# WAVE EQUATIONS (July 2019)

Pdf generated August 24, 2019.

# 1 Wave equations

## 1.1 Optics

The scalar wave equation for light in an inhomogeneous medium is (cf p. 310 of (1e) of ref. [1])

$$\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} - \frac{1}{n(\mathbf{x})^2}\nabla^2\Phi = 0 \tag{1.1}$$

Assuming a monochromatic ansatz $\Phi(\mathbf{x}, t) = \mathrm{e}^{-i\omega t}\phi(\mathbf{x})$ we obtain

$$\nabla^2\phi + \left[\frac{\omega}{c}n(\mathbf{x})\right]^2\phi = 0 \tag{1.2}$$

The index of refraction can also depend on frequency so this becomes

$$\nabla^2\phi + k^2(\mathbf{x}, \omega)\phi = 0, \qquad k(\mathbf{x}, \omega) \equiv \frac{\omega}{c}n(\mathbf{x}, \omega) \tag{1.3}$$

## 1.2 Nonrelativistic matter

The Schrödinger equation for a nonrelativistic particle in a potential is

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m^2}\nabla^2\Psi + V(\mathbf{x})\Psi \tag{1.4}$$

An energy eigenstate $\Psi(\mathbf{x}, t) = \psi(\mathbf{x})e^{-iEt/\hbar}$ obeys

$$-\frac{\hbar^2}{2m^2}\nabla^2\psi + V(\mathbf{x})\psi = E\psi \tag{1.5}$$

which we can rewrite as

$$\nabla^2\psi + k^2(\mathbf{x}, E)\psi = 0, \qquad k(\mathbf{x}, E) = \frac{\sqrt{2m(E - V(\mathbf{x}))}}{\hbar} \tag{1.6}$$

## 1.3 Relativistic matter

The Klein-Gordon equation in gravitational background ($c = 1$) is

$$\begin{aligned} 0 &= g^{\mu\nu}D_\mu D_\nu\Phi + \frac{m^2}{\hbar^2}\Phi \\ &= \frac{1}{\sqrt{g}}g^{\mu\nu}\partial_\mu\left(\sqrt{g}\partial_\nu\Phi\right) + \frac{m^2}{\hbar^2}\Phi \end{aligned} \tag{1.7}$$

Now consider the Schwarzschild metric in isotropic coordinates (cf. d'Inverno, section 14.7)

$$ds^2 = A(r)dt^2 - B(r)d\mathbf{x}^2, \tag{1.8}$$

where

$$\begin{aligned} A(r) &= \left(\frac{1+\frac{1}{2}\phi}{1-\frac{1}{2}\phi}\right)^2 \approx (1+2\phi) \\ B(r) &= (1-\tfrac{1}{2}\phi)^4 \approx (1-2\phi) \\ \phi &\equiv -GM/r \end{aligned} \tag{1.9}$$

Since the metric is independent of time, we may use the Ansatz $\Phi(x^\mu) = \mathrm{e}^{-iEt/\hbar}\Phi(\mathbf{x})$ to obtain

$$-g^{00}\frac{E^2}{\hbar^2}\Phi + \frac{1}{\sqrt{g}}g^{ij}\partial_i\left(\sqrt{g}\partial_j\Phi\right) + \frac{m^2}{\hbar^2}\Phi = 0 \tag{1.10}$$

or (assuming a mostly minus metric)

$$-\frac{1}{\sqrt{g}}g^{ij}\partial_i\left(\sqrt{g}\partial_j\Phi\right) + k^2(\mathbf{x},E)\Phi = 0 \qquad\qquad k(\mathbf{x},E) \equiv \frac{\sqrt{g^{00}E^2 - m^2}}{\hbar} \tag{1.11}$$

