Renormalization of QED (March 2017)

Pdf file generated on April 6, 2017.
To do: finish up (2.15) –massless.
To do: complete calc for (3.4).

1 Lagrangian and Feynman rules

In these notes, we'll use $\eta_{00} = 1$.

Consider the bare QED Lagrangian with gauge fixing term

$$\mathcal{L}_0 = \bar{\psi}_0 \left(i\partial\!\!\!/ - m_0 - e_0 A\!\!\!/_0\right) \psi_0 - \tfrac{1}{4} F_{0\mu\nu} F_0^{\mu\nu} - \tfrac{1}{2}\lambda_0 (\partial_\mu A_0^\mu)^2 \tag{1.1}$$

Let $\psi_0 = Z_2^{1/2}\psi$ and $A_0 = Z_3^{1/2} A$, so that

$$\mathcal{L}_0 = Z_2 \bar{\psi}\left(i\partial\!\!\!/ - m_0 - e_0\sqrt{Z_3} A\!\!\!/\right)\psi - \tfrac{1}{4} Z_3 F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2}\lambda_0 Z_3 (\partial_\mu A^\mu)^2 \tag{1.2}$$

Define

$$\begin{aligned} &Z_2 = 1 + \delta Z_2 \qquad Z_2 m_0 = m + \delta m \qquad Z_2\sqrt{Z_3} e_0 = \mu^{\frac{4-d}{2}} Z_1 e = \mu^{\frac{4-d}{2}}(e + \delta Z_1 e) \\ &Z_3 = 1 + \delta Z_3 \qquad Z_3 \lambda_0 = \lambda + \delta\lambda \end{aligned} \tag{1.3}$$

to write $\mathcal{L}_0 = \mathcal{L} + \mathcal{L}_{\text{c.t.}}$ with

$$\begin{aligned} \mathcal{L} &= \bar{\psi}\left(i\partial\!\!\!/ - m - e\mu^{\frac{4-d}{2}} A\!\!\!/\right)\psi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2}\lambda(\partial_\mu A^\mu)^2 \\ \mathcal{L}_{\text{c.t.}} &= \bar{\psi}\left(i\delta Z_2 \partial\!\!\!/ - \delta m - \delta Z_1 e\mu^{\frac{4-d}{2}} A\!\!\!/\right)\psi - \tfrac{1}{4}\delta Z_3 F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2}\delta\lambda(\partial_\mu A^\mu)^2 \end{aligned} \tag{1.4}$$

The fermion and photon propagators, photon-electron vertex and the associated counterterms are

$$\begin{aligned} &\frac{i}{p\!\!\!/ - m + i\varepsilon} && - i\delta m + i\delta Z_2\, p\!\!\!/ \\ &\frac{-iP_{\mu\nu}}{k^2 + i\epsilon} + \frac{-iR_{\mu\nu}}{\lambda k^2 + i\epsilon} && - i\delta Z_3(k^2\eta_{\mu\nu} - k_\mu k_\nu) - i\delta\lambda k_\mu k_\nu = -i\delta Z_3 k^2 P_{\mu\nu} - i\delta\lambda k^2 R_{\mu\nu} \\ &- ie\mu^{\frac{4-d}{2}}\gamma^\mu && - i\,\delta Z_1 e\,\mu^{\frac{4-d}{2}}\gamma^\mu \end{aligned} \tag{1.5}$$

We use Feynman gauge ($\lambda = 1$) in all one-loop calculations.

Also note Peskin and Schroeder (7.89)

$$\begin{aligned} \{\gamma^\mu, \gamma^\nu\} &= 2\eta^{\mu\nu} \\ \gamma^\mu\gamma^\nu\gamma_\mu &= (2-d)\gamma^\nu \\ \gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu &= 4\eta^{\nu\rho} + (d-4)\gamma^\nu\gamma^\rho \\ \gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu &= -2\gamma^\sigma\gamma^\rho\gamma^\nu + (4-d)\gamma^\nu\gamma^\rho\gamma^\sigma \end{aligned} \tag{1.6}$$

2 Fermion two-point function

We now repeat the discussion in the renormalization notes.

The fermion two-point Green function with self energy $-i\Sigma$ *is*

$$\langle T(\psi\bar{\psi})\rangle = \frac{i}{\not{p}-m} + \frac{i}{\not{p}-m}\left(-i\Sigma(p) - i\delta m + i\delta Z\,\not{p}\right)\frac{i}{\not{p}-m} + \cdots = \frac{i}{\not{p}-m-\Sigma(p)-\delta m+\delta Z\,\not{p}}$$

