.

The fact that the scattering is always concentrated near the forward direction enables us to carry out the integration of Eq. (162) much as we did for the case of scattering by a fixed potential. We replace the integration over the sphere $|\vec{k}'| = k$ by an integration over a plane tangent to the sphere at the point $\vec{k}$, which represents forward scattering, i.e.,

$$d\Omega_{k'} \longrightarrow \frac{d^{(2)}k}{k^2} .$$

In this way we find

$$\sigma_{fi} = \left(\frac{k}{2\pi}\right)^2 \iint e^{i(\vec{k}-\vec{k}')\cdot(\vec{b}-\vec{b}')} d^{(2)}b\, d^{(2)}b' \iint u_i^*(q') u_f(q') \left[e^{-i\chi^*(\vec{b}-\vec{s}')} - 1\right] \cdot$$

$$\left[e^{i\chi(\vec{b}-\vec{s})} - 1\right] u_f^*(q)\, u_i(q)\, dq'dq\, \frac{d^{(2)}k'}{k^2} \tag{163}$$

$$= \int d^{(2)}b \iint u_i^*(q') u_f(q') \left[e^{-i\chi^*(\vec{b}-\vec{s}')} - 1\right]\left[e^{i\chi(\vec{b}-\vec{s})} - 1\right] u_f^*(q) u_i(q)\, dq'dq ,$$

where the representation Eq. (95) of the delta-function has been used in reaching the latter expression.

To find the total cross section for all types of collisions in which the target particle begins in the state i, we must sum (163) over all states f. The summation is easily carried out by using the completeness relation

$$\sum_f u_f(\vec{q}') u_f^*(\vec{q}) = \delta(\vec{q}' - \vec{q}) , \tag{164}$$

so that we find

$$\sigma_{tot.} = \sum_f \sigma_{fi}$$

$$= \int d^{(2)}b \int |u_i(\vec{q})|^2 \left| e^{i\chi(\vec{b}-\vec{s})} - 1 \right|^2 d\vec{q} . \tag{165}$$

Now, as long as there are no interactions which allow inelastic processes other than those we have described, the phase shift function $\chi(\vec{b} - \vec{s})$ will be real (or, more generally, a hermitian operator), so that Eq. (165) may be written as

$$\sigma_{tot.} = 2 \int d^{(2)}b \int |u_i(\vec{q})|^2 \left[1 - \mathrm{Re}\, e^{i\chi(\vec{b}-\vec{s})}\right] d\vec{q} . \tag{166}$$

The latter expression when multiplied by $k/4\pi$ is seen to be just the elastic scattering amplitude in the forward direction, as given by Eq. (159). We have therefore verified the optical theorem

$$\sigma_{tot.} = \frac{4\pi}{k} \mathrm{Im}\, F_{ii}(\vec{k}, \vec{k}) . \tag{167}$$

An expression for the purely inelastic part of the total cross section may be obtained by subtracting the elastic cross section σ_{el}, as given by Eq. (163), from the expression (165) for σ_{tot}. In this way we obtain

$$\sigma_{\text{inelast.}} = \sum_{f\neq i} \sigma_{fi}$$

$$= \int d^{(2)}b \left\{ \int |u_i(q)|^2 |e^{i\chi(b-s)}-1|^2 dq - \left| \int |u_i(q)|^2 [e^{i\chi(b-s)}-1] dq \right|^2 \right\} \quad (168)$$

$$= \int d^{(2)}b \left\{ \int |u_i(q)|^2 |e^{i\chi(b-s)}|^2 dq - \left| \int |u_i(q)|^2 e^{i\chi(b-s)} dq \right|^2 \right\}.$$

If the phase shift function is real this expression reduces to

$$\sigma_{\text{inelast.}} = \int d^{(2)}b \left\{ 1 - \left| \int |u_i(q)|^2 e^{i\chi(b-s)} dq \right|^2 \right\}. \quad (169)$$

It is a bit premature to discuss the conditions under which the phase shift function may take on complex values, but it is worth mentioning that this may be the case when the interactions we have considered are not sufficient to describe the physical situation in all detail. If other interactions are present which lead to transitions of a type not accounted for above (e.g., meson production or particle-antiparticle annihilation), the phase shifts for simple scattering processes will in general be complex. Transitions in which particle numbers are not conserved have been omitted from the analysis, but the effect of their existence on the transitions which do conserve particle numbers will be felt through the alteration of the phase shift functions.

Scattering by a Many-Particle System

The conditions for the validity of the approximation we have been discussing do not, in fact, restrict the number of particles making up the target system. To use the same method to treat collisions with many-body systems we have only to require that the interactions between the target particles and the incident particle be of a form which may be treated by the high-energy approximation, and place the restrictions mentioned earlier on the bound state energies. The treatment of a many-body target then becomes a simple extension of that of a single bound particle. We have only to replace the single particle wave function by the many-particle wave function

$$u(\vec{q}) \longrightarrow u(\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_N) \quad (170)$$

and the single-particle phase shift function (161) by the phase shift suffered by the incident particle in passing through a configuration of target particles. If the incident particle interacts with the target particles by means of two-body forces this phase shift is simply the sum of the phase shifts caused by the individual particles

$$\chi(\vec{b}-\vec{s}) \longrightarrow \sum_{j=1}^{N} \chi_j(\vec{b}-\vec{s}_j) \,. \quad (171)$$

The index j has been placed on the functions χ since the interactions need not all be the same.

By using these more general expressions, we may write the amplitude for a scattering process in which the target system makes a transition from the state i to the state f as

$$F_{fi}(\vec{k}',\vec{k}) = \frac{k}{2\pi i} \int e^{i(\vec{k}-\vec{k}')\cdot\vec{b}} d^{(2)}b \cdot$$

$$\int u_f^*(\vec{q}_1, \ldots \vec{q}_N) \left[e^{i\sum_{j=1}^{N} \chi_j(\vec{b}-\vec{s}_j)} - 1 \right] u_i(\vec{q}_1, \ldots \vec{q}_N) \prod_j d\vec{q}_j \,. \quad (172)$$

There is only one limitation on the validity of this expression which is of a type not already encountered in dealing with single-particle targets. We must require that the particles of the target system not be too far apart to prevent the use of the approximate wave functions on which the method is based. As we remarked in connection with Eq. (74), the approximate wave functions used are not intended to be accurate at large distances from the force centers. One may see, in fact, that their accuracy breaks down at distances of order $a^2/\chi = ka^2$, where a is the force range. (A simple example which illustrates this is the breakdown of the geometrical shadow cast by a black sphere.) For this reason one must require that the target particles be contained within a radius R of one another obeying the inequality

$$R \ll ka^2 \,. \quad (173)$$

Since the entire procedure is only applicable for $ka \gg 1$, this condition is ordinarily satisfied for scattering by nuclei.

From the standpoint of the logic involved, no great innovation has been required to treat many-particle as well as single-particle targets. There is, however, a variety of ways of looking at the many-particle problem, and according to one of the more popular ones, what we have done can seem quite remarkable indeed. The method we refer to is one which views the incident particle as undergoing a succession of individual collisions with the different particles of the target system. The difficulties this method encounters stem from the close proximity of the target particles, and the obvious mathematical complication of treating multiple scattering. The closeness of the target particles means that the scattering amplitudes required for such calculations are not the ones measured experimentally. Furthermore, because of the evident difficulty of treating even double scattering, nearly all such work has been confined to the discussion of single-scattering, i.e., the scattering amplitude has been written as the sum of the amplitudes due to each of the target particles considered individually.

Now the expression (172) is no such simple superposition. We have added together the phase shifts produced by the individual target particles, not the scattering amplitudes. This means that multiple scattering effects are taken into account by Eq. (172). We shall presently demonstrate that it treats correctly even scatterings by all N particles in succession. Of course this is only to say that, under the conditions stated, the multiple scattering problem is less formidable than one usually imagines. The high-energy approximation, as we have noted, is, in fact, favored by the close proximity of the nucleons.

Eclipse-Effect in Deuterium

One of the simplest illustrations we can make of the theory outlined is the discussion of collisions between a single particle and a deuteron.[12] When the energy of the incident particle is so high that its wavelength is much smaller than the deuteron radius one might presume, rather naively as it develops, that the neutron and proton scatter independently and that the deuteron cross section is simply the sum of their two cross sections. Experiments have shown, however, that even at extremely high energies the various cross sections of the deuteron deviate measurably from the assumed additive rule. Such deviations have been found in experiments involving beams of incident nucleons, mesons, and antiprotons, and appear to be dramatically large in the latter case. A very crude model may serve to illustrate the way in which this comes about.

At energies of the order of 1 bev or higher colliding nucleons engage in meson production about as often as they scatter elastically. That means that if we confine our attention to collisions which conserve particle numbers, we may not be too far wrong in thinking of nucleons as being quite absorptive. We might think of their region of interaction, say, as a black sphere having a total cross section which is twice its absorption cross section.

Now the absorption cross section of the deuteron will deviate from the sum of the two absorption cross sections simply because one of its particles will sometimes lie in the shadow cast by the other. One has no trouble in correcting the absorption cross section since it is clear that the incident particle cannot be absorbed more than once. Let us say that the absorption cross sections of the neutron and proton are $\sigma_{n_{abs.}}$ and $\sigma_{p_{abs.}}$ respectively, and that their interaction radii are much smaller than the dueteron radius. Then to correct the absorption cross section for the deuteron we need only perform a simple geometrical calculation. The solid angle subtended by one particle as seen from the other, and the two configurations in which the correction is required, show us that the absorption cross section of the deuteron is

$$\sigma_{d_{abs.}} = \sigma_{p_{abs.}} + \sigma_{n_{abs.}} - 2\,\sigma_{p_{abs.}}\sigma_{n_{abs.}}\left\langle \frac{1}{4\pi r^2} \right\rangle_d , \qquad (174)$$

where in the correction term the inverse square of the neutron-proton distance is averaged in the deuteron ground state.

The correction to the absorption cross section comes from an effect which, as we have seen, is very close to a classical picture of an eclipse. The correction to the scattering cross section, however, is not at all so simple in its interpretation. There is quite a variety of effects which all contribute corrections of similar magnitudes. Such effects are: interference of single scattering amplitudes for the two particles, double scattering, interference of single and double scattering, and scattering by one particle followed by absorption by the other. These corrections are all significant but their separate consideration may be avoided by a simple stratagem.

At an earlier point we noted that, regardless of the shape of a black obstacle, its total cross section is twice its absorption cross section. That means that the total cross section of the deuteron may be obtained simply by multiplying Eq. (174) by two. In doing that we introduce the neutron and proton total cross sections $\sigma_n = 2\,\sigma_{n_{abs}}$ and $\sigma_p = 2\,\sigma_{p_{abs.}}$ to find

$$\sigma_d = \sigma_n + \sigma_p - \frac{\sigma_n \sigma_p}{4\pi}\left\langle \frac{1}{r^2} \right\rangle_d . \qquad (175)$$

Now this expression, simple as it is to derive, takes proper account of all the long list of corrections mentioned above, i.e., interference, etc. It furnishes an illustration of the advantage of treating interactions with the target system as a whole, wherever possible, rather than encounters between individual particles.

The correction term in Eq. (175) is indeed of the order of magnitude of the cross section differences observed. The proportionality of the

correction to the product of the total cross sections means that the effect becomes much more noticeable as the cross sections increase. A very large effect of this type must be present, for example, in antiproton-deuteron scattering, if the cross section of the free neutron is of the same order of magnitude as the rather large antiproton-proton cross section discovered recently. In that case analysis of the present sort becomes quite necessary in order to reach a reasonable estimate of the neutron cross section from a knowledge of the deuteron and proton cross sections. For the numbers actually observed in that case the indicated correction increases the estimated antiproton-neutron cross section by some forty to fifty per cent, and thereby shows it to be approximately as large as the antiproton-proton cross section.[13]

Since the effect we are considering occurs under a wide variety of conditions, and is not necessarily a small one, it is of interest to generalize the model we have used. We shall remove the assumptions that the nucleons are effectively black, and that the interaction ranges are much smaller than the deuteron radius. The latter, in fact, is a rather poor approximation.

Consider first the scattering of the incident particle by a single nucleon, say a neutron. The scattering amplitude is given by

$$f_n(\vec{k}',\vec{k}) = \frac{k}{2\pi i}\int e^{i(\vec{k}-\vec{k}')\cdot\vec{b}}\left\{e^{i\chi_n(\vec{b})}-1\right\}d^{(2)}b. \tag{176}$$

Note that no account is taken of recoil, i.e., the neutron might as well be fixed. (For small-angle scattering, recoil corrections are proportional to θ^2 and hence are unimportant.) A similar expression yields the amplitude for scattering by the proton.