# 2 Helmholtz equations

In all these cases, we obtained a generalized Helmholtz equation

$$\nabla^2\phi + k^2(\mathbf{x},\omega)\phi = 0 \tag{2.1}$$

When $k(\mathbf{x},\omega)$ is spatially homogeneous

$$k_0 = \begin{cases} \omega/c, & \text{optics} \\ \sqrt{2mE}/\hbar, & \text{nonrelativistic} \\ \sqrt{E^2-m^2}/\hbar, & \text{relativistic} \end{cases} \tag{2.2}$$

the Helmholtz equation has solutions

$$\phi(\mathbf{x}) = Ae^{i\mathbf{k}\cdot\mathbf{x}}, \qquad\qquad \mathbf{k} = k_0\hat{k}, \tag{2.3}$$

with arbitrary (const) direction $\hat{k}$ and amplitude $A$.

When $k(\mathbf{x},\omega)$ is not spatially homogeneous, we may proceed to solve the equation in one of two ways. First, we can separate off a perturbation and write

$$\nabla^2\phi + k_0^2\phi = \left[k_0^2 - k^2(\mathbf{x},\omega)\right]\phi \tag{2.4}$$

where

$$k_0^2 - k^2(\mathbf{x},\omega) = \begin{cases} \left[1-n^2(\mathbf{x},\omega)\right](\omega/c)^2, & \text{optics} \\ 2mV(\mathbf{x})/\hbar^2, & \text{nonrelativistic} \\ (1-g^{00})E^2/\hbar^2, & \text{relativistic} \end{cases} \tag{2.5}$$

Second, we can try an eikonal ansatz, as we do in the next section.

## 3 Eikonal equation

Consider the eikonal ansatz [1–3]

$$\phi(\mathbf{x}) = A\mathrm{e}^{i\chi(\mathbf{x})} \tag{3.1}$$

where we will refer to $\chi(\mathbf{x})$ as the "eikonal phase." If the spatial inhomogeneity of $k(\mathbf{x},\omega)$ is slowly varying with respect to $\lambda$, the generalized Helmholtz equation (2.1) simplifies to the eikonal equation

$$\boxed{(\nabla\chi)^2 = k^2(\mathbf{x},\omega)} \tag{3.2}$$

In specific situations, the eikonal phase is given by

$$\chi = \begin{cases} (\omega/c)L, & \text{optics} \\ W/\hbar, & \text{particles} \end{cases} \tag{3.3}$$

where $L$ is known as the "eikonal" and $W$ is Hamilton's characteristic function, and the eikonal equation becomes

$$\begin{cases} (\nabla L)^2 = n^2(\mathbf{x},\omega), & \text{optical eikonal equation} \\ (\nabla W)^2 = p^2(\mathbf{x},E), & \text{Hamilton-Jacobi equation} \end{cases} \tag{3.4}$$

where $p(\mathbf{x},E) = \hbar k(\mathbf{x},E)$. We could instead use the eikonal ansatz

$$\Psi(\mathbf{x},t) = Ae^{iS(\mathbf{x},t)/\hbar} \tag{3.5}$$

in the time-dependent Schrödinger equation to obtain the (time-dependent) Hamilton-Jacobi equation for Hamilton's principal function $S$:

$$\frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla S)^2 + V(\mathbf{x}) = 0 \tag{3.6}$$

Then

$$S(\mathbf{x},t) = -Et + W(\mathbf{x})$$

gives the (time-independent) Hamilton-Jacobi equation

$$(\nabla W)^2 = 2m(E - V(\mathbf{x})) \tag{3.7}$$

In the relativistic case, we could employ the eikonal ansatz

$$\Phi(x^\mu) = A\mathrm{e}^{iS(x^\mu)/\hbar} \tag{3.8}$$

to obtain the relativistic Hamilton-Jacobi equation (pp. 28 and 306 of ref. [3], also ref. [4])