Peskin and Schroeder (p. 220) point out that the matrix structure of Σ *contains only* $\not{p}$ *and* $\mathbb{1}$*, with coefficients depending on* $p^2 = \not{p}\not{p}$*, so it can be considered a function of* $\not{p}$*. Thus we can expand* $\Sigma(\not{p})$ *in* $\not{p}$ *near the physical mass* m_p

$$\Sigma(\not{p}) = \Sigma(m_p) + \Sigma'(m_p)(\not{p}-m_p) + \cdots\,, \qquad \text{where} \qquad \Sigma'(\not{p}) \equiv \frac{\partial\Sigma}{\partial\not{p}} \tag{2.1}$$

we find that the denominator of the two-point Green function is

$$[m_p - m - \Sigma(m_p) - \delta m + \delta Z\, m_p] + [1 - \Sigma'(m_p) + \delta Z]\,(\not{p} - m_p) + \mathcal{O}((\not{p}-m)^2)\,. \tag{2.2}$$

The physical mass is determined by the solutions of

$$m_p - m - \Sigma(m_p) - \delta m + \delta Z\, m_p = 0 \tag{2.3}$$

and

$$\langle T(\psi\bar{\psi})\rangle = \frac{iR}{\not{p}-m_p}, \qquad \text{where} \qquad R = \frac{1}{1-\Sigma'(m_p)+\delta Z} \tag{2.4}$$

In the one-loop approximation, set $m_p = m$ *in the terms that are already first order to find*

$$m_p = m + \Sigma(m) + \delta m - \delta Z m \qquad \text{and} \qquad R = 1 + \Sigma'(m) - \delta Z\,. \tag{2.5}$$

The one-loop self energy is

$$\begin{aligned} -i\Sigma(\not{p}) &= \left(-ie\mu^{\frac{4-d}{2}}\right)^2 \int \frac{d^d\ell}{(2\pi)^d}\gamma^\mu \frac{i}{\not{p}+\not{\ell}-m+i\varepsilon}\gamma^\nu\left(\frac{-i\eta_{\mu\nu}}{\ell^2-m_\gamma^2+i\varepsilon}\right) \\ &= \;-\; e^2\mu^{4-d}\int\frac{d^d\ell}{(2\pi)^d}\frac{\gamma^\mu\left(\not{p}+\not{\ell}+m\right)\gamma_\mu}{[(\ell+p)^2-m^2+i\varepsilon]\left[\ell^2-m_\gamma^2+i\varepsilon\right]} \end{aligned} \tag{2.6}$$

where we have added a small photon mass m_γ to regulate the IR divergence. Now using $\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu} \longrightarrow \gamma^\mu\gamma_\mu = d$ and $\gamma^\mu\gamma^\nu\gamma_\mu = (2-d)\gamma^\nu$ this becomes

$$-i\Sigma(\not{p}) = \;-\; e^2\mu^{4-d}\int\frac{d^d\ell}{(2\pi)^d}\frac{(2-d)(\not{p}+\not{\ell})+dm}{[(\ell+p)^2-m^2+i\varepsilon]\left[\ell^2-m_\gamma^2+i\varepsilon\right]} \tag{2.7}$$

We now turn to our loop integral notes, and use the α representation with $P_1 = p$, $P_2 = 0$, $M_1 = m$, and $M_2 = m_\gamma$. In doing so, the loop momentum gets shifted to

$\ell = \bar{\ell} - \sum_j (\alpha_j/\alpha) P_j = \bar{\ell} - (\alpha_1/\alpha) p$. The $\bar{\not{\ell}}$ term will vanish upon integration over $\bar{\ell}$ leaving us with a term $(2-d)(1-\alpha_1/\alpha)\not{p} + dm$ in the integrand. Next we convert to Feynman parametrization (choosing $c_i = 1$) $\alpha_j = \alpha\beta_j$ so the integrand now contains $(2-d)(1-\beta_1)\not{p} + dm$. Setting $\beta_1 = x$, we use the results in the loop integral notes to write

$$-i\Sigma(\not{p}) = - e^2 \mu^{4-d} \frac{i}{(4\pi)^{d/2}} \Gamma\left(\tfrac{4-d}{2}\right) \int_0^1 dx \frac{(2-d)(1-x)\not{p} + dm}{\left[xm^2 + (1-x)m_\gamma^2 - x(1-x)p^2\right]^{\frac{4-d}{2}}} \tag{2.8}$$

We evaluate near four dimensions ($d = 4 - 2\epsilon$)

$$\begin{aligned}
\Sigma(\not{p}) &= \frac{e^2\mu^{2\epsilon}}{(4\pi)^{2-\epsilon}} \Gamma(\epsilon) \int_0^1 dx \frac{[-2(1-\epsilon)(1-x)\not{p} + (4-2\epsilon)m]}{\left[xm^2 + (1-x)m_\gamma^2 - x(1-x)p^2\right]^\epsilon} \\
&= \frac{e^2}{(4\pi)^2} \left(\frac{4\pi\mu^2}{m^2}\right)^\epsilon \frac{\Gamma(1+\epsilon)}{\epsilon} \int_0^1 dx \frac{[-2(1-\epsilon)(1-x)\not{p} + (4-2\epsilon)m]}{\left[x + (1-x)\frac{m_\gamma^2}{m^2} - x(1-x)\frac{p^2}{m^2}\right]^\epsilon} \\
&= \frac{e^2}{(4\pi)^2} \left(\frac{4\pi\mu^2}{m^2}\right)^\epsilon \frac{\Gamma(1+\epsilon)}{\epsilon} \Bigg(- (1-\epsilon)\not{p} + (4-2\epsilon)m \\
&\qquad + 2\epsilon \int_0^1 dx\, [(1-x)\not{p} - 2m] \log\left[x + (1-x)\frac{m_\gamma^2}{m^2} - x(1-x)\frac{p^2}{m^2}\right] \Bigg)
\end{aligned} \tag{2.9}$$

See my notes on Sterman for a slightly different derivation. The first line agrees with Peskin and Schroeder (10.41) with $x \to 1-x$.