Now the amplitude for elastic scattering by the deuteron may, according to the results of the preceding section, be constructed as follows: Let $\vec{q}$ be the neutron-proton separation, $\vec{q} = \vec{r}_n - \vec{r}_p$, and let $u(\vec{q})$ be the ground state wave function of a deuteron at rest. The projection of $\vec{q}$ on the plane perpendicular to $\vec{k}$ will be written as $\vec{s}$. Then the elastic scattering amplitude is

$$F(\vec{k}',\vec{k}) = \tag{177}$$

$$\frac{k}{2\pi i}\int e^{i(\vec{k}-\vec{k}')\cdot\vec{b}}d^{(2)}b\int|u(\vec{q})|^2\left[e^{i\chi(\vec{b},\vec{s})}-1\right]d\vec{q}$$

where the phase shift $\chi(\vec{b},\vec{s})$ is the sum of the phase shifts produced by the neutron and proton in their instantaneous positions,

$$\chi(\vec{b},\vec{s}) = \chi_n\left(\vec{b}-\tfrac{\vec{s}}{2}\right) + \chi_p\left(\vec{b}+\tfrac{\vec{s}}{2}\right). \tag{178}$$

The scattering amplitudes are Fourier transforms of a set of functions which we may write as

$$\Gamma_n(\vec{b}) = 1 - e^{i\chi_n(\vec{b})}, \quad \Gamma_p(\vec{b}) = 1 - e^{i\chi_p(\vec{b})} \tag{179}$$

$$\Gamma(\vec{b},\vec{s}) = 1 - e^{i\left[\chi_n\left(\vec{b}-\frac{\vec{s}}{2}\right)+\chi_p\left(\vec{b}+\frac{\vec{s}}{2}\right)\right]}.$$

These functions vanish when $\vec{b}$ lies outside the appropriate interaction ranges. Now, the last of these functions may be expressed in terms of the first two by means of the simple identity

$$\Gamma(\vec{b},\vec{s}) = 1 - \left[1-\Gamma_n\left(\vec{b}-\tfrac{\vec{s}}{2}\right)\right]\left[1-\Gamma_p\left(\vec{b}+\tfrac{\vec{s}}{2}\right)\right] \tag{180}$$

$$= \Gamma_n\left(\vec{b}-\tfrac{\vec{s}}{2}\right) + \Gamma_p\left(\vec{b}+\tfrac{\vec{s}}{2}\right) - \Gamma_n\left(\vec{b}-\tfrac{\vec{s}}{2}\right)\Gamma_p\left(\vec{b}+\tfrac{\vec{s}}{2}\right).$$

On substituting this expression into Eq. (177) and shifting the origin of the $\vec{b}$-plane in the first two integrals we express the elastic scattering amplitude in the form

$$F(\vec{k}',\vec{k}) = f_n(\vec{k}',\vec{k})\int e^{i(\vec{k}-\vec{k}')\cdot\frac{\vec{s}}{2}}|u(\vec{q})|^2 d\vec{q} + f_p(\vec{k}',\vec{k})\int e^{-i(\vec{k}-\vec{k}')\cdot\frac{\vec{s}}{2}}|u(\vec{q})|^2 d\vec{q} \tag{181}$$

$$+ \frac{k}{2\pi i}\int e^{i(\vec{k}-\vec{k}')\cdot\vec{b}}d^{(2)}b\int|u(\vec{q})|^2\,\Gamma_n\left(\vec{b}-\tfrac{\vec{s}}{2}\right)\Gamma_p\left(\vec{b}+\tfrac{\vec{s}}{2}\right)d\vec{q}.$$

This expression explicitly separates the effects of single and double scattering. The first two terms are the amplitudes for single scattering by each of the two target particles. The third term contributes only for deuteron configurations in which the incident particle can pass through the two regions of interaction surrounding the neutron and proton, i.e., it represents double scattering. Triple and higher order scattering does not occur in this calculation since the scattering is assumed to take place predominantly through small angles.

The elastic scattering amplitude in the forward direction is given by

$$F(\vec{k},\vec{k}) = f_n(\vec{k},\vec{k}) + f_p(\vec{k},\vec{k}) + \frac{k}{2\pi i}\int |u(\vec{q})|^2 \Gamma_n(\vec{b}-\tfrac{\vec{s}}{2})\Gamma_p(\vec{b}+\tfrac{\vec{s}}{2})\,d^{(2)}b\,d\vec{q}\ , \tag{182}$$

and from this, by means of the optical theorem, we immediately find the total cross section,

$$\sigma_d = \sigma_n + \sigma_p - 2\,\mathrm{Re}\int |u(\vec{q})|^2 \Gamma_n(\vec{b}-\tfrac{\vec{s}}{2})\Gamma_p(\vec{b}+\tfrac{\vec{s}}{2})\,d^{(2)}b\,d\vec{q} \tag{183}$$

The last term here is evidently the general expression for the cross section defect. To evaluate it accurately one must know the functions Γ, or equivalently, the phase shift functions for interactions with both the neutron and proton, as well as the deuteron wave function, particularly for small neutron-proton separations. Once again, in evaluating the total cross section we have implicitly summed over all the effects which decrease the incident beam strength. That means that corrections of all the types detailed earlier, i.e., interference, etc., are properly taken into account by Eq. (183).

To show the correspondence of this result with our earlier geometrical analysis, we have only to assume that the interaction ranges are small compared with the deuteron radius. In that case the correction term of Eq. (183) may be integrated approximately, so that we have

$$\sigma_d = \sigma_n + \sigma_p + \frac{4\pi}{k^2}\left\{f_n(0)\,f_p(0)\right\}\left\langle\frac{1}{r^2}\right\rangle_d . \tag{184}$$

If the neutron and proton are simply absorptive, i.e., their phase shifts are purely imaginary, their scattering amplitudes become imaginary as well, so that an application of the optical theorem reduces Eq. (184) to Eq. (175). It should be noted that the condition that the phase shifts be imaginary may correspond to any degree of grayness so that the present calculation is much more general than that for the black-sphere model we began with. In practice the interaction radii of nucleons and mesons are not small enough compared with the deuteron size for Eq. (184) to furnish more than a crude estimate. The correct result may be found from the general expression (183) by taking the finite size of the interaction regions into account.

Stripping of High-Energy Deuterons

Another type of problem, one which involves the deuteron as a high-energy projectile may be treated by methods similar to those we have just presented. One of the earliest observations made with high-energy deuteron beams was of their decomposition on passing through a target into well collimated beams of neutrons and protons of approximately half the incident energy. These emerging nucleons were described by Serber[14] as constituents of deuterons which had made grazing collisions with target nuclei and in which the other member, either a neutron or a proton, had been greatly deflected by a strongly inelastic collision.

If the target nucleus is represented schematically by a complex potential well (anticipating the results of the next section), the major difference between a deuteron-nucleus collision and the collisions we have treated, of an unspecified incident particle with the deuteron, lies in the frame of reference used. Formulae such as (177) for $F(\vec{k}',\vec{k})$ apply equally well to cases in which the deuteron is moving and the incident particle (i.e., a nucleus) assumes the role of a stationary target. The functions $u_i(q)$ and $u_f(q)$ still refer to the internal states of the deuteron, but $\hbar\vec{k}$ and $\hbar\vec{k}'$ become the initial and final momenta of the deuteron center of mass. This formulation of the stripping problem has been used, for example, to show that the deuteron may be dissociated in grazing encounters with nuclei in which neither the neutron nor the proton suffers any strong collision.[15] In effect the deuteron is disintegrated simply by the diffraction of particles about the edge of the nucleus. A more detailed treatment of deuteron-nucleus collisions based on essentially the present method has been published by Akhieser and Sitenko.[16]

The Optical Model for Elastic Scattering

Let us apply the approximation procedure described in the earlier sections to the collision of an incident particle with a nucleus. The coordinates of the particles which inhabit the nucleus will be written as

$$\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_N .$$

Now if the nucleus is initially in the state i, the elastic scattering amplitude is

$$F_{ii}(\vec{k}',\vec{k}) = \frac{k}{2\pi i}\int e^{i(\vec{k}-\vec{k}')\cdot\vec{b}}d^{(2)}b \cdot \tag{185}$$

$$\int |u_i(\vec{q}_1,\ldots,\vec{q}_N)|^2\left\{e^{i\chi(\vec{b},\vec{s}_1,\ldots\vec{s}_N)}-1\right\}\prod_{j=1}^{N}d\vec{q}_j \ ,$$

where the phase $\chi(\vec{b},\vec{s}_1,\ldots,\vec{s}_N)$ is the sum of the phase shifts contributed by the individual nucleons

$$\chi(\vec{b},\vec{s}_1,\ldots\vec{s}_N) = \sum_{j=1}^{N}\chi_j(\vec{b}-\vec{s}_j) \ . \tag{186}$$

In the above formulae the $\vec{s}_j$ are, as before, the components of the $\vec{q}_j$ perpendicular to $\vec{K}$.

Now it is clear from a comparison of initial and final states that the nucleus plays a relatively passive role in elastic scattering. It is reasonable to ask, therefore, whether a totally inert model of the nucleus can be constructed which yields precisely the same elastic scattering. A way of constructing such a static model would be to find a potential well which scatters the incident particle in the way desired. This procedure, if feasible, would reduce at least this aspect of the many-body problem to the consideration of an effective one-body problem.

Now the procedure we have described is, in fact, quite simple to carry out, using the high-energy approximation. Instead of solving directly for an effective potential, however, we shall first work with phase shift functions. We have only to compare the general expression (185) for a nucleus as target with the expression

$$f(\vec{k}',\vec{k}) = \frac{k}{2\pi i}\int e^{i(\vec{k}-\vec{k}')\cdot\vec{b}}\left\{e^{i\chi(\vec{b})}-1\right\}d^{(2)}b \ , \tag{187}$$

for the scattering by a static potential. We see immediately that if we let the static potential have a phase shift function

$$\chi_{opt.}(\vec{b})$$

given by

$$e^{i\chi_{opt}(\vec{b})} = \int |u_i(\vec{q}_1,\ldots,\vec{q}_N)|^2 e^{i\sum_{j=1}^{N}\chi_j(\vec{b}-\vec{s}_j)}\prod_i d\vec{q}_i \ , \tag{188}$$

we secure the desired equality

$$f(\vec{k}',\vec{k}) = F_{ii}(\vec{k}',\vec{k}) \ . \tag{189}$$

The relation defining $\chi_{opt.}(\vec{b})$ may be abbreviated a bit by writing

$$e^{i\chi_{opt.}(\vec{b})} = \left\langle e^{i\chi(\vec{b},\vec{s}_1,\ldots,\vec{s}_N)}\right\rangle_i \ , \tag{190}$$

where the phase shift $\chi(\vec{b},\vec{s}_1,\ldots,\vec{s}_N)$ is given by Eq. (186), and the brackets $\langle\ \rangle_i$ signify that an average is to be taken over all configurations of nucleons in the i-th state.

Now, as we mentioned earlier, as long as the interactions we are considering furnish a complete description of the problem, the phase shift produced by each of the nuclear particles will be real. Hence their sum, $\chi(\vec{b},\vec{s}_1,\ldots,\vec{s}_N)$ will be real as well. The expression (190) for the optical phase shift shows, however, that $\chi_{opt}(\vec{b})$ must, in general, be complex. Mathematically, this can be seen by noting that the values of

$$\exp\left[i\chi(\vec{b},\vec{s}_1,\ldots,\vec{s}_N)\right]$$

are unit vectors in the complex plane. An average taken over such unit vectors must have absolute magnitude less than or equal to unity. Furthermore, the case of equality would be realized only if $\chi(\vec{b},\vec{s}_1,\ldots\vec{s}_N)$ were independent of the configuration of nuclear particles. Since no such independence holds in general, we have

$$\left| e^{i\chi_{opt}(\vec{b})}\right| < 1 \ . \tag{191}$$

i.e., the optical phase shift function $\chi_{opt}(\vec{b})$ has a positive (and, in

general, non-vanishing) imaginary part.