$$g^{\mu\nu}(\partial_\mu S)(\partial_\nu S) = m^2 \tag{3.9}$$

Letting $S = -Et + W$, this becomes

$$g^{00}E^2 + g^{ij}(\partial_i W)(\partial_j W) = m^2 \tag{3.10}$$

Using isotropic coordinates, this gives

$$A^{-1}E^2 - B^{-1}(\nabla W)^2 = m^2 \tag{3.11}$$

or

$$(\nabla W)^2 \equiv p^2(\mathbf{x}, E) = \frac{B}{A}[E^2 - m^2 A] \tag{3.12}$$

In the weak field limit

$$\begin{aligned} p^2(\mathbf{x}, E) &= (1 - 4\phi)\left[E^2 - (1 + 2\phi)m^2\right] \\ &= (E^2 - m^2) - \phi(4E^2 - 2m^2) \end{aligned} \tag{3.13}$$

# 4 Constructing a solution to the eikonal equation

To solve the eikonal equation, we need to determine the direction of $\nabla\chi$ in addition to its magnitude, which is given by $k(\mathbf{x}, \omega)$. (From now on, we will suppress the $\omega$ dependence of $k$.) As discussed by Weinberg [5], this requires a boundary condition; namely, specifying some initial surface on which $\chi = 0$. Define a unit normal vector field $\hat{n}(\mathbf{x})$ on this surface directed perpendicular to the surface. For example, we could take the surface to be a plane passing through the origin, $\hat{n} \cdot \mathbf{x} = 0$, for some constant $\hat{n}$. Then $\hat{n}(\mathbf{x}) = \hat{n}$ along the surface. Alternative we could take the surface to be a sphere of radius $R$, with $\hat{n} = \hat{r}$ on the sphere. Then define (on the initial surface) the vector field

$$\mathbf{k}(\mathbf{x}) \equiv k(\mathbf{x})\hat{n}(\mathbf{x}), \qquad\qquad \nabla\chi = \mathbf{k}(\mathbf{x}) \tag{4.1}$$

We can integrate this vector field to obtain other surfaces along which $\chi$ is const, not necessarily parallel to the original surface. This then allows us to define $\hat{n}(\mathbf{x})$ (and therefore $\mathbf{k}(\mathbf{x})$) along and perpendicular to the new surface. By construction, the vector field $\mathbf{k}(\mathbf{x})$ has zero curl

$$\nabla \times \mathbf{k}(\mathbf{x}) = 0, \qquad\qquad \chi(\mathbf{x}) = \int \mathbf{k}(\mathbf{x}) \cdot d\mathbf{x} \tag{4.2}$$

We now define *rays* $\mathbf{r}(s)$ as the integral curves of the vector field $\hat{n}(\mathbf{x})$. If $s$ measures the length along the curve, then $d\mathbf{r}(s)/ds = \hat{n}(\mathbf{x})$. Integrating $\mathbf{k}(\mathbf{x})$ *along a ray*

$$\chi(\mathbf{x}) = \int \mathbf{k}(\mathbf{x}) \cdot \frac{d\mathbf{r}}{ds} ds = \int_{\text{ray}} k(\mathbf{x}) ds \tag{4.3}$$

We can now see that if we integrate $\mathbf{k}(\mathbf{x})$ along any path other than a ray, then

$$\chi(\mathbf{x}) = \int \mathbf{k}(\mathbf{x}) \cdot \frac{d\mathbf{r}}{ds} ds \leq \int k(\mathbf{x}) ds \tag{4.4}$$

since $\mathbf{k}(\mathbf{x}) \cdot (d\mathbf{r}/ds) \leq 1$. That is, for *any* path, the integral $\int k(\mathbf{x})ds$ is greater than or equal to $\chi(\mathbf{x})$. Thus we have proved that $\int k(\mathbf{x})ds$ is minimized (and is equal to $\chi(\mathbf{x})$) along a ray connecting two points.