We evaluate this near the physical mass m_p. At the order to which we are working $m_p = m$ so

$$\begin{aligned}
\Sigma(\not{p} = m) &= \frac{e^2}{(4\pi)^2} \left(\frac{4\pi\mu^2}{m^2}\right)^\epsilon \frac{\Gamma(1+\epsilon)}{\epsilon} \left((3-\epsilon)m - 2m\epsilon \int_0^1 dx (1+x) \log\left[x^2 + (1-x)\frac{m_\gamma^2}{m^2}\right] \right) \\
&= \frac{me^2}{(4\pi)^2} \left(\frac{4\pi\mu^2}{m^2}\right)^\epsilon \frac{\Gamma(1+\epsilon)}{\epsilon} (3+4\epsilon) \qquad \text{using} \qquad \int_0^1 dx (1+x) \log x^2 = -\frac{5}{2} \\
&= \frac{3me^2}{(4\pi)^2} \left(\frac{1}{\epsilon} - \gamma + \log\left[\frac{4\pi\mu^2}{m^2}\right] + \frac{4}{3} \right)
\end{aligned} \tag{2.10}$$

We set $m_\gamma = 0$ in the integral because it converges without an IR cutoff.

Also

$$\begin{aligned}
\Sigma'(m) =& \frac{e^2}{(4\pi)^2} \left(\frac{4\pi\mu^2}{m^2}\right)^\epsilon \frac{\Gamma(1+\epsilon)}{\epsilon} \times \\
& \left(-1 + \epsilon + 2\epsilon \int_0^1 dx (1-x) \log\left[x^2 + (1-x)\frac{m_\gamma^2}{m^2}\right] + 2\epsilon \int_0^1 dx \frac{2x(1-x^2)}{x^2 + (1-x)\frac{m_\gamma^2}{m^2}} \right) \\
=& - \frac{e^2}{(4\pi)^2} \left(\frac{4\pi\mu^2}{m^2}\right)^\epsilon \frac{\Gamma(1+\epsilon)}{\epsilon} \left(1 - \epsilon - 2\epsilon \int_0^1 dx (1-x) \log x^2 - 2\epsilon \int_0^1 dx \frac{2x(1-x^2)}{x^2 + \frac{m_\gamma^2}{m^2}} \right) \\
=& - \frac{e^2}{(4\pi)^2} \left(\frac{1}{\epsilon} - \gamma + \log\left[\frac{4\pi\mu^2}{m^2}\right] + 4 + 2\log\left[\frac{m_\gamma^2}{m^2}\right] \right)
\end{aligned} \tag{2.11}$$

We have simplified the integrals in the second line assuming that m_γ is very small, and then used

$$\int_0^1 dx(1-x)\log x^2 = -\frac{3}{2}$$
$$\int_0^1 dx\, \frac{2x(1-x^2)}{(x^2+a^2)} = -1-(a^2+1)\log\left(\frac{a^2}{a^2+1}\right) = -1-\log\left(a^2\right)+\cdots \tag{2.12}$$

courtesy of Mathematica. Choosing

$$\delta Z_2 = -\frac{e^2}{(4\pi)^2}\left(\frac{1}{\epsilon}+c_2\right) \qquad \text{and} \qquad \delta m = -\frac{4me^2}{(4\pi)^2}\left(\frac{1}{\epsilon}+c_m\right) \tag{2.13}$$

we have

$$R_2 = 1+\Sigma'(m)-\delta Z_2 = 1 \;-\; \frac{e^2}{(4\pi)^2}\left(-c_2-\gamma+\log\left[\frac{4\pi\mu^2}{m^2}\right]+4+2\log\left[\frac{m_\gamma^2}{m^2}\right]\right)$$
$$m_p = m+\Sigma(m)+\delta m-\delta Z_2 m = m+\frac{3me^2}{(4\pi)^2}\left(\frac{1}{3}c_2-\frac{4}{3}c_m-\gamma+\log\left[\frac{4\pi\mu^2}{m^2}\right]+\frac{4}{3}\right) \tag{2.14}$$

For massless fermions (and photons), the self energy is

$$\begin{aligned}\Sigma(\not{p}) &= \frac{e^2}{(4\pi)^{d/2}}\left(\frac{\mu^2}{-p^2}\right)^{\frac{4-d}{2}}\Gamma\left(\tfrac{4-d}{2}\right)(2-d)\not{p}\int_0^1 dx(1-x)^{\frac{d-2}{2}}x^{\frac{d-4}{2}} \\ &= \;-\;\frac{e^2}{(4\pi)^{d/2}}\left(\frac{\mu^2}{-p^2}\right)^{\frac{4-d}{2}}\frac{\Gamma\left(\frac{4-d}{2}\right)\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d-2}{2}\right)}{\Gamma(d-2)}\not{p} \\ &\longrightarrow \;-\;\frac{e^2}{(4\pi)^2}\frac{1}{\epsilon}\not{p}\end{aligned} \tag{2.15}$$

Calculate the subleading piece for this!