The physical interpretation of the imaginary part of $\chi_{opt.}(\vec{b})$ is, of course, quite simple. Not all of the particles colliding with the actual nucleus undergo elastic scattering. Those which undergo inelastic collisions must be absent from the purely elastic scattering described by the optical model. The optical potential describes their removal in terms of an effective absorption. To see this in somewhat more detail we note that the exponential

$$\exp[i\chi(\vec{b},\vec{s}_1,\ldots,\vec{s}_N)]$$

is a unitary operator on the nuclear states

$$u_j(\vec{q}_1,\ldots,\vec{q}_N)\ ,$$

and the expression (188) is one of its diagonal matrix elements. Such a diagonal matrix element must, in general, have absolute value less than unity simply because the state

$$e^{i\chi(\vec{b},\vec{s}_1,\ldots,\vec{s}_N)}u_i(\vec{q}_1,\ldots,\vec{q}_N)$$

contains nuclear states other than the initial state u_i. These other components are the final states of inelastic scattering processes. To formulate the inequality more precisely, we use the relation

$$\sum_f\left|\int u_f^*\, e^{i\chi(\vec{b},\vec{s}_1,\ldots,\vec{s}_N)}u_i\prod_j d\vec{q}_j\right|^2 =$$

$$= \left|e^{i\chi_{opt}(\vec{b})}\right|^2+\sum_{f\neq i}\left|\int u_f^*\, e^{i\chi(\vec{b},\vec{s}_1,\ldots,\vec{s}_N)}u_i\prod_j d\vec{q}_j\right|^2 \qquad (192)$$

$$= 1\ ,$$

which follows from the definition of a unitary operator, and the definition (188) of

$$\exp[i\chi_{opt.}(\vec{b})]\ .$$

From this relation we see once again

$$\left|e^{i\chi_{opt.}(\vec{b})}\right|\leq 1\ .$$

(The case of equality arises only when $\chi(\vec{b},\vec{s}_1,\ldots,\vec{s}_N)$ is diagonal in the initial nuclear state and there are thus no inelastic transitions.)

A simple way of gaining insight into the structure of the optical model is to calculate the phase shift function $\chi_{opt.}(\vec{b})$ as a series expansion in powers of $\chi(\vec{b},\vec{s}_1,\ldots,\vec{s}_N)$, i.e., we write

$$\chi_{opt}(\vec{b}) = -i\log\left\langle e^{i\chi(\vec{b},\vec{s}_1,\ldots,\vec{s}_N)}\right\rangle_i$$

$$= -i\log\left\{1+i\langle\chi\rangle_i-\tfrac{1}{2}\langle\chi^2\rangle_i+\cdots\right\}\ , \qquad (193)$$

which yields

$$\chi_{opt.}(\vec{b}) = \langle\chi(\vec{b},\vec{s}_1,\ldots,\vec{s}_N)\rangle_i$$

$$+\frac{i}{2}\left\{\langle\chi^2(\vec{b},\vec{s}_1,\ldots,\vec{s}_N)\rangle-\langle\chi(\vec{b},\vec{s}_1,\ldots,\vec{s}_N)\rangle^2\right\}+\cdots\ . \qquad (194)$$

To first order we see that the expression for $\chi_{opt}(\vec{b})$ is simply the average of the function $\chi(\vec{b},\vec{s}_1,\ldots,\vec{s}_N)$, i.e., the phase shift for any particular set of nucleon positions, averaged over all nuclear configurations. If the interaction takes place by means of ordinary potentials the first order term of χ_{opt} is evidently real. An imaginary part for $\chi_{opt.}$ appears only as a second order perturbation since it is due to the possibility of nuclear excitation. The second order term of $\chi_{opt.}$ is evidently purely absorptive (i.e., positive imaginary) in character, since it involves just the mean variance of the phase shift function. The fluctuation of the phase shift function arises simply from the fact that the nucleons of the target nucleus are not fixed in position. Indeed were the nucleons fixed in position, the imaginary part of $\chi_{opt.}$ would vanish, in accord with the purely elastic nature of the scattering in that case.

The power series expansion of the expression (193) for $\chi_{opt}(\vec{b})$ is of a type which is well known in statistical theory. To develop its higher order terms we may write it as

$$i\chi_{opt}(\vec{b}) = \log\left\langle e^{i\chi}\right\rangle_i = i\langle\chi\rangle_i + \log\left\langle e^{i(\chi-\langle\chi\rangle_i)}\right\rangle_i . \tag{195}$$

Now the expansion in powers of a parameter λ of the function

$$\log\left\langle e^{i\lambda(\chi-\langle\chi\rangle_i)}\right\rangle_i$$

is easily obtained. Its coefficients are simple homogeneous polynomials in the various central moments of the function $\chi(\vec{b},\vec{s}_1,\ldots,\vec{s}_N)$. The expansion which we find for $\chi_{opt}(\vec{b})$ may be written as

$$i\chi_{opt}(\vec{b}) = \sum_{n=1}^{\infty}\frac{i^n}{n!}K_n(\vec{b}) , \tag{196}$$

where the $K_n(\vec{b})$ are given by

$$\begin{aligned}
K_1(\vec{b}) &= \langle\chi(\vec{b},\vec{s}_1,\ldots\vec{s}_N)\rangle_i \\
K_2(\vec{b}) &= \left\langle\left(\chi(\vec{b},\vec{s}_1,\ldots\vec{s}_N)-\langle\chi(\vec{b},\vec{s}_1,\ldots\vec{s}_N)\rangle_i\right)^2\right\rangle_i \\
K_3(\vec{b}) &= \langle(\chi-\langle\chi\rangle_i)^3\rangle_i \\
K_4(\vec{b}) &= \langle(\chi-\langle\chi\rangle_i)^4\rangle_i - 3\langle(\chi-\langle\chi\rangle_i)^2\rangle_i^2 \\
K_5(\vec{b}) &= \langle(\chi-\langle\chi\rangle_i)^5\rangle_i - 10\langle(\chi-\langle\chi\rangle_i)^2\rangle_i\langle(\chi-\langle\chi\rangle_i)^3\rangle_i \\
K_6(\vec{b}) &= \langle(\chi-\langle\chi\rangle_i)^6\rangle_i - 16\langle(\chi-\langle\chi\rangle_i)^2\rangle_i\langle(\chi-\langle\chi\rangle_i)^4\rangle_i \\
&\quad - 10\langle(\chi-\langle\chi\rangle_i)^3\rangle_i^2 - 15\langle(\chi-\langle\chi\rangle_i)^2\rangle_i^3 \\
&\cdots
\end{aligned} \tag{197}$$

The coefficients K_n are the cumulants of the distribution function for $\chi(\vec{b},\vec{s}_1,\ldots,\vec{s}_N)$, or the semi-invariants of Thiele,[17] as they are also known to statisticians.

Once the optical phase shift function $\chi_{opt.}(\vec{b})$ has been evaluated either by means of the integral (188) for the series (196), it may be used in a number of ways to reduce the many-particle problem to one involving, in effect, only the incident particle. In particular the scattering and total cross sections are given in terms of $\chi_{opt}(\vec{b})$ by means of the single-particle formulae (96) and (98). That the latter formula holds for the total cross section follows from the generalized form of the optical theorem (167) which was shown to hold for collisions with bound particles. In that discussion of the optical theorem an expression (169) was written for the inelastic part of the total cross section which we may generalize to the case of many bound particles as

$$\sigma_{inelast.} = \int d^{(2)}b\left\{1-\left|\int|u_i(\vec{q}_1,\ldots,\vec{q}_N)|^2 e^{i\sum\chi_j(\vec{b}-\vec{s}_j)}\prod_j d\vec{q}_j\right|^2\right\} . \tag{198}$$

This expression may be written in terms of $\chi_{opt}(b)$ as

$$\sigma_{inelast.} = \int d^{(2)}b\left\{1-\left|e^{i\chi_{opt}(\vec{b})}\right|^2\right\} ,$$

which is simply the absorption cross section (98) for the equivalent single particle problem. This observation illustrates the self-consistency of our formalism; particles which are described as absorbed in traversing a nucleus indeed account correctly for those which appear in the inelastic scattering.

Although we have illustrated the derivation of the function $\chi_{opt}(\vec{b})$ for the particular case of a many-particle nucleus it is well to bear in mind that the concepts used make no particular reference to the number of degrees of freedom involved. The scattering system need only be capable of a response which permits inelastic collisions. The effective phase shift function is equally well defined (and a great deal simpler to calculate) if the scatterer consists only of a single bound particle. It is also well defined for schematic models of the nucleus which ignore all but a few degrees of freedom. In discussing rotational states, for example, it is convenient to represent the nucleus as a homogeneous ellipsoid having only the Euler angles as degrees of freedom. The calculation of $\chi_{opt.}(b)$ for such a system describes the effect of rotational inelasticity on the elastic scattering.

Solution for the Optical Potential

The optical model, as it is conventionally represented, replaces the actual interaction of an incident particle with the nucleus by an effective potential which is complex. In the preceding section, however, we have derived an optical representation of the nucleus on the basis of a complex effective phase shift function, $\chi_{opt}(\vec{b})$, which may be found from a knowledge of the force laws between particles and the nuclear ground state wave function. The two representations are clearly equivalent under conditions in which our approximations hold. An effective potential $V_{opt}(\vec{r})$ corresponds to a unique phase shift function $\chi_{opt}(\vec{b})$ and vice versa. In calculations making use of these approximations the phase shift function is usually the more useful of the two, but there is no difficulty in providing an explicit evaluation of the optical potential.

If the optical potential is assumed to be spherically symmetric ($V_{opt} = V_{opt}(|\vec{r}|)$), the effective phase shift function is related to it by means of the equation

$$\chi_{opt}(b) = -\frac{1}{\hbar v}\int_{-\infty}^{\infty} V_{opt}(\sqrt{b^2+z^2})\,dz\ , \tag{199}$$

or

$$\chi_{opt}(b) = -\frac{2}{\hbar v}\int_{b}^{\infty} \frac{V_{opt}(r)\,r\,dr}{\sqrt{r^2-b^2}}\ . \tag{200}$$

If $\chi_{opt}(b)$ is regarded as known and $V_{opt}(\vec{r})$ as unknown, the latter relation becomes an Abel integral equation, a type which may be solved in elementary terms. The solution, in fact, is

$$V_{opt}(r) = \frac{\hbar v}{\pi}\frac{1}{r}\frac{d}{dr}\int_{r}^{\infty}\frac{\chi_{opt}(b)\,b\,db}{\sqrt{b^2-r^2}} \tag{201}$$

This expression completes, in principle, the construction of the optical potential from a knowledge of the elementary interactions.

Relation of the Optical Potential to the Scattering Amplitudes (for a Simple Model)

The foregoing formalism may be illustrated in a simple way by considering an independent particle model of the nucleus. The effects of nucleon-nucleon correlations which such a model neglects will be considered explicitly in the next section. It will be helpful, therefore, to phrase the problem in general terms before introducing the simplifying features of the model.

We assume that the phase shift functions $\chi_j(b)$ which describe the

interaction of the incident particle with the individual target nucleons ($j = 1,\dots,N$) are known, and introduce the functions

$$\Gamma_j(\vec{b}) = 1 - e^{i\chi_j(\vec{b})}\ . \tag{202}$$

These functions vanish for $\vec{b}$ outside the nuclear force range. The scattering amplitude of the incident particle by the l-th nucleon, if it were isolated from the others ($j \neq l$), would be

$$f_l(\vec{k}',\vec{k}) = \frac{ik}{2\pi}\int e^{i(\vec{k}-\vec{k}')\cdot\vec{b}}\,\Gamma_l(\vec{b})\,d^{(2)}b\ . \tag{203}$$

We are, however, dealing with a situation in which multiple scattering involving all of the particles occurs. Once again we let $\vec{q}_j$ be the nucleon coordinates and $\vec{s}_j$ their projections on a plane perpendicular to the momentum of the incident particle. Now if we let $u_i(\vec{q}_1,\dots,\vec{q}_N)$ be the nuclear ground state wave function and make use of the identity

$$e^{i\sum_j \chi_j(\vec{b}-\vec{s}_j)} = \prod_{j=1}^{N}\left(1-\Gamma_j(\vec{b}-\vec{s}_j)\right), \tag{204}$$

we may write the elastic scattering amplitude of the nucleus as

$$F_{ii}(\vec{k}',\vec{k}) = \tag{205}$$

$$\frac{k}{2\pi i}\int e^{i(\vec{k}-\vec{k}')\cdot\vec{b}}\left\{\int |u_i(\vec{q}_1,\dots,\vec{q}_N)|^2 \prod_j (1-\Gamma_j(\vec{b}-\vec{s}_j))\,d\vec{q}_j - 1\right\} d^{(2)}b\ .$$

This expression is, in effect, simply an extension of Eqs. (177) and (181), for the scattering amplitude of the deuteron, to the case of a many-particle system. The product (204) may be expanded to the form

$$\prod_j (1-\Gamma_j) = 1 - \sum_j \Gamma_j + \sum_{j\neq l} \Gamma_j \Gamma_l - \cdots + (-1)^N \prod_{j=1}^N \Gamma_j \quad . \tag{206}$$

When this expansion is substituted in Eq. (205), the terms containing a single function Γ_j may be expressed immediately in terms of the scattering amplitudes (203). If all further terms were dropped, the sum of these terms linear in the Γ_j would constitute the familiar single-scattering approximation

$$F_{il}(\vec{k}',\vec{k}) \approx \sum_{j=1}^N f_j(\vec{k}',\vec{k}) \int e^{i(\vec{k}-\vec{k}')\cdot\vec{s}_j} |u_i(\vec{q}_1,\ldots,\vec{q}_N)|^2 \prod_j d\vec{q}_j \quad . \tag{207}$$

The further terms of the expansion (206) provide corrections for effects of multiple scattering and interference which become important in configurations in which nucleons lie in the "shadows" cast by one another.