The solution of the wave equation in the eikonal approxiatmion is therefore

$$\Phi(\mathbf{x}, t) = A \exp\left(i \int \mathbf{k}(\mathbf{x}) \cdot d\mathbf{x} - i\omega t\right) \tag{4.5}$$

from which we can infer the phase speed along a ray to be

$$v_{\text{phase}} = \frac{\omega}{k(\mathbf{x})} \tag{4.6}$$

In the context of optics $k(\mathbf{x}) = (\omega/c)n(\mathbf{x})$, so

$$v_{\text{phase}} = \frac{c}{n(\mathbf{x})} \tag{4.7}$$

The "optical length"' of a path is defined by

$$\int n(\mathbf{x}) ds = \int \frac{c}{v_{\text{phase}}} ds = c \int dt \tag{4.8}$$

and is proportional to the "time" it would take the wave front to travel along the given path (assuming that the wave front remains perpendicular to the path, which will not be the case for any path other than a ray). Since a ray is therefore the path of minimal optical length between two points, it is also the path of "least time" between those points.

In the context of a nonrelativistic particle, we integrate the Hamilton-Jacobi equation to obtain a curl-free vector field

$$\mathbf{p}(\mathbf{x}) = \hbar \mathbf{k}(\mathbf{x}) = \nabla W = \hat{n}(\mathbf{x}) \sqrt{2m(E - V(\mathbf{x})} \tag{4.9}$$

(cf. eq. (9-12) of ref. [1]). The solution to the eikonal equation is the WKB solution

$$\Phi(\mathbf{x}, t) = \exp\left(\frac{i}{\hbar} \int \mathbf{p}(\mathbf{x}) \cdot d\mathbf{x} - \frac{iEt}{\hbar}\right) \tag{4.10}$$

and the phase speed is thus (section 9-8, pp. 307ff of ref. [1])

$$v_{\text{phase}} = \frac{E}{p(\mathbf{x})} \tag{4.11}$$

but is physically meaningless since $E$ depends on an arbitrary reference point.

For massless particles in an isotropic Schwarzchild metric,

$$p(\mathbf{x}, E) = \sqrt{\frac{B}{A}} E \approx (1 - 2\phi)E \tag{4.12}$$

therefore the phase velocity of light is (see p. 127 of ref. [6])

$$v_{\text{phase}} = \frac{E}{p(\mathbf{x})} = \sqrt{\frac{A}{B}} \approx 1 + 2\phi = 1 - \frac{2GM}{r} \tag{4.13}$$

# 5 Ray equation

Following Born and Wolf (p. 122 of ref. [2], see also refs. [7,8]), we compute

$$\frac{d}{ds}\left(k\frac{d\mathbf{r}}{ds}\right) = \frac{d}{ds}\mathbf{k} = \left(\frac{d\mathbf{r}}{ds}\cdot\nabla\right)\mathbf{k} = \left(\frac{1}{k}\nabla\chi\cdot\nabla\right)\nabla\chi = \frac{1}{2k}\nabla\left(\nabla\chi\right)^2 = \frac{1}{2k}\nabla(k^2) \tag{5.1}$$

thus obtaining the "ray equation"

$$\boxed{\frac{d}{ds}\left(k\frac{d\mathbf{r}}{ds}\right) = \nabla k} \tag{5.2}$$

In the context of optics this is just the generalization of Snell's law for a slowly-varying medium

$$\frac{d}{ds}\left(n\frac{d\mathbf{r}}{ds}\right) = \nabla n \tag{5.3}$$

In the context of nonrelativistic particles, the ray equation is

$$\frac{d}{ds}\mathbf{p} = \frac{1}{2p}\nabla(p^2) = -\frac{m}{p}\nabla V \tag{5.4}$$

and for a relativistic spin zero particle

$$\frac{d}{ds}\mathbf{p} = \frac{1}{2p}\nabla(p^2) = \frac{1}{2}\sqrt{\frac{E^2-m^2A}{AB}}\,\nabla B - \frac{E^2}{2A^2}\sqrt{\frac{AB}{E^2-m^2A}}\,\nabla A \tag{5.5}$$

which in the weak field limit goes to

$$\frac{d}{ds}\mathbf{p} \approx \frac{1}{2\sqrt{E^2-m^2}}\left[(E^2-m^2)\,\nabla B - E^2\,\nabla A\right] = -\frac{2E^2-m^2}{\sqrt{E^2-m^2}}\nabla\phi = -\left(\frac{E^2+p_0^2}{p_0}\right)\nabla\phi \tag{5.6}$$

where $p_0 = \sqrt{E^2-m^2}$.