3 Photon two-point function

Recall from the renormalization notes (renn.tex) the definitions of the projection operators

$$P^{\mu\nu} = \eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}, \qquad R^{\mu\nu} = \frac{k^\mu k^\nu}{k^2} \tag{3.1}$$

which obey $P^2 = P$, $R^2 = R$, $PR = RP = 0$, and $P + R = 1$. Recall the propagator and counterterms

$$\frac{-iP_{\mu\nu}}{k^2} + \frac{-iR_{\mu\nu}}{\lambda k^2}, \qquad -i\delta Z_3 k^2 P_{\mu\nu} - i\delta\lambda k^2 R_{\mu\nu} \tag{3.2}$$

Denote the self-energy diagram by

$$i\Pi_{\mu\nu}(k^2) = ik^2\Pi(k^2)P_{\mu\nu}$$

then the two-point function is

$$\begin{aligned}
\langle T(A_\mu A_\nu)\rangle &= \left(\frac{-iP_{\mu\nu}}{k^2} + \frac{-iR_{\mu\nu}}{\lambda k^2}\right) \\
&+ \left(\frac{-iP_{\mu\alpha}}{k^2} + \frac{-iR_{\mu\alpha}}{\lambda k^2}\right)(ik^2\Pi(k^2)P_{\alpha\beta} - i\delta Z_3 k^2 P_{\alpha\beta} - i\delta\lambda k^2 R_{\alpha\beta})\left(\frac{-iP_{\beta\nu}}{k^2} + \frac{-iR_{\beta\nu}}{\lambda k^2}\right) + \cdots \\
&= \frac{-iP_{\mu\nu}}{k^2}\left(1 + (\Pi(k^2) - \delta Z_3) + \cdots\right) + \frac{-iR_{\mu\nu}}{\lambda k^2}\left(1 + \frac{-\delta\lambda}{\lambda} + \cdots\right) \\
&= \frac{-iP_{\mu\nu}}{(1 - \Pi(k^2) + \delta Z_3)k^2} + \frac{-iR_{\mu\nu}}{(\lambda + \delta\lambda)k^2}
\end{aligned} \tag{3.3}$$

Thus we do not require a counterterm $\delta\lambda$ for the gauge-fixing term so $Z_3\lambda_0 = \lambda$.

The photon wave function is

$$i\Pi_{\mu\nu}(k^2) = \left(-ie\mu^{\frac{4-d}{2}}\right)^2(-1)\int\frac{d^d\ell}{(2\pi)^d}\,\mathrm{tr}\left(\gamma_\mu\frac{i}{\not k + \not\ell - m + i\varepsilon}\gamma_\nu\frac{i}{\not\ell - m + i\varepsilon}\right) \tag{3.4}$$

Then ....

$$\begin{aligned}
\Pi(k^2) &= -\frac{8e^2\mu^{4-d}}{(4\pi)^{d/2}}\Gamma\left(\tfrac{4-d}{2}\right)\int_0^1 dx\,\frac{x(1-x)}{[m^2 - x(1-x)k^2]^{\frac{4-d}{2}}} \\
&= -\frac{8e^2}{(4\pi)^2}\left(\frac{4\pi\mu^2}{m^2}\right)^\epsilon\frac{\Gamma(1+\epsilon)}{\epsilon}\int_0^1 dx\,\frac{x(1-x)}{\left[1 - x(1-x)\frac{k^2}{m^2}\right]^\epsilon} \\
&= -\frac{8e^2}{(4\pi)^2}\left(\frac{4\pi\mu^2}{m^2}\right)^\epsilon\frac{\Gamma(1+\epsilon)}{\epsilon}\left[\frac{1}{6} - \epsilon\int_0^1 dx\; x(1-x)\log F\right] \\
&= -\frac{4}{3}\frac{e^2}{(4\pi)^2}\left(\frac{1}{\epsilon} - \gamma + \log\left[\frac{4\pi\mu^2}{m^2}\right] - 6\int_0^1 dx\; x(1-x)\log F\right)
\end{aligned} \tag{3.5}$$

Since $F(k^2 = 0) = 1$, we obtain

$$\Pi(0) = \; - \; \frac{4}{3}\frac{e^2}{(4\pi)^2}\left(\frac{1}{\epsilon} - \gamma + \log\left[\frac{4\pi\mu^2}{m^2}\right]\right) \tag{3.6}$$