The integrations required to find the nuclear scattering amplitude are simplified considerably by using an independent particle model for the nucleus. We assume, as a first approximation, that the nuclear ground state wave function factorizes so that $|u_i(\vec{q}_1,\ldots,\vec{q}_N)|^2$ may be written as

$$|u_i(\vec{q}_1,\ldots\vec{q}_N)|^2 = \prod_j \rho_j(\vec{q}_j) \quad , \tag{208}$$

where $\rho_j(\vec{q}_j)$ is the normalized density for the j-th particle. The integral (188) which defines the effective phase shift function is then given by

$$\begin{aligned} e^{i\chi_{opt}(\vec{b})} &= \int |u_i(\vec{q}_1,\ldots\vec{q}_N)|^2 \prod_{j=1}^N (1-\Gamma_j(\vec{b}-\vec{s}_j))\, d\vec{q}_j \\ &= \prod_{j=1}^N \{1 - \int \rho_j(\vec{q}_j)\Gamma_j(\vec{b}-\vec{s}_j)\, d\vec{q}_j\} \quad . \end{aligned} \tag{209}$$

The function $\chi_{opt}(\vec{b})$ is thus given by

$$\chi_{opt}(\vec{b}) = -i \sum_j \log\{1 - \int \rho_j(\vec{q}_j)\Gamma_j(\vec{b}-\vec{s}_j)\, d\vec{q}_j\} \quad , \tag{210}$$

where in each term the branch of the logarithm which approaches zero as $b \to \infty$ is to be chosen.

A further simplification which is convenient at this point is the assumption that the range a of the interactions between the incident particle and the target nucleons is much smaller than the nuclear radius R. In that case, since the functions Γ_j assume values at most of order unity within the force range and vanish outside it, the integrals

$$\int \rho_j(\vec{q}_j)\Gamma_j(\vec{b}-\vec{s}_j)\, d\vec{q}_j \tag{211}$$

must be quite small in absolute value. In fact they must vanish for values of b appreciably larger than R, and for smaller b take on values at most of order a^2/R^2. We may therefore expand the arguments of the logarithms in (210) to find

$$\chi_{opt.}(\vec{b}) = i \sum_j \int \rho_j(\vec{q}_j)\Gamma_j(\vec{b}-\vec{s}_j)\, d\vec{q}_j + \cdots \tag{212}$$

The higher order terms, which have been dropped for simplicity are at most of order a^2/R^2 relative to those retained.

Since there are only two types of nucleons within the nucleus, neutrons and protons, only two different functions Γ_j actually enter the problem. We shall carry the simplification one stage further, however, by assuming only one kind of particle to be present in the model nucleus. The actual case may be described by a self-evident extension of the notation.

If all the functions $\Gamma_j(b)$ take the same form, $\Gamma(b)$, the expression (212) for $\chi_{opt}(b)$ may be written in terms of the average particle density

$$\rho(\vec{q}) = \frac{1}{N} \sum_{j=1}^N \rho_j(\vec{q}) \tag{213}$$

as

$$\chi_{opt}(\vec{b}) = iN \int \rho(\vec{q})\Gamma(\vec{b}-\vec{s})\, d\vec{q} \quad . \tag{214}$$

The average density function $\rho(\vec{q})$ presumably varies fairly smoothly over the nuclear volume. We may therefore take advantage once more of the smallness of the force range to carry out the integration (214) in an approximate way. For this purpose it is convenient to write the argument of the function $\rho(\vec{q})$ in terms of its components z, parallel to $\vec{k}$, and $\vec{s}$, perpendicular to $\vec{k}$, i.e.,

$$\rho(\vec{q}) \equiv \rho(\vec{s},z).$$

We then find, again by neglecting terms of relative order α^2/R^2,

$$\chi_{opt}(\vec{b}) = i N \int \Gamma(\vec{s}) d^{(2)}\vec{s} \int_{-\infty}^{\infty} \rho(\vec{b},z)\, dz . \tag{215}$$

Now, according to Eq. (203), the amplitude for forward scattering by a single nucleon is

$$f(0) = \frac{ik}{2\pi} \int \Gamma(\vec{b}) d^{(2)}b. \tag{216}$$

Hence the effective phase shift function may be written as

$$\chi_{opt}(\vec{b}) = \frac{2\pi N}{k} f(0) \int \rho(\vec{b},z)\, dz . \tag{217}$$

Since the function ρ is, in general, spherically symmetric, it is clear that this phase shift function corresponds to an optical potential which has the same form as the density. The optical potential is given by

$$\begin{aligned} -\frac{1}{\hbar v} V_{opt}(r) &= \frac{2\pi N}{k} f(0)\, \rho(r) \\ &= N\left[\frac{2\pi}{k} \mathrm{Re} f(0) + \frac{i}{2}\sigma\right] \rho(r) , \end{aligned} \tag{218}$$

where σ is the total cross section of a single nucleon. An equivalent description of the scattering may be made in terms of a complex nuclear refractive index $n(r)$ given by

$$n(r) - 1 = \frac{2\pi N}{k^2} f(0)\, \rho(r) . \tag{219}$$

This is an expression frequently quoted in the literature, but rarely with much indication of the conditions under which it holds. The present derivation besides requiring conditions favorable to the high energy approximation, has assumed the nucleons to move independently of one another and has neglected terms of order α^2/R^2.

The foregoing considerations are easily generalized to include spin-orbit interactions between the incident particle and the target nucleons. The functions $\Gamma(b)$, which we assume for simplicity remain the same for all nucleons, are then given by

$$\Gamma(\vec{b}) = \Gamma_c(\vec{b}) + i\vec{\sigma}\cdot\left(\frac{\vec{b}}{b} \times \hat{K}\right)\Gamma_s(\vec{b}) , \tag{220}$$

where Γ_c and Γ_s are the axially symmetric functions defined by (131) and (132). The optical phase shift function may be evaluated as before by substituting this expression in (214) and approximating the required integral. The treatment of the non-spin-dependent term to order α^2/R^2 is the same as in the previous case. The spin-dependent term, however, requires a bit more attention. To the extent that the function

$$\int_{-\infty}^{\infty} \rho(\vec{s},z)\, dz$$

may be regarded as a constant independent of $\vec{s}$ within the nucleon interaction range, the integral of the spin-dependent term vanishes. To evaluate this term, therefore, the function

$$\int_{-\infty}^{\infty} \rho(\vec{s},z)\, dz$$

must be expanded in a power series about the point $\vec{s} = \vec{b}$, i.e., by introducing the vector $\vec{l} = \vec{b} - \vec{s}$ we may write

$$\begin{aligned} &\int \rho(\vec{q})\Gamma(\vec{b}-\vec{s})\, d\vec{q} = \\ &\int \rho(\vec{b}-\vec{l},z)\left\{\Gamma_c(\vec{l}) + i\vec{\sigma}\cdot\left(\frac{\vec{l}}{l}\times\hat{K}\right)\Gamma_s(\vec{l})\right\} d^{(2)}l\, dz \\ &\approx \int \Gamma_c(\vec{l})\, d^{(2)}l \int_{-\infty}^{\infty} \rho(\vec{b},z)\, dz + \\ &i\vec{\sigma}\cdot\left\{\hat{K}\times\int \frac{\vec{l}}{l}\Gamma_s(\vec{l})\,\vec{l}\cdot\nabla_b \int_{-\infty}^{\infty} \rho(\vec{b},z)\, dz\, d^{(2)}l\right\} . \end{aligned} \tag{221}$$

If the spin-orbit and central forces are of comparable magnitudes the spin-dependent term above is smaller than the spin-independent one by a factor of order a/R. Although spin-dependence must therefore be viewed as a small effect it may be consistently retained in an expansion which neglects terms of order a^2/R^2.

The integrals of the functions Γ_c and Γ_s which occur in (221) may be related to the two terms $f(\vec{k}';\vec{k})$ and $g(\vec{k}';\vec{k})$ of the spin-dependent scattering amplitude of a nucleon defined by (135) and (136). Clearly the integral of Γ_c is related to the forward value of the spin-independent amplitude f,

$$f(0) = f(\vec{k},\vec{k}) = \frac{ik}{2\pi}\int \Gamma_c(\vec{b})\, d^{(2)}b \,. \tag{222}$$

Advantage may be taken of the axial symmetry of the function Γ_s by writing the integral involving it as

$$\int \frac{l_i l_j}{l}\,\Gamma_s(l)\, d^{(2)}l = \frac{1}{2}\delta_{ij}\int l\,\Gamma_s(l)\, d^{(2)}l \,, \tag{223}$$

where the indices i and j label cartesian axes. The latter integral may be expressed in terms of the derivative with respect to the scattering angle ϑ of the spin-dependent amplitude g near the forward direction

$$\begin{aligned}\frac{1}{2}\int b\,\Gamma_s(b)\, d^{(2)}b &= \frac{2\pi i}{k^2}\frac{d}{d\vartheta}\, g(\vartheta)\Big|_{\vartheta=0} \\ &= \frac{2\pi i}{k^2}\, g'(0) \,.\end{aligned} \tag{224}$$

Hence, with the presence of a spin-orbit interaction, we find the effective phase shift function

$$\begin{aligned}\chi_{opt}(\vec{b}) &= iN\int \rho(\vec{q})\,\Gamma(\vec{b}-\vec{s})\, d\vec{q} \\ &= \frac{2\pi N}{k}\left\{f(0) - \frac{i}{k}\, g'(0)\,\vec{\sigma}\cdot(\hat{k}\times\nabla_b)\right\}\int_{-\infty}^{\infty}\rho(b,z)\, dz \,.\end{aligned} \tag{225}$$

An optical potential may be derived from this expression by writing it in the equivalent form

$$\chi_{opt}(\vec{b}) = \frac{2\pi N}{k}\int\left\{f(0) - \frac{i}{k}\, g'(0)\,\vec{\sigma}\cdot(\hat{k}\times\nabla_q)\right\}\rho(\vec{q})\, dz\Big|_{\vec{q}-\hat{k}(\hat{k}\cdot\vec{q})=\vec{b}} \,. \tag{226}$$

This form makes it evident that a suitable potential is given by

$$-\frac{1}{\hbar v}\, V_{opt}(\vec{r}) = \frac{2\pi N}{k}\left\{f(0) - \frac{i}{k}\, g'(0)\,\vec{\sigma}\cdot(\hat{k}\times\nabla_r)\right\}\rho(r) \,. \tag{227}$$

Since the average density function is spherically symmetric we have

$$\nabla_r\,\rho(r) = \frac{\vec{r}}{r}\frac{d}{dr}\rho(r) \,.$$

We must now remember that $\hat{k}$ is a unit vector in the direction of the incident momentum $\hbar\vec{k}$ (or, the nearly parallel direction given by (89)). The vector product $(\vec{r}\times\vec{k})$ is of course the orbital angular momentum (in units of $\hbar$) of the incident particle. Hence the optical potential takes a form such that

$$-\frac{1}{\hbar v}\, V_{opt}(r) = \frac{2\pi N}{k}\left\{f(0)\,\rho(r) + \frac{i}{k}\, g'(0)\frac{1}{r}\frac{d}{dr}\rho(r)\,\vec{\sigma}\cdot\vec{L}\right\} \,. \tag{228}$$

It is of considerable interest to note that the spin-orbit portion of this potential falls naturally into a form bearing an analogy to the Thomas interaction of atomic theory. Such a form was employed by Fermi in an early analysis of polarization experiments, and has been widely used since. We have shown that this form follows naturally from a power series expansion in the ratio a/R. Furthermore the coefficient of the spin-orbit potential, $(i/k^2)g'(0)$, is complex and may be established directly from suitable scattering and polarization measurements on single-nucleon targets.