# 6 Snell's law

We can use eq. (5.3) to derive the usual form of Snell's law [8]. Suppose $n$ varies only along the $z$ direction. Let $\theta$ define the direction of the ray with respect to the $z$-axis:

$$\hat{z}\cdot d\mathbf{r}/ds = \cos\theta \tag{6.1}$$

Then

$$\frac{d}{ds}\left(n\frac{d\mathbf{r}}{ds}\right)\cdot\hat{z} = \cos\theta\frac{d}{dz}\left(n\cos\theta\right) = \frac{dn}{dz} \tag{6.2}$$

Multiply both sides by $\frac{1}{2}n$ to find

$$\frac{d}{dz}\left[(n\cos\theta)^2 - n^2\right] = 0 \tag{6.3}$$

or

$$n \sin\theta = \text{independent of } z \tag{6.4}$$

Another way to derive Snell's law is to consider "Lagrange's integral invariant" [2, 8]

$$\int_A^B \mathbf{k}(\mathbf{x}) \cdot d\mathbf{x} \tag{6.5}$$

which is independent of the path from $A$ to $B$. Suppose we compute this integral along each edge of a discontinuity in the index of refraction. Then

$$\begin{aligned} k_1 \hat{n}_1 \cdot d\mathbf{x} &= k_2 \hat{n}_2 \cdot d\mathbf{x} \\ k_1 \sin\theta_1 &= k_2 \sin\theta_2 \end{aligned} \tag{6.6}$$

where $\theta_i$ is the angle that $\hat{n}$ (parallel to the ray) makes with respect to the normal to the discontinuity.

Since $k = (\omega/c)n = \omega/v_{\text{phase}}$, we have the usual Snell's law

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \tag{6.7}$$

or

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2} \tag{6.8}$$

across a discontinuity.

A particle's momentum $\mathbf{p}$ (whether nonrelativistic or relativistic) is proportional to $\mathbf{k}$, and therefore

$$p_1 \sin\theta_1 = p_2 \sin\theta_2 \tag{6.9}$$

i.e., the component of momentum parallel to the discontinuity is conserved. (The force $\nabla V$ is normal to the discontinuity.)

If we were to make the **mistake** that a photon's velocity is proportional to momentum ($\mathbf{p} = m\mathbf{v}$), we might we might conclude that the transverse component of velocity is conserved across a discontinuity

$$v_1 \sin\theta_1 = v_2 \sin\theta_2 \tag{6.10}$$

which would be incorrect [7].

# 7 Deflection of a beam

Suppose that a beam travels along the $z$ axis, nearly undeflected. Then

$$\frac{d\mathbf{r}}{ds} = \cos\theta(z)\hat{z} + \sin\theta(z)\hat{x} \approx \hat{z} + \theta(z)\hat{x} \tag{7.1}$$

where $\theta(-\infty) = 0$. Then the ray equation (5.2) gives

$$\begin{aligned} \frac{d}{dz}\left(k\frac{d\mathbf{r}}{ds}\right) &= \nabla k \\ k\frac{d\mathbf{r}}{ds}\bigg|_{-\infty}^{z} &= \int_{-\infty}^{z} dz\ \nabla k \\ [k(z) - k(-\infty)]\hat{z} + [k(z)\theta(z) - 0]\hat{x} &= \int_{-\infty}^{z} dz\left[\frac{\partial k}{\partial z}\hat{z} + \frac{\partial k}{\partial x}\hat{x}\right] \end{aligned} \tag{7.2}$$