Thus

$$\begin{aligned} \delta Z_3 = & \; - \; \frac{4}{3}\frac{e^2}{(4\pi)^2}\left(\frac{1}{\epsilon} + c_A\right) = \; - \; \frac{\alpha}{3\pi}\left(\frac{1}{\epsilon} + c_A\right) \\ R_3 = 1 + \Pi(0) - \delta Z_3 = & \; 1 \; - \; \frac{4}{3}\frac{e^2}{(4\pi)^2}\left(-c_A - \gamma + \log\left[\frac{4\pi\mu^2}{m^2}\right]\right) \end{aligned} \tag{3.7}$$

4 Electron-photon vertex

The unrenormalized and renormalized 1PI three-point functions are related by

$$\Gamma^{\mu}_{\bar\psi\psi A,0} = Z_2^{-1} Z_3^{-1/2} \Gamma^{\mu}_{\bar\psi\psi A} \tag{4.1}$$

The unrenormalized function is computed using bare parameters

$$\Gamma^{\mu}_{\bar\psi\psi A,0} = -ie_0\gamma^\mu + e_0^3(\text{one-loop-diagram}) \tag{4.2}$$

The renormalized function is computed using renormalized parameters and a counterterm

$$\Gamma^{\mu}_{\bar\psi\psi A} = -ie\mu^{\frac{4-d}{2}}\gamma^\mu - i\delta Z_1\, e\mu^{\frac{4-d}{2}}\gamma^\mu + e^3\mu^{\frac{3(4-d)}{2}}(\text{one-loop-diagram}) \tag{4.3}$$

Let's strip off a factor of $-ie$ from the 1PI diagrams

$$\begin{aligned} \Gamma^{\mu}_{\bar\psi\psi A,0} &= -ie_0\Gamma^\mu_0(p',p) \qquad & \Gamma^\mu_0(p',p) &= \gamma^\mu + e_0^2(\text{one-loop-diagram}) \\ \Gamma^{\mu}_{\bar\psi\psi A} &= -ie\mu^{\frac{4-d}{2}}\Gamma^\mu(p',p) \qquad & \Gamma^\mu(p',p) &= \gamma^\mu + \delta Z_1\gamma^\mu + e^2\mu^{4-d}(\text{one-loop-diagram}) \end{aligned} \tag{4.4}$$

Since $e^2 = e_0^2$+(higher-order), it is evident that

$$\Gamma^\mu_0(p',p) = Z_1^{-1}\Gamma^\mu(p',p) \tag{4.5}$$

as can be seen directly from eq. (4.1) by using $Z_2\sqrt{Z_3}e_0 = \mu^{\frac{4-d}{2}}Z_1 e$. This is essentially the content of Peskin and Schroeder (7.47).

Thus

$$\Gamma^\mu(p',p) = \gamma^\mu + \delta Z_1\gamma^\mu + \delta\Gamma^\mu(p',p) \tag{4.6}$$

where p is incoming, p' is outgoing, $q = p' - p$ is the incoming photon momentum, and the one-loop contribution is

$$\begin{aligned} \delta\Gamma^\mu(p',p) &= \left(-ie\mu^{\frac{4-d}{2}}\right)^2 \int \frac{d^d\ell}{(2\pi)^d}\left(\frac{-i\eta_{\nu\lambda}}{\ell^2+i\varepsilon}\right)\gamma^\nu \frac{i}{\not p' + \not\ell - m + i\varepsilon}\gamma^\mu \frac{i}{\not p + \not\ell - m + i\varepsilon}\gamma^\lambda \\ &= -ie^2\mu^{4-d}\int \frac{d^d\ell}{(2\pi)^d}\frac{\gamma^\nu(\not p' + \not\ell + m)\gamma^\mu(\not p + \not\ell + m)\gamma_\nu}{([\ell+p]^2 - m^2 + i\varepsilon)([\ell+p']^2 - m^2 + i\varepsilon)(\ell^2 - \mu^2 + i\varepsilon)} \end{aligned} \tag{4.7}$$

We can simplify this by sandwiching between spinors and using $\{\not a, \not b\} = 2a\cdot b$ together with $(\not p - m)u(p) = \bar u(p')(\not p' - m) = 0$ to find

$$\begin{aligned} \bar u(p')\delta\Gamma^\mu(p',p)u(p) &= -ie^2\mu^{4-d}\int \frac{d^d\ell}{(2\pi)^d}\frac{\bar u(p')\gamma^\nu(\not p' + \not\ell + m)\gamma^\mu(\not p + \not\ell + m)\gamma_\nu u(p)}{([\ell+p]^2 - m^2 + i\varepsilon)([\ell+p']^2 - m^2 + i\varepsilon)(\ell^2 - \mu^2 + i\varepsilon)} \\ &= -ie^2\mu^{4-d}\int \frac{d^d\ell}{(2\pi)^d}\frac{\bar u(p')\left(4p\cdot p'\gamma^\mu + 2\not p\not\ell\gamma^\mu + 2\gamma^\mu\not\ell\not p' + \gamma^\nu\not\ell\gamma^\mu\not\ell\gamma_\nu\right)u(p)}{([\ell+p]^2 - m^2 + i\varepsilon)([\ell+p']^2 - m^2 + i\varepsilon)(\ell^2 - \mu^2 + i\varepsilon)} \end{aligned} \tag{4.8}$$

We could simplify the last term using $\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu + (4-d)\gamma^\nu\gamma^\rho\gamma^\sigma$.