The expression (228) for the optical spin-orbit potential is not an unfamiliar one, but previous derivations of it appear to have been confined to the single-scattering approximation analogous to (207). Such derivations, of course, do not justify the use of the potential (228) in calculations to any accuracy higher than the first order of perturbation theory. The actual discussion of polarization experiments, however, has required

a much more accurate treatment than perturbation theory, and hence the implicit assumption that the interaction (228) is correct to higher order as well. The present derivation, by implicitly summing the effects of multiple scattering, goes considerably further to justify the use of this interaction in the high-energy limit. The particular form found for the spin-orbit interaction depends on the assumption that the force range is much smaller than the nuclear radius.

Where this is not so, as in the α particle, for example, the spin-orbit potential may assume a rather different shape. A key point in the derivation has been the assumption that the nucleons are uncorrelated in position within the nucleus. The presence of correlations introduces corrections to the optical potential which we shall discuss in the next section.

Influence of Nucleon Correlations on the Optical Model [18]

Various types of correlations in position and spin may exist between the nucleons of an actual nucleus. Attractive or repulsive forces between the particles will in general lead to such deviations from independent-particle behavior, but correlations must be expected even in the absence of interaction effects. A certain correlation is introduced simply by the circumstance that nucleons are Fermi particles. This may be seen from the fact that an antisymmetric wave function vanishes whenever two similar particles have the same space and spin coordinates.

All of the information we require about the correlations is contained, of course, in the square of the absolute value of the nuclear ground state wave function, $u_i(\vec{q}_1, \ldots, \vec{q}_N)$. To develop a more convenient notation we shall define an N-particle density function $\rho^{(N)}(\vec{q}_1, \ldots, \vec{q}_N)$ as

$$\rho^{(N)}(\vec{q}_1, \ldots, \vec{q}_N) = |u_i(\vec{q}_1, \ldots, \vec{q}_N)|^2 , \tag{229}$$

which is symmetric in its N variables. A succession of density functions involving N-1, N-2, . . . coordinates may be defined by integrating over the remaining coordinates

$$\begin{aligned} \rho^{(N-1)}(\vec{q}_1, \ldots, \vec{q}_{N-1}) &= \int \rho^{(N)}(\vec{q}_1, \ldots, \vec{q}_N)\, d\vec{q}_N \\ &\cdots \\ \rho^{(2)}(\vec{q}_1, \vec{q}_2) &= \int \rho^{(3)}(\vec{q}_1, \vec{q}_2, \vec{q}_3)\, d\vec{q}_3 \\ \rho^{(1)}(\vec{q}_1) \equiv \rho(\vec{q}_1) &= \int \rho^{(2)}(\vec{q}_1, \vec{q}_2)\, d\vec{q}_2 \ . \end{aligned} \tag{230}$$

Each of these functions is symmetric and normalized to unity when integrated over all of its coordinates. The symbol ρ is introduced as an abbreviation for $\rho^{(1)}$. This function will be seen to play the same role as the independent-particle density ρ of the preceding section.

According to the definition (188), the optical phase shift function may be expressed in the form

$$\begin{aligned} \chi_{opt}(\vec{b}) &= -i \log \int \rho^{(N)}(\vec{q}_1, \ldots, \vec{q}_N)\, e^{i \sum_j \chi_j(\vec{b} - \vec{s}_j)} \prod_j d\vec{q}_j \\ &= -i \log \left\langle e^{i \sum_j \chi_j(\vec{b} - \vec{s}_j)} \right\rangle_i \ . \end{aligned} \tag{231}$$

If we again introduce the functions Γ, according to (204), and assume all the nucleons identical, we have

$$\chi_{opt}(\vec{b}) = -i \log \left\langle \prod_{j=1}^{N} (1 - \Gamma(\vec{b} - \vec{s}_j)) \right\rangle_i . \tag{232}$$

To illustrate a way in which this expression may be evaluated, it is convenient to simplify the notation a bit and introduce a certain parametric dependence on a new variable λ. Let us consider the function

$$\chi(\lambda) = \log \left\langle \prod_j (1 + \lambda S(q_j)) \right\rangle , \tag{233}$$

where the function $S(q_j)$ obviously plays the role of $-\Gamma(b - s_j)$ in (232). We now expand the function $\chi(\lambda)$ in a power series about $\lambda = 0$.

$$\chi(0) = 0 , \tag{234}$$

$$\begin{aligned} \left. \frac{d\chi}{d\lambda} \right|_{\lambda=0} &= \left. \frac{\sum_i \left\langle S(q_i) \prod_{j \neq i} (1 + \lambda S(q_j)) \right\rangle}{\left\langle \prod_j (1 + \lambda S(q_j)) \right\rangle} \right|_{\lambda=0} = \sum_i \langle S(q_i) \rangle \\ &= N \langle S(q_1) \rangle = N \int S(\vec{q}) \rho(\vec{q})\, d\vec{q} \ , \end{aligned} \tag{235}$$

$$\left.\frac{d^2\mathcal{L}}{d\lambda^2}\right|_{\lambda=0} = \left[\frac{\sum_{l\neq k}\langle S(q_l)S(q_k)\prod_{j\neq l,k}(1+\lambda S(q_j))\rangle}{\langle\prod_j(1+\lambda S(q_j))\rangle} - \frac{\sum_{l,k}\langle S(q_l)\prod_{j\neq l}(1+\lambda S(q_j))\rangle\langle S(q_k)\prod_{j\neq k}(1+\lambda S(q_j))\rangle}{\langle\prod_j(1+\lambda S(q_j))\rangle^2}\right]_{\lambda=0} \tag{236}$$

$$= \sum_{l\neq k}\langle S(q_l)S(q_k)\rangle - \sum_{l,k}\langle S(q_l)\rangle\langle S(q_k)\rangle$$

$$= N(N-1)\langle S(q_1)S(q_2)\rangle - N^2\langle S(q_1)\rangle^2$$

$$= \int S(\vec{q}_1)S(\vec{q}_2)\{N(N-1)\rho^{(2)}(\vec{q}_1,\vec{q}_2) - N^2\rho(\vec{q}_1)\rho(\vec{q}_2)\}d\vec{q}_1 d\vec{q}_2 .$$

$$\left.\frac{d^3\mathcal{L}}{d\lambda^3}\right|_{\lambda=0} = \int S(\vec{q}_1)S(\vec{q}_2)S(\vec{q}_3)\{N(N-1)(N-2)\rho^{(3)}(\vec{q}_1,\vec{q}_2,\vec{q}_3) - \tag{237}$$

$$3N^2(N-1)\rho^{(2)}(\vec{q}_1,\vec{q}_2)\rho(\vec{q}_3) + 2N^3\rho(\vec{q}_1)\rho(\vec{q}_2)\rho(\vec{q}_3)\}d\vec{q}_1 d\vec{q}_2 d\vec{q}_3 .$$

These averages may be expressed a bit more briefly in terms of the actual particle densities, which may be written as

$$n(\vec{q}) = N\rho(\vec{q}) \quad , \quad n^{(2)}(\vec{q}_1,\vec{q}_2) = N(N-1)\rho^{(2)}(\vec{q}_1,\vec{q}_2) \, , \cdots$$

$$n^{(r)}(\vec{q}_1,\ldots,\vec{q}_r) = \frac{N!}{(N-r)!}\rho^{(r)}(\vec{q}_1,\ldots,\vec{q}_r) . \tag{238}$$

The expansion of $\mathcal{L}(\lambda)$ is then

$$\mathcal{L}(\lambda) = \lambda\int S(q)n(q)dq + \tag{239}$$

$$\frac{\lambda^2}{2}\int S(q_1)S(q_2)\{n^{(2)}(q_1,q_2) - n(q_1)n(q_2)\}dq_1 dq_2 + \cdots .$$

The expression (232) for the optical phase shift function may be evaluated by means of the expansion (239) by substituting $S(\vec{q}_j) = -\Gamma(\vec{b}-\vec{s}_j)$ and $\lambda = 1$. The phase shift function is then found to be

$$\chi_{opt.}(\vec{b}) = i\int n(\vec{q})\Gamma(\vec{b}-\vec{s})d\vec{q}$$

$$- \frac{i}{2}\int[n^{(2)}(\vec{q}_1,\vec{q}_2) - n(\vec{q}_1)n(\vec{q}_2)]\Gamma(\vec{b}-\vec{s}_1)\Gamma(\vec{b}-\vec{s}_2)d\vec{q}_1 d\vec{q}_2 \tag{240}$$

$$+ \frac{i}{6}\int[n^{(3)}(\vec{q}_1,\vec{q}_2,\vec{q}_3) - 3n^{(2)}(q_1,q_2)n(\vec{q}_3) +$$

$$2n(\vec{q}_1)n(\vec{q}_2)n(\vec{q}_3)]\Gamma(\vec{b}-\vec{s}_1)\Gamma(\vec{b}-\vec{s}_2)\Gamma(\vec{b}-\vec{s}_3)d\vec{q}_1 d\vec{q}_2 d\vec{q}_3 + \cdots .$$

The first term of this expansion is of precisely the form discussed in the preceding section. The function $n(q) = N\rho(q)$ plays in it the same role as the density in the independent-particle model. The succeeding terms of the expansion represent the effects of pair correlations, three-particle correlations, etc. To illustrate these effects further we shall discuss the second term of the series in a simple limit.

If the number of particles, N, is quite large, it becomes convenient to write the two-particle density $n^{(2)}$ in the form

$$n^{(2)}(\vec{q}_1,\vec{q}_2) = n(\vec{q}_1)n(\vec{q}_2)\left\{g(\vec{q}_1,\vec{q}_2) + \frac{h(\vec{q}_1,\vec{q}_2)}{N} + O\left(\frac{1}{N^2}\right)\right\} . \tag{241}$$

An expression of this type is useful since the function $n^{(2)}(\vec{q}_1,\vec{q}_2)$ will not in general deviate appreciably from the product $n(\vec{q}_1)n(\vec{q}_2)$ unless the points $\vec{q}_1$ and $\vec{q}_2$ are fairly close together. The function $h(\vec{q}_1,\vec{q}_2)/N$ is added within the parentheses so that the pair density function $g(\vec{r},\vec{r}')$ may be defined to take on the asymptotic value unity as either $\vec{r}$ or $\vec{r}'$ recedes to infinity. The function h takes on values of order unity and so the correction h/N may be neglected in integrals over small volumes. If the system being considered is spatially uniform, the pair density $g(\vec{r},\vec{r}')$ will depend only on the distance $|\vec{r}-\vec{r}'|$. With this assumption we may write

$$n^{(2)}(\vec{q}_1,\vec{q}_2) - n(\vec{q}_1)n(\vec{q}_2) \approx n(\vec{q}_1)n(\vec{q}_2)\{g(|\vec{q}_1-\vec{q}_2|) - 1\} \tag{242}$$

An idea of the magnitude and nature of the effects due to pair correlations may be obtained by assuming once again that the interaction range of a nucleon is small. In this limit the second term of the expansion (240) may be approximated as

$$-\frac{i}{2}\int\left[n^{(2)}(\vec{q}_1,\vec{q}_2)-n(\vec{q}_1)n(\vec{q}_2)\right]\Gamma(\vec{b}-\vec{s}_1)\Gamma(\vec{b}-\vec{s}_2)\,d\vec{q}_1\,d\vec{q}_2$$

$$\approx -\frac{i}{2}\left(\frac{2\pi}{ik}f(0)\right)^2\int_{-\infty}^{\infty}n^2(\vec{b},z)\,dz\cdot 2\int_0^{\infty}\left[g(|r|)-1\right]d|r|\,, \tag{243}$$

where the notation $n(\vec{s},z)\equiv n(\vec{q})$ has been used. The pair density function $g(|r|)$ is defined to approach unity as $|\vec{r}|\to\infty$. The distance within which $g-1$ undergoes a substantial decrease may be termed the correlation length l. It should be noted that the expression (243) is based on the assumption that this distance considerably exceeds the range of nucleon-nucleon forces, i.e., $l \gg a$. Furthermore the assumption of homogeneity which makes g a function of a single distance requires that l be much smaller than the nuclear radius R. Because R is not vastly larger than a, these conditions cannot both be satisfied very well for actual nuclei. Since the correlation length is not too different in magnitude from the force range, the short-range approximation (243) should only be used for rough estimates.