The $z$ component is satisfied automatically, and the $x$ component gives

$$\theta(z) = \frac{1}{k(z)}\int_{-\infty}^{z} dz\frac{\partial k}{\partial x} \tag{7.3}$$

In the optics context, we ahve

$$\theta(z) = \frac{1}{n(z)}\int_{-\infty}^{z} dz\frac{\partial n}{\partial x} \tag{7.4}$$

For nonrelativistic particles, assuming that $p(z) \approx p_0$, we have

$$\theta(z) = \frac{1}{2p_0^2}\int_{-\infty}^{z} dz\frac{\partial}{\partial x}p^2(z) = \frac{m}{p_0^2}\int_{-\infty}^{z} dz\left(-\frac{\partial V}{\partial x}\right) = \frac{1}{p_0}\int_{-\infty}^{z} dt\left(-\frac{\partial V}{\partial x}\right) \tag{7.5}$$

that is, the transverse change in momentum is given by the impulse. For relativistic scattering,

$$\theta(z) = \frac{1}{2p_0^2}\int_{-\infty}^{z} dz\frac{\partial}{\partial x}p^2(z) = \left(\frac{E^2 + p_0^2}{p_0^2}\right)\int_{-\infty}^{z} dz\left(-\frac{\partial \phi}{\partial x}\right) = \left(\frac{E^2 + p_0^2}{p_0^2}\right)\frac{2GM}{b} \tag{7.6}$$

# 8 Extremal phase

Alternatively [4], we could find the equation for the path that extremizes the phase

$$\chi = \int k(\mathbf{x}) ds \tag{8.1}$$

For optics this is the path that minimizes the optical length

$$L = \int n(\mathbf{x}) ds = \int ds \frac{c}{v_{\text{phase}}} \tag{8.2}$$

For a nonrelativistic particle that satisfies energy conservation, the motion minimizes

$$W = \int p(\mathbf{x}) ds = \int ds \sqrt{2m(E-V)} = \int ds \frac{E}{v_{\text{phase}}} \tag{8.3}$$

where $\int p(\mathbf{x})ds$ is the "abbreviated action" (Landau, *Mechanics*; Goldstein, 3e); this is sometimes referred to as Maupertuis's principle (p. 132 of ref. [3], p. 231 of (1e) of ref. [1]) or Jacobi's form of the least action principle (p. 233 of (1e) of ref. [1]). Note that $W$ is parametrization independent and makes no reference to time. (See both sections 7-4 and 7-5 of Goldstein, (1e) and sections 8-5 and 8-6 of Goldstein, (3e) for discussion of the relation to the usual Hamilton's principle, involving time.) Choosing an arbitrary parametrization $\sigma$ of the path, and letting $\dot{\mathbf{x}} = d\mathbf{x}/d\sigma$, we have

$$\chi = \int k(\mathbf{x}) \sqrt{\dot{\mathbf{x}}^2} d\sigma \tag{8.4}$$

Euler's equation then gives

$$\frac{d}{d\sigma} \left( k \frac{\dot{\mathbf{x}}}{\sqrt{\dot{\mathbf{x}}^2}} \right) = \sqrt{\dot{\mathbf{x}}^2} \, \nabla k \tag{8.5}$$

If we choose $d\sigma$ as the path length $|d\mathbf{x}|$, then $|\dot{\mathbf{x}}| = 1$ and we recover the ray equation

$$\frac{d}{ds} \left( k \frac{d\mathbf{x}}{ds} \right) = \nabla k \tag{8.6}$$

If we instead choose $d\sigma$ to denote $|d\mathbf{x}|/k$, then $|\dot{\mathbf{x}}| = k$ and

$$\frac{d^2\mathbf{x}}{d\sigma^2} = k \nabla k = \tfrac{1}{2} \nabla (k^2) \tag{8.7}$$