Peskin and Schroeder argue that symmetry restricts the electron/photon vertex to the form

$$\Gamma_\mu(p', p) = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m} F_2(q^2) \tag{4.9}$$

The contribution $F_2(q^2)$ comes entirely from $\delta\Gamma^\mu$ and is UV- and IR-finite, whereas

$$F_1(q^2) = 1 + \delta Z_1 + \delta F_1(q^2) \tag{4.10}$$

and $\delta F_1(q^2)$ is both UV- and IR-divergent. In (10.45), Peskin and Schroeder compute the shift in form factor in dimensional regularization obtaining (I'm also including the μ term)

$$\begin{aligned}\delta F_1(q^2) = \frac{e^2\mu^{4-d}}{(4\pi)^{d/2}} \int_0^1 dx\, dy\, dz\, \delta(1-x-y-z) \Bigg[\frac{\Gamma(\frac{4-d}{2})}{\Delta^{\frac{4-d}{2}}} \frac{(2-2\epsilon)^2}{2} \\ + \frac{\Gamma(\frac{6-d}{2})}{\Delta^{\frac{6-d}{2}}} \left(\left[2(1-x)(1-y) - 2\epsilon xy\right] q^2 + \left[2(1-4z+z^2) - 2\epsilon(1-z)^2\right] m^2 \right) \Bigg]\end{aligned} \tag{4.11}$$

where $\Delta = (1-z)^2 m^2 + z m_\gamma^2 - zyq^2$. Expanding around $d = 4 - 2\epsilon$, we have

$$\begin{aligned}\delta F_1(q^2) = \frac{e^2}{(4\pi)^2} \left(\frac{4\pi\mu^2}{m^2}\right)^\epsilon \int_0^1 dx\, dy\, dz\, \delta(1-x-y-z) \Bigg[\frac{2\Gamma(1+\epsilon)(1-2\epsilon)}{\epsilon\left[(1-z)^2 + z(m_\gamma^2/m^2) - zy(q^2/m^2)\right]^\epsilon} \\ + \frac{2(1-x)(1-y)q^2 + 2(1-4z+z^2)m^2}{(1-z)^2 m^2 + z m_\gamma^2 - zyq^2} \Bigg]\end{aligned} \tag{4.12}$$

Also see Srednicki (63.19). To determine the counterterm let's look at the $1/\epsilon$ pole.

$$\delta F_1(q^2) = \frac{e^2}{(4\pi)^2} \int_0^1 dx\, dy\, dz\, \delta(1-x-y-z) \frac{2}{\epsilon} = \frac{e^2}{(4\pi)^2} \frac{1}{\epsilon} \tag{4.13}$$

so

$$\delta Z_1 = -\frac{e^2}{(4\pi)^2} \left(\frac{1}{\epsilon} + c_1 \right). \tag{4.14}$$

We see that $\delta Z_1 = \delta Z_2$ (provided $c_1 = c_2$) as expected by the Ward identity.

Next consider the form factor at $q^2 = 0$. Since the integrand is independent of x and y we obtain

$$\begin{aligned}\delta F_1 =& \frac{e^2}{(4\pi)^2} \left(\frac{4\pi\mu^2}{m^2}\right)^\epsilon \frac{\Gamma(1+\epsilon)}{\epsilon} \int_0^1 dz\, (1-z) \left[\frac{2(1-2\epsilon)}{\left[(1-z)^2 + z(m_\gamma^2/m^2)\right]^\epsilon} + \frac{2\epsilon(1-4z+z^2)m^2}{(1-z)^2 m^2 + z m_\gamma^2} \right] \\ =& \frac{e^2}{(4\pi)^2} \left(\frac{4\pi\mu^2}{m^2}\right)^\epsilon \frac{\Gamma(1+\epsilon)}{\epsilon} \times \\ & \left(1 - 2\epsilon - 2\epsilon \int_0^1 dz\, (1-z) \log\left[(1-z)^2 + z\frac{m_\gamma^2}{m^2}\right] + 2\epsilon \int_0^1 dz\, \frac{(1-z)(1-4z+z^2)}{(1-z)^2 + z(m_\gamma^2/m^2)} \right)\end{aligned} \tag{4.15}$$

where we drop m_γ from the first term and set $z = 1$ in front of m_γ^2 in the second. Then using

$$\int_0^1 dz\,(1-z)\log(1-z) = -\frac{1}{4}$$
$$\int_0^1 dz\,\frac{(1-z)(1-4z+z^2)}{(1-z)^2+(m_\gamma^2/m^2)} = \frac{5}{2} + \log\left[\frac{m_\gamma^2}{m^2}\right] \tag{4.16}$$

courtesy of Mathematica (and taking the small m_γ limit of the result), we obtain

$$\delta F_1(0) = \frac{e^2}{(4\pi)^2}\left(\frac{1}{\epsilon} - \gamma + \log\left[\frac{4\pi\mu^2}{m^2}\right] + 4 + 2\log\left[\frac{m_\gamma^2}{m^2}\right]\right) \tag{4.17}$$