When the short-range approximation is used the expression (240) for the optical phase shift becomes

$$\chi_{opt.}(\vec{b}) = \frac{2\pi}{k}f(0)\int_{-\infty}^{\infty}n(\vec{b},z)\,dz + i\left(\frac{2\pi}{k}f(0)\right)^2\int_0^{\infty}\left[g(|r|)-1\right]d|r|\int_{-\infty}^{\infty}n^2(\vec{b},z)\,dz + \cdots \tag{244}$$

This form for the phase shift function is evidently equivalent to an optical potential given by

$$-\frac{1}{\hbar v}V_{opt}(r) = \frac{2\pi N}{k}f(0)\rho(r) + i\left(\frac{2\pi N}{k}f(0)\right)^2\rho^2(r)\int_0^{\infty}\left[g(|r|)-1\right]dr + \cdots . \tag{245}$$

The first term of this expansion is of a form we have already encountered in discussing the independent-particle model. The succeeding terms,

which represent the effects of nucleon-nucleon correlations, correspond to an expansion of the optical potential in powers of the nucleon density. The series is equally well an expansion in powers of the forward scattering amplitude; it possesses the dimensionless expansion parameter $2\pi\lambda l f(0) n$.

Since the forward scattering amplitude $f(0)$ is complex, the correlation term in (245) corrects both the real and imaginary parts of the optical potential. By using the optical theorem these parts may be written as

$$-\frac{1}{\hbar v}\mathrm{Re}\,V_{opt}(r) = \frac{2\pi}{k}n(r)\,\mathrm{Re}\,f(0)\left\{1-\sigma n(r)\int_0^{\infty}\left(g(r)-1\right)dr\right\} \tag{246}$$

$$-\frac{1}{\hbar v}\mathrm{Im}\,V_{opt.}(r) = n(r)\frac{1}{2}\left\{\sigma+\frac{8\pi^2}{k^2}\left[(\mathrm{Re}\,f(0))^2-(\mathrm{Im}\,f(0))^2\right]n(r)\int_0^{\infty}(g(r)-1)\,dr\right.$$

$$\left. = n(r)\frac{1}{2}\left\{\sigma+\frac{1}{2}\left[\left(\frac{4\pi}{k}\mathrm{Re}\,f(0)\right)^2-\sigma^2\right]n(r)\int_0^{\infty}\left(g(r)-1\right)dr\right\}. \right. \tag{247}$$

The correction terms due to correlations are not necessarily small in magnitude. If, for example, the correlation length is $l = 2\times10^{-13}$ cm, the total cross section of an isolated nucleon is $\sigma = 40$ millibarns, and the particle density is that of actual nuclei, the correction to the real part of the optical potential may be of the same magnitude as the independent-particle term. Larger cross sections, such as occur for incident antiprotons, will tend proportionally to increase the correction. Other factors, however, will tend to diminish the correlation effect to some extent. Correlations which keep particles apart, such as those due to repulsive forces and the exclusion principle, lead to negative values of the function $g(r)-1$. Any attractive correlations present will tend to compensate the repulsive ones by leading to positive values of $g(r)-1$. The correlations of actual nucleons may be expected to depend considerably on their spin and charge states. A further consideration which tends to diminish the correlation effect somewhat is the removal of the short-range approximation for the force between the incident particle and the nucleons. This approximation overestimates the corrections due to nucleons which lie at distances apart smaller than the force range.

The technique we have used in discussing correlations may easily be extended in various directions. We may, for example, seek the effect of correlations in the presence of spin-orbit forces. Since the functions $\Gamma(\vec{b}-\vec{s}_j)$ are operators in this case, and the particle density no longer

factorizes, it is necessary to pay careful attention to the ordering of factors. The result is simplest to state for the case in which the correlation length is much smaller than the nuclear radius. It is

$$-\frac{1}{\hbar v}V_{opt}(r) = \frac{2\pi}{k}\left\{f(0)n(r) + \frac{i}{k^2}g'(0)\frac{1}{r}\frac{d}{dr}n(r)\vec{\sigma}\cdot\vec{L}\right\}$$

$$+ i\left(\frac{2\pi}{k}\right)^2 n^2(r)\left\{f^2(0)\int_0^\infty (g(r)-1)dr + \left(\frac{g'(0)}{k}\right)^2\int_0^\infty \frac{z}{r}\frac{d}{dr}g(r)dr\right\}. \tag{248}$$

The function g'(o) which appears here is the derivative of the spin-dependent scattering amplitude (224) evaluated for $\vartheta = 0$. The function g(r) occurring in the integrands is again the pair density, defined by (241). The correlation term in (248) has been evaluated under the assumption that the nucleon density varies very slowly within a correlation length. The contributions from the nuclear surface which are thereby neglected may easily be constructed by the same means. The short-range approximation $a \ll l$ has also been used in deriving (248).

The Influence of the Exclusion Principle on the Optical Potential

The constraint of antisymmetry which the exclusion principle places upon a nuclear wave function produces certain correlations in the positions of the individual nucleons. The most important consequences of the exclusion principle may be described in terms of these correlations; their effects are present whatever is the nature of the incident particle.

We may assume, for the moment, in order to separate two problems, that the incident particle is not a nucleon. Then our main task in estimating the correlation effect is the evaluation of the pair density function g(r). For the simple model of a gas of many non-interacting Fermi particles, this function may be evaluated quite readily. Its derivation is usually discussed in connection with treatments of the electron gas in conductors. We define the Fermi momentum $\hbar k_F$ via the relation

$$\frac{1}{6\pi^2}k_F^3 = n\,, \tag{249}$$

where n is the density of nucleons (of similar charge and spin variables). Then the pair density function may be written as

$$g(r) = 1 - \frac{9\pi}{2}\frac{1}{(k_F r)^3}\left[J_{3/2}(k_F r)\right]^2$$

$$= 1 - 9\left\{\frac{\sin k_F r - k_F r\cos k_F r}{k_F^3 r^3}\right\}^2. \tag{250}$$

The pair density (250) may be seen to vanish for r = 0, and to approach unity for distances large compared to the Fermi wavelength, $1/k_F$. This behavior expresses the tendency of fermions of similar charge and spin to remain separated from one another; i.e., their positions are anticorrelated.

To evaluate the correlation effect in the short-range approximation we require the integral

$$\int_0^\infty (g(r)-1)dr = -\frac{9\pi}{2k_F}\int_0^\infty \frac{1}{x^3}\left(J_{3/2}(x)\right)^2 dx$$

$$= -\frac{3\pi}{5k_F}\,. \tag{251}$$

By using the definition, (249), of the Fermi momentum we may write

$$n(r)\int_0^\infty (g(r)-1)dr = -\frac{k_F^2}{10\pi}\,. \tag{252}$$

The optical potential may now be evaluated by means of equations (246) and (247). The real part of the optical potential is simply the real part found on the basis of the independent-particle model multiplied by the correction factor

$$1 + \frac{1}{10\pi}\sigma k_F^2\,. \tag{253}$$

For a cross section σ = 40 millibarns, the correction is some 20%. The imaginary part of the optical potential may be obtained by replacing the total cross section σ, which occurs in the expression for the independent-particle model, by the quantity

$$\sigma + \frac{k_F^2}{20\pi}\left[\sigma^2 - \left(\frac{4\pi}{k}\mathrm{Re}f(0)\right)^2\right] \tag{254}$$

which evidently represents an effective cross section for a nucleon in a many-particle nucleus.

The effective cross section (254) has a number of interesting properties. For incident particles of sufficiently high energy the expression $(4\pi/k)\,\mathrm{Re}\,f(0)$ may be expected to be considerably smaller in magnitude than the total cross section σ of a free nucleon. (This will be particularly true when absorptive, i.e., incoherent, processes play a large role. Experimental results to date for mesons and nucleons in the Bev range are consistent with assuming $\mathrm{Re}\,f(0) = 0$.) We must therefore expect that at such energies the effective cross section of a nucleon in the nucleus will

exceed the free particle cross section.[18] This prediction contradicts an assumption that is widely made in the literature and so deserves further discussion.

It is well known that the exclusion principle leads to the suppression of certain scattering processes, those in which the final states are already occupied by similar particles. It has usually been assumed, therefore, that the effective cross section of a particle within the nucleus is smaller than that of a free particle. Many calculations have been performed, either with the Born approximation or the single-scattering approximation (impulse approximation), to bear this point out. It would, in fact, be a simple matter to calculate the cross section of a Fermi gas on the basis of the single-scattering approximation (207). The effective cross section per nucleon that results is an expression similar to the effective cross section implicit in (247). The expression differs, however, through the replacement of $\mathrm{Re}\, f^2(0) = (\mathrm{Re}\, f(0))^2 - (\mathrm{Im}\, f(0))^2$ by $|f(0)|^2 = (\mathrm{Re}\, f(0))^2 + (\mathrm{Im}\, f(0))^2$. The latter result simply exhibits an inconsistency in the single-scattering approximation which grows in importance with the imaginary part of the forward scattering amplitude. The effects of double scattering and the interference of double and single scattering are responsible for changing the sign of the correlation term at high energies, i.e. where $(\mathrm{Im}\, f(0))^2$ is the dominant term.

The influence of the correlation effect may be understood in a fairly simple way, geometrically. For this purpose we recall that in the absence of correlations, i.e., in the independent particle model, the effective cross section of a nucleon is just the free particle cross section. The simplest influence of correlation upon the scattering is felt through interference effects. In a Fermi gas each particle is surrounded by a hole in the distribution of similar particles. Since a uniform distribution would lead to no interference effects, the scattering by the particles absent from the hole may be seen to interfere destructively with the scattering by the nucleon at its center. This destructive interference effect is precisely what is described by the single-scattering approximation. If the correlation length l greatly exceeds the wavelength, the interference correction to the cross section of a nucleon is negative and of the order of

$$\left(\frac{4\pi}{kl}\right)^2 |f(0)|^2 = \left(\frac{4\pi}{kl}\right)^2 \left\{(\mathrm{Re}\, f(0))^2 + (\mathrm{Im}\, f(0))^2\right\} = \left(\frac{4\pi}{kl}\mathrm{Re}\, f(0)\right)^2 + \frac{\sigma^2}{l^2}\,, \tag{255}$$

for each of the $\sim nl^3$ particles correlated with it.

In extending the picture to include multiple collisions we must remember that double and triple scattering, etc., are already present in the independent-particle model. Their effects are simply accounted for by use of the free nucleon cross section in calculating the imaginary part

of the optical potential. (I.e., the phase shift function for an independent-particle nucleus is proportional to $f(0)$ and is given by (217). The formula for the nuclear scattering amplitude contains the exponential of this function. If the exponential is expanded in a power series, its successive terms represent the contributions of successive orders of multiple collisions.) Since the effect of multiple scattering is already accounted for implicitly when the nucleon positions are distributed randomly, we have only to account for the changes introduced by correlations. Consider a particular pair of nucleons a correlation length apart. Double scattering effects (more generally including shadowing effects as well) will change their combined total cross section by an amount of order σ^2/l^2, i.e., by an amount precisely comparable to the contribution of $\mathrm{Im}\, f(0)$ to the interference effect (255). Thus double scattering and interference effects must be considered together when the forward scattering amplitude is predominantly imaginary.

The sign of the correlation effect and a number of its properties are made clear by recalling the "eclipse" effect in the deuteron, which was discussed in an earlier section. The total cross section of a deuteron at high energies is always found to be smaller than the sum of the two free nucleon cross sections. The reason for this was shown to lie in the way shadowing and double scattering effects dominate the corrections to the cross section. The treatment of the spatial correlation of nucleons within a Fermi gas poses a problem which is in a sense the reverse of the deuteron problem. Particles which are held together by an attractive force in the deuteron are kept apart by the exclusion principle in a Fermi gas. The anticorrelation of the Fermi particles simply reverses the sign of the effect. The tendency of particles to remain apart from one another in a Fermi gas means that they tend to remain outside the regions of "shadow" cast by their neighbors. They thus present a more effective target area to the incident beam than if their relative positions were random. It is for this reason that the effective cross section of a nucleon exceeds the free particle cross section at high energies.