If we choose $d\sigma$ to denote $dt = m|d\mathbf{x}|/p$, then $|\dot{\mathbf{x}}| = p/m$ and we obtain Newton's second law

$$m \frac{d^2\mathbf{x}}{dt^2} = \frac{\nabla(p^2)}{2m} = -\nabla V \tag{8.8}$$

# 9 Speed of wave packet in one dimension

Let's assume that the inhomogeneity is only in the $z$ direction, so that rays travelling in the $z$ direction will not be deflected. Consider a group of such rays with different frequencies centered around $\omega$. Let the phase vanish along $z = 0$ at $t = 0$. Then

$$\begin{aligned}\Phi(z,t,\omega) &= Ae^{i\Theta(z,t)} = Ae^{iS(z,t)/\hbar} \\ \Theta(z,t,\omega) &= \int_0^z k(z')dz' - \omega t \\ S(z,t,\omega) &= \int_0^z p(z')dz' - Et \end{aligned} \tag{9.1}$$

The stationary phase approximation $d\Theta/d\omega = 0$ or $dS/dE = 0$ implies that the peak of the wave packet occurs at

$$0 = \int_0^z \frac{\partial k}{\partial \omega} dz' - t \tag{9.2}$$

The peak is displaced by $\Delta z$ in time $\Delta t$ where

$$\frac{\partial k}{\partial \omega}\Delta z = \Delta t \tag{9.3}$$

Therefore, the speed of the peak (group speed) is

$$v_{\text{group}} = \frac{\Delta z}{\Delta t} = \frac{1}{\partial k/\partial \omega} \tag{9.4}$$

For nonrelativistic particles, the group speed is

$$v_{\text{group}} = \frac{1}{\partial p/\partial E} = \sqrt{\frac{2}{m}(E - V(z))} = \frac{p(z)}{m} \tag{9.5}$$

For relativistic particles

$$\begin{aligned} p^2 &= \frac{B}{A}\left(E^2 - m^2 A\right) \\ pdp &= \frac{B}{A}EdE \\ \frac{dE}{dp} &= \frac{A}{B}\frac{p}{E} = \sqrt{\frac{A}{B}\left(1 - \frac{m^2 A}{E^2}\right)} \end{aligned} \tag{9.6}$$

so that

$$v_{\text{group}} = \frac{1}{\partial p/\partial E} = \sqrt{\frac{A}{B}\left(1 - \frac{m^2 A}{E^2}\right)} \tag{9.7}$$

For massless particles

$$v_{\text{group}} = v_{\text{phase}} = \sqrt{\frac{A}{B}} \approx 1 + 2\phi = 1 - \frac{2GM}{r} \tag{9.8}$$

so there is (no dispersion). In general we have

$$v_{\text{group}} v_{\text{phase}} = \frac{A}{B} \tag{9.9}$$

In the context of optics, the stationary phase condition is

$$0 = \frac{1}{c}\int_0^z n dz' + \frac{\omega}{c}\int_0^z \frac{\partial n}{\partial \omega} dz' - t \tag{9.10}$$

The peak is displaced by $\Delta z$ in time $\Delta t$ where

$$\left[ n + \omega \frac{\partial n}{\partial \omega} \right] \Delta z = c \Delta t \tag{9.11}$$

Therefore, the group speed is

$$v_{\text{group}} = \frac{c}{n + \omega \frac{\partial n}{\partial \omega}} \tag{9.12}$$

# 10 Hamilton's principle

Lagrangian formulation
Integrate the infinitesimal invariant interval along the path in spacetime

$$S = \int d\tau = \int d\sigma \left(\frac{d\tau}{d\sigma}\right) = \int d\sigma \sqrt{g_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}} \tag{10.1}$$

Euler's equations then yield

$$\frac{d}{d\sigma}\left(\frac{g_{\mu\nu}(dx^\nu/d\sigma)}{(d\tau/d\sigma)}\right) = \tfrac{1}{2}\partial_\mu g_{\kappa\lambda}\frac{(dx^\kappa/d\sigma)(dx^\lambda/d\sigma)}{(d\tau/d\sigma)} \tag{10.2}$$

