Renormalization of ϕ^4 scalar field theory (January 2017)

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1 Lagrangian and Green functions in d dimensions

In these notes, we'll use $\eta_{00} = 1$.

1.1 Lagrangian, counterterms, and Feynman rules

Consider the scalar lagrangian with quartic interaction

$$\mathcal{L} = \tfrac{1}{2}\left(\partial\phi\right)^2 - \tfrac{1}{2}m^2\phi^2 - \tfrac{1}{24}\lambda\mu^{4-d}\phi^4 \tag{1.1}$$

where ϕ is the renormalized (finite) field, and m and λ renormalized (finite) parameters, but not directly identifiable as physical parameters. It also depends on d to regularize the loop amplitudes, and a scale μ to make the couplings dimensionless in d dimensions. The counterterm lagrangian

$$\mathcal{L}_{\text{c.t.}} = \tfrac{1}{2}\delta Z\left(\partial\phi\right)^2 - \tfrac{1}{2}\delta m^2\phi^2 - \tfrac{1}{24}\,\delta\lambda\;\mu^{4-d}\phi^4 \tag{1.2}$$

contains poles in ϵ and also depends on arbitrarily chosen finite constants c_i. Adding the counterterm Lagrangian to the original lagrangian, we obtain the full Lagrangian

$$\mathcal{L}_0 = \mathcal{L} + \mathcal{L}_{\text{c.t.}} = \tfrac{1}{2}(1+\delta Z)\left(\partial\phi\right)^2 - \tfrac{1}{2}(m^2+\delta m^2)\phi^2 - \tfrac{1}{24}(\lambda+\delta\lambda)\mu^{4-d}\phi^4 \tag{1.3}$$

Renormalized amplitudes are finite as $\epsilon \to 0$, but depend on m and λ, as well as μ and c_i. The choice of c_i is determined by the renormalization scheme. If we choose an on-shell scheme, m and λ will be directly related to physical properties. The c_i will involve $\log\mu$, and the amplitudes will be independent of μ. Alternatively, in an MS or $\overline{MS}$ scheme, the c_i only cancel poles and do not depend on μ. Physical couplings are functions of m, λ, and μ. We can define

$$\phi_0 = (1+\delta Z)^{1/2}\phi, \qquad m_0^2 = (1+\delta Z)^{-1}(m^2+\delta m^2), \qquad \lambda_0 = \mu^{4-d}(1+\delta Z)^{-2}(\lambda+\delta\lambda) \tag{1.4}$$

and then rewrite

$$\mathcal{L}_0 = \tfrac{1}{2}\left(\partial\phi_0\right)^2 - \tfrac{1}{2}m_0^2\phi_0^2 - \tfrac{1}{24}\lambda_0\phi_0^4 \tag{1.5}$$

showing that physical quantities depend only on two parameters m_0 and g_0. This then can be used to derive renormalization group equations.

The propagator and four-point vertex are given by

$$\frac{i}{p^2 - m^2 + i\varepsilon}, \qquad -i\lambda\mu^{4-d} \tag{1.6}$$

with counterterms

$$-i\delta m^2 + i\delta Z\,p^2, \qquad -i\delta\lambda\,\mu^{4-d} \tag{1.7}$$

1.2 Two-point function

We now construct the two-point Green function

$$
\begin{aligned}
\langle T(\phi\phi)\rangle &= \frac{i}{p^2-m^2} + \frac{i}{p^2-m^2}\left(-i\Sigma - i\delta m^2 + i\delta Z\, p^2\right)\frac{i}{p^2-m^2} + \cdots \\
&= \frac{i}{p^2-m^2}\left(1 + \frac{\Sigma + \delta m^2 - \delta Z\, p^2}{p^2-m^2} + \cdots\right) \\
&= \frac{i}{p^2-m^2}\left(1 - \frac{\Sigma + \delta m^2 - \delta Z\, p^2}{p^2-m^2}\right)^{-1} \\
&= \frac{i}{p^2 - m^2 - \Sigma - \delta m^2 + \delta Z\, p^2} \\
&= \frac{i}{\Gamma_2(p^2)} \qquad (1.8)
\end{aligned}
$$

where $-i\Sigma$ is the self energy diagram, and $\Gamma_2(p^2)$ the 1PI two-point function. Expanding

$$\Sigma(p^2) = \Sigma(m_p^2) + \Sigma'(m_p^2)(p^2 - m_p^2) + \mathcal{O}((p^2 - m_p^2)^2) \qquad (1.9)$$

about the physical mass m_p, we find that the denominator of the two-point Green function is

$$(m_p^2 - m^2 - \Sigma(m_p^2) - \delta m^2 + \delta Z\, m_p^2) + (1 - \Sigma'(m_p^2) + \delta Z)(p^2 - m_p^2) + \mathcal{O}((p^2 - m_p^2)^2) \qquad (1.10)$$