Although the correction added to the free particle cross section is positive at high energies, it should be emphasized that the correction goes over into the more familiar negative one as the energy decreases. As the incident energy descends the imaginary part of the forward scattering amplitude, $\mathrm{Im}\, f(0) = k\sigma/4\pi$, decreases in magnitude until it is dominated by $|\mathrm{Re}\, f(0)|$. In that limit the long-standing estimates based upon the single-scattering approximation become more nearly correct.

We have thus far assumed that the incident particle is not a nucleon in order to confine the application of the exclusion principle to the nuclear wave function. If the incident particle is a nucleon the entire system, nucleus plus incident particle, must be described by an antisymmetric wave function. Exchange processes then become possible; i.e., the nucleon which leaves the nucleus is different from the one that enters. Such processes are indistinguishable from other forms of elastic scattering and interfere coherently with them. Although exchange pro-

cesses may contribute significantly to the scattering at low energies, we shall show that they are of little importance in the high-energy limit.

Consider the scattering of a pair of (non-identical) nucleons in their center of mass system. Suppose that the scattering amplitude for one of them is $f(\vartheta)$. If the nucleons are now considered to be identical particles, antisymmetrization of the wave function yields the scattering amplitudes $f(\vartheta) \pm f(\pi-\vartheta)$ for the singlet and triplet states. The domain of the angular argument ϑ, for these amplitudes, must be restricted to a single hemisphere, say, $0 \le \vartheta \le \frac{\pi}{2}$. The superposition expresses our inability to say whether, speaking in classical terms, a scattering through the angle ϑ or $\pi-\vartheta$ actually takes place. For angles ϑ at which the amplitudes $f(\vartheta)$ and $f(\pi-\vartheta)$ are both large, their interference may be important. In the high-energy limit, however, the scattering is characterized by extreme anisotropy; $f(\vartheta)$ is very large for small ϑ and quite small for all other directions. The only angles in the range $0 \le \vartheta \le \pi/2$ for which $f(\vartheta) \pm f(\pi-\vartheta)$ takes on large values lie close to the forward direction. For such angles the exchange amplitude $f(\pi-\vartheta)$ is negligibly small; i.e., it is very much more probable that the scattering takes place through the small angle ϑ than through the angle $\pi-\vartheta \approx \pi$. The assumption that back-scattering is extremely weak, in fact, underlies most of our approximation methods.

Since the exchange amplitude is negligible, it is clear that the same result would have been obtained by foregoing the procedure of antisymmetrization altogether. This result may be stated a bit differently by noting that the strong anisotropy of the scattering means that large values of the orbital angular momentum are contained in the scattered wave. The effect of the exclusion principle disappears since we are at the limit in which the angular momenta may be treated semiclassically.

The exchange effects, which are negligible in the center of mass frame of two nucleons, are equally negligible in the laboratory system. They may with similar accuracy be neglected in treating collisions of the incident nucleon with a succession of nucleons within a nucleus. We conclude, therefore, that the results derived in this and earlier sections should hold to fair accuracy when the incident particles are nucleons, as well as any other types of particles.

The Optical Model for Nuclei of Arbitrary Radius

The expressions for the optical potential in the preceding sections have been evaluated under the assumption that the nuclear radius and correlation length are much greater than the range of interaction of a nucleon with the incident particle. Although this assumption has been convenient in simplifying the derivations, it is clearly not too accurately fulfilled in practice. By using it we have gained a certain geometrical insight into the optical model as well as the generalization of a number of results which have been widely discussed in connection with theories of (infinite) nuclear matter. In the present section we shall indicate briefly how the calculations may be generalized to treat nuclei of arbitrary dimensions.

The assumptions which restrict the radii were made as a means of simplifying certain integrations over the configuration space of the nucleons. In particular, it was assumed that the functions $\Gamma_j(b) = 1 - e^{i\chi_j(b)}$ are of such short range, compared to nuclear dimensions, that the variations of nuclear density functions, etc., within this range are negligible. All of the information required about the scattering by the individual nucleons was therefore contained in the integral

$$\int \Gamma_j(b)\, d^{(2)}b = (2\pi/ik) f_j(0).$$

In order to deal with nuclei of smaller radius, or with longer range interactions, more detailed knowledge of the functions $\Gamma_j(b)$ is clearly necessary. The values of these functions, however, are not, in general, directly available from experiment. The procedure by which they are to be found is essentially that of phase shift analysis. Within the scope of the present approximations this analysis is conveniently simple. The function $\Gamma_j(b)$ may be directly expressed as a fourier transform of the scattering amplitude. For this purpose we shall write the scattering amplitude $f(\vec{k}', \vec{k})$ hereafter simply as a function of the momentum transfer $\vec{k} - \vec{k}'$; i.e., we shall write $f(\vec{k} - \vec{k}')$ in place of $f(\vec{k}', \vec{k})$, omitting explicit reference to the energy parameter k, which remains constant throughout. With this notation the scattering amplitude is given in terms of $\Gamma_j(\vec{b})$ by the integral (203),

$$f_j(\vec{k}-\vec{k}') = \frac{ik}{2\pi} \int e^{i(\vec{k}-\vec{k}')\cdot\vec{b}}\, \Gamma_j(\vec{b})\, d^{(2)}b. \qquad (256)$$

The function Γ_j may be solved for in terms of f_j by means of a two-dimensional fourier transformation,

$$\Gamma_j(\vec{b}) = \frac{1}{2\pi i k} \int e^{-i\vec{\lambda}\cdot\vec{b}}\, f_j(\lambda)\, d^{(2)}\lambda . \qquad (257)$$

The vector variable of integration $\vec{\lambda}$ in this integral is conjugate to the impact vector $\vec{b}$. It is taken to vary over the plane perpendicular to the incident propagation direction $\vec{k}$.

Some practical limitations inherent in the above solution for Γ_j should be borne in mind. The scattering amplitude f_j upon which it is based is, in general, a complex function of the momentum transfer. The usual experiments, measuring scattered intensities, tell us only the modulus of this function (for non-vanishing momentum transfer). Knowledge of the phase of the scattering amplitude must therefore be procured from other sources. (E.g., if the interaction is purely absorptive f_j is pure imaginary.) Furthermore at any given energy only a finite range of momentum transfers, i.e., $0 \le \lambda \le 2k$, is physically available. Never-

theless the integration (257) extends over an infinite range of momentum transfers. The impossibility of measurement in the region $\lambda \geq 2k$ leads, in fact, to very little uncertainty under conditions suited to the high-energy approximation; i.e., the scattering amplitude is highly peaked in the forward direction and may, to the present accuracy, be considered to vanish for such large momentum transfers.

To evaluate the optical phase shift function for arbitrary radii we need only substitute the expression (257) for Γ_j into the general expressions (210) or (214) for $\chi_{opt}(\vec{b})$. In this way we find

$$\chi_{opt}(\vec{b}) = \frac{N}{2\pi k}\int e^{-i\vec{\lambda}\cdot(\vec{b}-\vec{s})} f(\vec{\lambda})\rho(\vec{q})\, d^{(2)}\lambda\, d\vec{q} \ . \qquad (258)$$

On introducing the form factor for the density ρ ,

$$S(\lambda) = \int e^{i\vec{\lambda}\cdot\vec{r}}\rho(r)\, d\vec{r}, \qquad (259)$$

the optical phase shift may be written as

$$\chi_{opt.}(\vec{b}) = \frac{N}{2\pi k}\int e^{-i\vec{\lambda}\cdot\vec{b}} f(\lambda) S(\lambda)\, d^{(2)}\lambda \qquad (260)$$

$$= \frac{N}{k}\int_0^\infty J_0(\lambda b) f(\lambda) S(\lambda)\, d^{(2)}\lambda \ . \qquad (261)$$

Since the functions f and S depend only on $|\vec{\lambda}|$, it is a simple matter to construct a spherically symmetric potential which leads to the phase shift function (260). Let us consider the optical potential given by the three-dimensional fourier integral

$$-\frac{1}{\hbar v} V_{opt.}(r) = \frac{N}{(2\pi)^2 k}\int e^{-i\vec{\lambda}\cdot\vec{r}} f(\lambda) S(\lambda)\, d^{(3)}\vec{\lambda} \qquad (262)$$

$$= \frac{N}{\pi k}\int_0^\infty \frac{\sin\lambda r}{r} f(\lambda) S(\lambda)\lambda\, d\lambda \ . \qquad (263)$$

This potential is clearly spherically symmetric. Furthermore integration along the z-axis shows directly its correspondence to the phase shift (260). As an alternative derivation of this expression we may substitute the phase shift function (261) in the explicit formula (201) for the solution of the Abel integral equation which defines the potential. The latter procedure shows the uniqueness of the solution. Contact may be made with our earlier work by noting the behavior of (262) as the nuclear radius be-

comes large compared with the interaction radius. In that limit the function S becomes so strongly peaked in the forward direction that the scattering amplitude $f(\lambda)$ in the integrand may be replaced by f(o). The expression then reduces to (218). The expression (263) is of just the form one would find by treating single scattering only, and using the Born approximation for $V_{opt.}$. Once again we must emphasize that the present treatment accounts fully for multiple scattering, and assumes the particles uncorrelated.

The construction of the optical phase shift function is equally simple in the presence of spin-orbit forces. For this purpose we need only substitute the expression (133) for the spin-dependent scattering amplitude in (260). We then find

$$\chi_{opt.}(b) = \frac{N}{2\pi k}\int e^{-i\vec{\lambda}\cdot\vec{b}}\left\{f(\lambda) + \vec{\sigma}\cdot\left(\frac{\vec{\lambda}\times\vec{k}}{\lambda k}\right) g(\lambda)\right\} S(\lambda)\, d^{(2)}\lambda \ . \qquad (264)$$

This phase shift function corresponds to the spin-orbit potential

$$-\frac{1}{\hbar v} V_{opt}(r) = \frac{N}{(2\pi)^2 k}\int e^{-i\vec{\lambda}\cdot\vec{r}}\left\{f(\lambda) + \vec{\sigma}\cdot\left(\frac{\vec{\lambda}\times\vec{k}}{\lambda k}\right) g(\lambda)\right\} S(\lambda)\, d^{(3)}\vec{\lambda} \ . \qquad (265)$$

Another point at which the nucleon interaction range was assumed relatively small occurred in the discussion of correlation effects. There it was assumed that the correlation length as well as the nuclear radius considerably exceeds the interaction range. Since nuclear forces will tend to produce correlations within distances of the order of their own ranges, it is most important that this restriction on the correlation length be relaxed. The calculation of the optical phase shift function may again be carried out by substitution of (257) in a general expression derived earlier, this time Eq. (240). We shall quote the result for the case in which the nuclear radius remains large compared to both the correlation length l and the interaction radius a, but the latter lengths may be arbitrarily related to one another. The change of the optical phase shift function due to pair correlations is then

$$\delta[\chi_{opt}(b)]_{corr.} = \frac{i}{2k^2}\int e^{i\vec{\lambda}\cdot\vec{s}}[g(q)-1] f^2(\lambda)\, d^{(2)}\lambda\, d\vec{q} \cdot \int_{-\infty}^{\infty} n^2(b,z)\, dz \ . \qquad (266)$$

If we define the form factor, $C(\lambda)$, of the correlation function as

$$C(\lambda) = \int e^{i\vec{\lambda}\cdot\vec{r}}[g(r)-1]\,d\vec{r}\ ,$$

the phase shift correction reduces to

$$\delta[\chi_{opt}(b)]_{corr.} = \frac{i}{2k^2}\int f^2(\lambda)\,C(\lambda)\,d^{(2)}\lambda \cdot \int_{-\infty}^{\infty} n^2(b,z)\,dz\ . \tag{267}$$

Since the density function n(b, z) is spherically symmetric, the correlation correction to the optical potential is given by

$$-\frac{1}{\hbar v}\delta[V_{opt}(r)]_{corr.} = \frac{i}{2k^2}\,n^2(r)\int f^2(\lambda)C(\lambda)d^{(2)}\lambda\ . \tag{268}$$

In the limit of large correlation lengths this expression reduces to (245).