Since the Schwarzchild metric in isotropic coordinates

$$ds^2 = A(r)(dx^0)^2 - B(r)(d\mathbf{x})^2 \tag{10.3}$$

is independent of $x^0$, the $\mu = 0$ Euler equation implies

$$\frac{A(dx^0/d\sigma)}{(d\tau/d\sigma)} = \text{const} \tag{10.4}$$

or

$$A dx^0 = \gamma d\tau \tag{10.5}$$

where we have evaluated the constant at $r \to \infty$ where $A \to 1$. We can also write this as

$$dx^0 = \frac{E}{A}\frac{d\tau}{m} \tag{10.6}$$

Then

$$\begin{aligned} d\tau^2 &= A(dx^0)^2 - B(d\mathbf{x})^2 \\ B(d\mathbf{x})^2 &= \left[\frac{E^2}{m^2 A} - 1\right](d\tau)^2 \\ |d\mathbf{x}| &= \sqrt{\frac{E^2 - m^2 A}{AB}}\frac{d\tau}{m} \end{aligned} \tag{10.7}$$

from which we have

$$\begin{aligned} \frac{dx^0}{|d\mathbf{x}|} &= \frac{E}{A}\sqrt{\frac{AB}{E^2 - m^2 A}} \\ \frac{d\tau}{|d\mathbf{x}|} &= m\sqrt{\frac{AB}{E^2 - m^2 A}} \end{aligned} \tag{10.8}$$

The $\mu = i$ Euler equation is then

$$\frac{d}{d\sigma}\left(\frac{-B(d\mathbf{x}/d\sigma)}{(d\tau/d\sigma)}\right) = \tfrac{1}{2}\left[\frac{(dx^0/d\sigma)^2}{(d\tau/d\sigma)}\nabla A - \frac{(d\mathbf{x}/d\sigma)^2}{(d\tau/d\sigma)}\nabla B\right] \tag{10.9}$$

If we choose $d\sigma = dx^0 = dt$, then

$$\frac{d}{dt}\left(\frac{B}{A}\frac{d\mathbf{x}}{dt}\right) = \frac{1}{2B}\left(1 - \frac{m^2 A}{E^2}\right)\nabla B - \frac{1}{2A}\nabla A \tag{10.10}$$

Can we also get this from?

$$S = \int dt \sqrt{A - B\left(\frac{d\mathbf{x}}{dt}\right)^2} \tag{10.11}$$

Choosing $d\sigma = |d\mathbf{x}|$, we obtain

$$\frac{d}{d\sigma}\left(\sqrt{\frac{B(E^2 - m^2 A)}{A}}\frac{d\mathbf{x}}{d\sigma}\right) = \frac{1}{2}\sqrt{\frac{E^2 - m^2 A}{AB}}\nabla B - \frac{E^2}{2A^2}\sqrt{\frac{AB}{E^2 - m^2 A}}\nabla A \tag{10.12}$$

# References


[1] H. Goldstein, *"Classical Mechanics"*, Addison Wesley (1950).

[2] M. Born and E. Wolf, *"Principles of Optics"*, Pergamon Press (1980).

[3] L. D. Landau and E. M. Lifshitz, *"The Classical Theory of Fields"*, Pergamon Press (1975).

[4] H. Murayama, *"Miscellaneous lecture notes: Classical Mechanics"*, Berkeley Lecture notes for 221A (2006).

[5] S. Weinberg, *"Lectures on Quantum Mechanics"*, Cambridge University Press (2013).

[6] H. Ohanian, *"Gravitation and Spacetime (1st ed.)"*, Norton (1976).

[7] M. Silverman, *"Waves and Grains"*, Princeton University Press (1998).

[8] C. A. Brau, *"Modern Problems in Classical Electrodynamics"*, Oxford University Press (2004).