Observe that $\delta F_1(0) + \Sigma'(m) = 0$. It is also possible to see this earlier in the calculation by following the approach of Peskin and Schroeder on p. 222 but in the context of dimensional regularization, as they suggest on p. 334. The term in parentheses in the first line of eq. (2.11) is, setting $z = 1 - x$ is

$$-1 + \epsilon + 2\epsilon\int_0^1 dz\;z\log\left[(1-z)^2 + z\frac{m_\gamma^2}{m^2}\right] + 2\epsilon\int_0^1 dz\frac{(1-z)(4z-2z^2)}{(1-z)^2+z(m_\gamma^2/m^2)} \tag{4.18}$$

The term in parentheses in the last line of eq. (4.15) is

$$1 - 2\epsilon - 2\epsilon\int_0^1 dz\,(1-z)\log\left[(1-z)^2 + z\frac{m_\gamma^2}{m^2}\right] + 2\epsilon\int_0^1 dz\,\frac{(1-z)(1-4z+z^2)}{(1-z)^2+z(m_\gamma^2/m^2)} \tag{4.19}$$

Adding these we get

$$\epsilon - 2\epsilon - 2\epsilon\int_0^1 dz\,(1-2z)\log\left[(1-z)^2 + z\frac{m_\gamma^2}{m^2}\right] + 2\epsilon\int_0^1 dz\frac{(1-z)^2(1+z)}{(1-z)^2+z(m_\gamma^2/m^2)} \tag{4.20}$$

Integrating by parts

$$-2\epsilon\int_0^1 dz\,(1-2z)\log\left[(1-z)^2 + z\frac{m_\gamma^2}{m^2}\right] = 2\epsilon\int_0^1 dz\,z(1-z)\frac{2(z-1)+(m_\gamma^2/m^2)}{(1-z)^2+z(m_\gamma^2/m^2)} \tag{4.21}$$

so the sum is

$$\epsilon - 2\epsilon + 2\epsilon\int_0^1 dz\,(1-z) = 0 \tag{4.22}$$

The result $\delta F_1(0) + \Sigma'(m) = 0$ is a consequence of a Ward identity (see next section). This also dictates that the UV divergent counterterms obey $Z_1 = Z_2$. Hence

$$F_1(0) = 1 + \frac{e^2}{(4\pi)^2}\left(-c_1 - \gamma + \log\left[\frac{4\pi\mu^2}{m^2}\right] + 4 + 2\log\left[\frac{m_\gamma^2}{m^2}\right]\right) \tag{4.23}$$

Also recall from earlier that

$$R_2 = 1 \;-\; \frac{e^2}{(4\pi)^2}\left(-c_2 - \gamma + \log\left[\frac{4\pi\mu^2}{m^2}\right] + 4 + 2\log\left[\frac{m_\gamma^2}{m^2}\right]\right) \tag{4.24}$$

We must choose the finite counterterms equal $c_1 = c_2$ to preserve gauge invariance. Then $F_1(0) = 1$ implies $R_2 = 1$ and vice versa.

Sterman also computes the vertex diagram for a massless electron exactly (p. 400), finding

$$F_1(q^2) = 1 - \frac{e^2}{8\pi^2} \frac{\Gamma(1+\epsilon)\,\Gamma(1-\epsilon)^2}{\Gamma(1-2\epsilon)} \left(\frac{4\pi\mu^2}{-q^2}\right)^{\epsilon} \left[\frac{1}{\epsilon^2} + \frac{3}{2\epsilon} + 4\right] \tag{4.25}$$

This includes UV as well as IR divergences. We can imagine minimally subtracting the UV pole. Then we compute the self-energy for massless electrons, but this vanishes, being scaleless. By the Ward identity the UV counterterms for the self-energy and the vertex cancel out. Therefore the total form factor is just

$$F_1(q^2) = 1 - \frac{e^2}{8\pi^2} \frac{\Gamma(1+\epsilon)\,\Gamma(1-\epsilon)^2}{\Gamma(1-2\epsilon)} \left(\frac{4\pi\mu^2}{-q^2}\right)^{\epsilon} \left[\frac{1}{\epsilon^2} + \frac{3}{2\epsilon} + 4\right] \tag{4.26}$$