Our discussion of the correlations brought about by the exclusion principle may now be extended by finding the fourier transform, $C(\lambda)$, of the correlation function (250). This calculation corresponds simply to finding the common volume in momentum space of two Fermi spheres whose centers are displaced by a distance λ relative to one another. Its result is

$$C(\lambda) = \begin{cases} -\frac{9\pi^2}{k_F^3}\left\{\frac{2}{3} - \frac{1}{2}\frac{\lambda}{k_F} + \frac{1}{24}\frac{\lambda^3}{k_F^3}\right\} & \text{for } 0 \le \lambda \le 2k_F \\ 0 \quad \text{for } 2k_F \le \lambda\ . \end{cases} \tag{269}$$

Hence the effect of the exclusion principle on the optical potential is given by the simple integral

$$-\frac{1}{\hbar v}\delta[V_{opt}(r)]_{\substack{excl.\\ princ.}} = -\frac{i}{4\pi k^2}\int_0^{2k_F} f^2(\lambda)\left\{\frac{2}{3}k_F^3 - \frac{1}{2}k_F^2\lambda + \frac{1}{24}\lambda^3\right\}\lambda\,d\lambda\ . \tag{270}$$

In the preceding section we noted the close correspondence between the correlation effect in nuclei and the "eclipse" effect in the deuteron. In the discussion of the eclipse effect the calculations were also

simplified at one point by the assumption that the interaction range is much smaller than the deuteron radius. This assumption may likewise be removed by making use of the expression (257) for Γ_j. The generalization of the expression (184) for the deuteron total cross section, which we reach in this way, is

$$\sigma_d = \sigma_n + \sigma_p + \frac{2}{k^2}\int S(\lambda)\,\mathrm{Re}\{f_n(\lambda)f_p(\lambda)\}\,d^{(2)}\lambda\ , \tag{271}$$

where $S(\lambda)$ is the deuteron form factor

$$S(\lambda) = \int e^{i\vec{\lambda}\cdot\vec{r}}|u(r)|^2 d\vec{r}\ . \tag{272}$$

The expression (271) reduces to (184) in the limit of large deuteron radii. It should be a considerably more accurate expression to use in analyzing experimental data, but its use requires some knowledge of the angular distributions of scattering.

Inelastic Scattering

The optical model, as we have discussed it thus far, represents an approach to the problem of elastic scattering. In the present section we shall indicate briefly how it may be employed in treating inelastic transitions as well as elastic ones. We shall then devote a few remarks to some of the more general properties of inelastic scattering at high energies.

We have already derived a general expression for the inelastic scattering amplitude of an incident particle by a many-particle system, e.g., a nucleus. As long as the energy difference between the initial nuclear state, i, and the final state, f, is small compared to the incident energy, the scattering amplitude is given by (172),

$$F_{fi}(\vec{k}',\vec{k}) = \frac{k}{2\pi i}\int e^{i(\vec{k}-\vec{k}')\cdot\vec{b}}\,d^{(2)}b\ \cdot \int u_f^*(\vec{q}_1,\ldots,\vec{q}_N)\left\{e^{i\sum_j \chi_j(\vec{b}-\vec{s}_j)}-1\right\}u_i(\vec{q}_1,\ldots,\vec{q}_N)\prod_l d\vec{q}_l\ . \tag{172}$$

Since the nucleus changes its state in the course of a collision for $f \neq i$, quantities which enter the integrand of (172) can no longer be described entirely in terms of static density and correlation functions. In practice a model of some sort must always be used to provide suitable initial and final state wave functions. Many such models have been devised and we shall not enter into their details here, other than to mention a simplifying feature which most of them share. It appears that many transitions may

be described as involving a change of state of only a very small number of nucleons. The remaining nucleons, e.g., those in closed shells, play no active role in the transition. It is not difficult to show that the treatment of such transitions may be carried out by explicitly describing the collision of the incident particle with the small group of nucleons, and allowing the collision to take place within an optical potential well which is produced by the remaining (inert) nucleons.

The simplest such model is one in which a single nuclear particle is assumed to make a transition. We separate its wave function from that of the remaining nucleons by writing

$$u_i(\vec{q}_1,\dots,\vec{q}_N) = v_i(\vec{q}_1)u(\vec{q}_2,\dots,\vec{q}_N) \ ,$$
$$u_f(\vec{q}_1,\dots,\vec{q}_N) = v_f(\vec{q}_1)u(\vec{q}_2,\dots,\vec{q}_N) \ . \tag{271}$$

An optical phase shift function

$$\chi_{opt}^{(N-1)}(\vec{b})$$

may then be defined for the system of $N-1$ inert particles by means of the relation

$$e^{i\chi_{opt}^{(N-1)}(\vec{b})} = \int |u(\vec{q}_2,\dots,\vec{q}_N)|^2 e^{i\sum_{j=2}^{N}\chi_j(\vec{b}-\vec{s}_j)} \prod d\vec{q}_j \ . \tag{272}$$

With this definition, the scattering amplitude (172) may be written as

$$\begin{aligned} F_{fi}(\vec{k}',\vec{k}) &= \frac{k}{2\pi i}\int e^{i(\vec{k}-\vec{k}')\cdot\vec{b}}d^{(2)}b \cdot \\ &\int v_f^*(\vec{q}_1)\left\{e^{i\chi_{opt}^{(N-1)}(\vec{b})+i\chi_1(\vec{b}-\vec{s})}-1\right\}v_i(\vec{q}_1)d\vec{q}_1 \\ &= \int v_f^*(\vec{q}_1)e^{i(\vec{k}-\vec{k}')\cdot\vec{s}_1}v_i(\vec{q}_1)\cdot \\ &\frac{k}{2\pi i}\int e^{i(\vec{k}-\vec{k}')\cdot(\vec{b}-\vec{s}_1)}\left\{e^{i\chi_{opt}^{(N-1)}(\vec{b})+i\chi_1(\vec{b}-\vec{s}_1)}-1\right\}d^{(2)}b\,d\vec{q}_1 \ . \end{aligned} \tag{273}$$

This expression falls into the general form

$$F_{fi}(\vec{k}',\vec{k}) = \int v_f^*(\vec{q}_1)e^{i(\vec{k}-\vec{k}')\cdot\vec{s}_1}\bar{f}(\vec{k}',\vec{k},\vec{s})v_i(\vec{q}_1)d\vec{q}_1 \ , \tag{274}$$

where $\bar{f}(\vec{k}',\vec{k},\vec{s})$ represents the scattering amplitude of the first nucleon, regarded not as an isolated particle, but as lying at an impact coordinate $\vec{s}$ within a region containing an optical potential determined by the remaining N - 1 nucleons. These considerations are easily generalized to treat transitions of small groups of particles. Transitions of a more highly organized character, however, such as those between rotational states, require somewhat simpler models. To treat such processes the nucleus might be regarded as a rigid or fluid continuum, and the corresponding phase shift functions employed.

Since energy discrimination poses a difficult experimental problem in dealing with highly energetic particles, it probably will not be practical to observe individually very many examples of particular inelastic transitions. It will be much easier, instead, to observe angular distributions of the total inelastic scattering. In the high-energy limit, as we have noted, forward scattering will be intense and the energy transfers in inelastic scattering will be predominantly quite small. The scattered intensities may then be summed over all final states to good approximation by using the completeness relation (164). To find the angular distribution of inelastic or incoherent scattering, we subtract from this sum the elastic scattering intensity. The resulting inelastic differential cross's section may be written as

$$\sum_{f\neq i}|F_{fi}(\vec{k}',\vec{k})|^2 = \left(\frac{k}{2\pi}\right)^2\int e^{i(\vec{k}-\vec{k}')\cdot(\vec{b}-\vec{b}')}\left\{\left\langle e^{i\sum_j\chi_j(\vec{b}-\vec{s}_j)}e^{-i\sum_l\chi_l^*(\vec{b}'-\vec{s}_j)}\right\rangle_i\right. \tag{275}$$
$$\left. -\left\langle e^{i\sum_j\chi_j(\vec{b}-\vec{s}_j)}\right\rangle_i\left\langle e^{-i\sum_l\chi_l^*(\vec{b}'-\vec{s}_l)}\right\rangle_i\right\}d^{(2)}b\,d^{(2)}b'$$

$$= \left(\frac{k}{2\pi}\right)^2\int e^{i(\vec{k}-\vec{k}')\cdot(\vec{b}-\vec{b}')}\left\{\left\langle\prod_j(1-\Gamma_j(\vec{b}-\vec{s}_j))(1-\Gamma_j^*(\vec{b}'-\vec{s}_j))\right\rangle_i\right. \tag{276}$$
$$\left. -\left\langle\prod_j(1-\Gamma_j(\vec{b}-\vec{s}_j))\right\rangle_i\left\langle\prod_l(1-\Gamma_l^*(\vec{b}'-\vec{s}_l))\right\rangle_i\right\}d^{(2)}b\,d^{(2)}b' \ ,$$

where the brackets $\langle\ \rangle_i$ again signify the expectation value in the initial nuclear state. The expression (276) contains the expectation values of products rather similar in form to those which occurred in the discussion of the optical model. We shall omit the details of their evaluation here and simply quote the form to which the inelastic angular distribution may be reduced, when certain assumptions we have discussed earlier are introduced. If we assume that the interaction range of a nucleon is much smaller than the nuclear radius, and that the nucleons are independent particles, we find

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{inelast.}} = \sum_{f\neq i}\left|F_{fi}(\vec{k}',\vec{k})\right|^2 = \left(\frac{k}{2\pi}\right)^2\int d^{(2)}B\, e^{-\sigma\int_{-\infty}^{\infty}n(\vec{B},z)dz}\int d^{(2)}\beta\, e^{i(\vec{k}-\vec{k}')\cdot\vec{\beta}}\cdot \left\{e^{\frac{1}{k^2}\int e^{-i\vec{\lambda}\cdot\vec{\beta}}|f(\lambda)|^2 d\lambda\int_{-\infty}^{\infty}n(\vec{B},z)dz} - 1\right\}, \tag{277}$$

where the vector variables of integration are related to the variables of (276) by

$$\vec{B} = \tfrac{1}{2}(\vec{b}+\vec{b}')\quad,\quad \vec{\beta} = \vec{b}-\vec{b}'\ . \tag{278}$$

Some insight into the nature of the expression (277) for the inelastic differential cross section may be obtained by expanding its integrand in powers of the nucleon cross section. To lowest order we find

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{inelast.}} = \frac{1}{(2\pi)^2}\int d^{(2)}B\int d^{(2)}\beta\, e^{i(\vec{k}-\vec{k}')\cdot\vec{\beta}}\int e^{-i\vec{\lambda}\cdot\vec{\beta}}|f(\lambda)|^2 d^{(2)}\lambda\cdot \int_{-\infty}^{\infty}n(\vec{B},z)dz + \cdots = N|f(k-k')|^2 + \cdots, \tag{279}$$

which is just the sum of the intensities of single scattering by the individual nucleons. The higher order terms correct for multiple collisions, which are important when the nucleus has appreciable opacity.

Another expression which illustrates the meaning of (277) is the total inelastic cross section. The angular distribution (277) may be integrated over directions of $\vec{k}'$ by means of the approximation (94). The result is

$$\sigma_{\text{inelast.}} = \int d^{(2)}B\, e^{-\sigma\int_{-\infty}^{\infty}n(\vec{B},z)dz}\left\{e^{-\int|f(\lambda)|^2\frac{d^{(2)}\lambda}{k^2}\int_{-\infty}^{\infty}n(\vec{B},z)dz} - 1\right\}. \tag{280}$$

Under the approximation (94) the integral

$$\frac{1}{k^2}\int|f(\lambda)|^2 d^{(2)}\lambda$$

is simply the total scattering cross section of a nucleon, $\sigma_{\text{scatt.}}$. If processes other than simple nucleon-nucleon scattering are present, their effect upon the scattering is described in terms of an absorption; i.e., we write $\sigma = \sigma_{\text{abs.}} + \sigma_{\text{scatt.}}$. Thus the inelastic cross section may be written as

$$\sigma_{\text{inelast.}} = \int d^{(2)}B\, e^{-\sigma_{\text{abs.}}\int_{-\infty}^{\infty}n(\vec{B},z)dz}\left\{1 - e^{-\sigma_{\text{scatt.}}\int_{-\infty}^{\infty}n(\vec{B},z)dz}\right\}. \tag{281}$$

This expression has a simple interpretation. It shows that the inelastic cross section is just the integral over impact parameters of the probability that the incident particle both escapes absorption and undergoes at least one scattering collision.

It is also possible to calculate the effect of nucleon position correlations on the angular distribution of inelastic scattering. In particular, the correlations due to the exclusion principle are of some interest in their effect on the inelastic scattering near the forward direction. This calculation is a simple application of the methods we have discussed, but would be a bit lengthy to present here.

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