5 Ward identity

The Ward identity is given by

$$S(p')\left[-ieq_\mu\Gamma^\mu(p',p)\right]S(p) = e\left[S(p) - S(p')\right] \tag{5.1}$$

where $S(p)$ is the full fermion propagator, and $q = p' = p$. Equivalently

$$q_\mu\Gamma^\mu(p',p) = i\left[S^{-1}(p') - S^{-1}(p)\right] \tag{5.2}$$

This is proven below diagrammatically to one loop when written in terms of bare quantities. Since

$$\begin{aligned} S_0(p) &= \langle T(\psi_0\bar{\psi}_0)\rangle = Z_2\langle T(\psi\bar{\psi})\rangle = Z_2 S(p) \\ S_0^{-1}(p) &= Z_2^{-1}S^{-1}(p) \\ \Gamma_0^\mu(p',p) &= Z_1^{-1}\Gamma^\mu(p',p) \end{aligned} \tag{5.3}$$

we see that it will also hold for renormalized quantities provided that

$$Z_1 = Z_2 \tag{5.4}$$

Now for the proof. Expanding the propagators and vertex function to first order

$$\begin{aligned} &S(p')\left[-ieq_\mu\Gamma^\mu(p',p)\right]S(p) \\ &= \left[\frac{i}{\not{p}'-m} + \frac{i}{\not{p}'-m}(-i\Sigma(p'))\frac{i}{\not{p}'-m}\right]\left[-ie\not{q} - ieq_\mu\delta\Gamma^\mu(p',p)\right]\left[\frac{i}{\not{p}-m} + \frac{i}{\not{p}-m}(-i\Sigma(p))\frac{i}{\not{p}-m}\right] \\ &= \frac{i}{\not{p}'-m}\left[-ie\not{q}\right]\frac{i}{\not{p}-m} + \frac{i}{\not{p}'-m}\left[-ie\not{q}\right]\frac{i}{\not{p}-m}(-i\Sigma(p))\frac{i}{\not{p}-m} \\ &+ \frac{i}{\not{p}'-m}\left[-ieq_\mu\delta\Gamma^\mu(p',p)\right]\frac{i}{\not{p}-m} + \frac{i}{\not{p}'-m}(-i\Sigma(p'))\frac{i}{\not{p}'-m}\left[-ie\not{q}\right]\frac{i}{\not{p}-m} \end{aligned} \tag{5.5}$$

Next we observe that

$$\frac{i}{\not{p}'-m}\left[-ie\not{q}\right]\frac{i}{\not{p}-m} = \frac{i}{\not{p}'-m}\left[-ie(\not{p}'-m)-(\not{p}-m)\right]\frac{i}{\not{p}-m} = e\left[\frac{i}{\not{p}-m} - \frac{i}{\not{p}'-m}\right] \tag{5.6}$$

Also

$$\begin{aligned} q_\mu\delta\Gamma^\mu(p',p) &= \left(-ie\mu^{\frac{4-d}{2}}\right)^2\int\frac{d^d\ell}{(2\pi)^d}\left(\frac{-i\eta_{\nu\lambda}}{\ell^2+i\varepsilon}\right)\gamma^\nu\frac{i}{\not{p}'+\not{\ell}-m+i\varepsilon}(\not{p}'-\not{p})\frac{i}{\not{p}+\not{\ell}-m+i\varepsilon}\gamma^\lambda \\ &= \left(-ie\mu^{\frac{4-d}{2}}\right)^2\int\frac{d^d\ell}{(2\pi)^d}\left(\frac{-i\eta_{\nu\lambda}}{\ell^2+i\varepsilon}\right)\gamma^\nu\left[\frac{i}{\not{p}+\not{\ell}-m+i\varepsilon} - \frac{i}{\not{p}'+\not{\ell}-m+i\varepsilon}\right]\gamma^\lambda \\ &= i\left[(-i\Sigma(p)) - (-i\Sigma(p'))\right] \end{aligned} \tag{5.7}$$

Thus

$$-ieq_\mu \delta\Gamma^\mu(p',p) = e\left[(-i\Sigma(p)) - (-i\Sigma(p'))\right] \tag{5.8}$$

Inserting eqs. (5.6) and (5.8) into eq. (5.5) we obtain the telescoping expression

$$\begin{aligned}
&S(p')\left[-ieq_\mu\Gamma^\mu(p',p)\right]S(p) \\
&= e\Bigg\{\frac{i}{\not p - m} - \frac{i}{\not p' - m} + \left[\frac{i}{\not p - m} - \frac{i}{\not p' - m}\right](-i\Sigma(p))\frac{i}{\not p - m} \\
&+ \frac{i}{\not p' - m}\Big[(-i\Sigma(p)) - (-i\Sigma(p'))\Big]\frac{i}{\not p - m} + \frac{i}{\not p' - m}(-i\Sigma(p'))\left[\frac{i}{\not p - m} - \frac{i}{\not p' - m}\right]\Bigg\} \\
&= e\Big[S(p) - S(p')\Big]
\end{aligned} \tag{5.9}$$

QED. (Pun intended.) This proof is actually for bare quantities. The proof will also be valid for renormalized quantities, provided that eq. (5.8) also works for the counterterms

$$-ie\delta Z_1 \not q = e\left[(-i\delta m + i\delta Z_2 \not p) - (-i\delta m + i\delta Z_2 \not p')\right] \tag{5.10}$$

Naturally, this requires $\delta Z_1 = \delta Z_2$ for both the divergent and the finite pieces.