The physical mass is determined by the solutions of

$$m_p^2 - m^2 - \Sigma(m_p^2) - \delta m^2 + \delta Z\, m_p^2 = 0 \qquad (1.11)$$

and the two-point function at the pole is given by

$$\langle T(\phi\phi)\rangle = \frac{iR}{p^2 - m_p^2} \qquad (1.12)$$

where the (finite) field renormalization is given by

$$R = \frac{1}{1 - \Sigma'(m_p^2) + \delta Z} \qquad (1.13)$$

Now make the one-loop approximation, setting $m_p^2 = m^2$ in the terms that are already first order, to find

$$m_p^2 = m^2 + \Sigma(m^2) + \delta m^2 - \delta Z\, m^2 \qquad (1.14)$$

$$R = 1 + \Sigma'(m^2) - \delta Z \qquad (1.15)$$

1.3 Self energy

The one-loop self energy is

$$-i\Sigma(p^2, m^2) = \left(-i\lambda\mu^{4-d}\right)\frac{1}{2}\int \frac{d^d\ell}{(2\pi)^d}\frac{i}{\ell^2 - m^2 + i\varepsilon}$$
$$\Sigma(p^2, m^2) = \frac{i\lambda\mu^{4-d}}{2}I_1 = \frac{\lambda m^2\Gamma(\frac{2-d}{2})}{2(4\pi)^{d/2}}\left(\frac{m}{\mu}\right)^{d-4} \tag{1.16}$$

UV divergent for $d = 2, 4, 6, \cdots$. Obviously

$$\frac{\partial\Sigma}{\partial p^2}(p^2, m^2) = 0 \tag{1.17}$$

In the massless limit

$$\Sigma(p^2, 0) = \begin{cases} 0, & \text{for } d > 2 \\ \infty, & \text{for } d \leq 2 \end{cases} \tag{1.18}$$

1.4 Four-point graph

The 1PI four-point function

$$i\Gamma_4 = i\left[-(\lambda + \delta\lambda)\mu^{4-d} + B(s) + B(t) + B(u)\right] \tag{1.19}$$

where

$$iB(s) = \left(-i\lambda\mu^{4-d}\right)^2\frac{1}{2}\int\frac{d^d\ell}{(2\pi)^d}\frac{i}{\ell^2 - m^2 + i\varepsilon}\frac{i}{(\ell + p_1 + p_2)^2 - m^2 + i\varepsilon}$$
$$= \left(\frac{\lambda^2\mu^{8-2d}}{2}\right)I_2(p_1 + p_2) \tag{1.20}$$

so that

$$B(s) = \frac{\lambda^2\mu^{4-d}\Gamma(\frac{4-d}{2})}{2(4\pi)^{d/2}}\left(\frac{m}{\mu}\right)^{d-4}\int_0^1 dx\left[1 - x(1-x)\frac{(p_1 + p_2)^2}{m^2}\right]^{\frac{d-4}{2}} \tag{1.21}$$

This is UV divergent for $d = 4, 6, \cdots$.

In the massless limit,

$$B(s) = \frac{\lambda^2\mu^{4-d}\Gamma(\frac{4-d}{2})}{2(4\pi)^{d/2}}\left[\frac{-(p_1 + p_2)^2}{\mu^2}\right]^{\frac{d-4}{2}}\int_0^1 dx\,[x(1-x)]^{\frac{d-4}{2}}$$
$$= \frac{\lambda^2\mu^{4-d}}{2(4\pi)^{d/2}}\left[\frac{-(p_1 + p_2)^2}{\mu^2}\right]^{\frac{d-4}{2}}\frac{\Gamma(\frac{4-d}{2})\Gamma(\frac{d-2}{2})^2}{\Gamma(d-2)} \tag{1.22}$$

This is IR divergent for $d \leq 2$.

1.5 Four-point scattering amplitude

The sum of all one-loop truncated diagrams is

$$i\Gamma_4 \tag{1.23}$$

The one-loop external leg corrections yield

$$\left(\prod_{i=1}^{4} \frac{iR(m^2)}{p_i^2 - m^2}\right) i\Gamma_4 \tag{1.24}$$

LSZ tells us to multiply this by

$$\prod_{i=1}^{4} \frac{p_i^2 - m^2}{\sqrt{R(m^2)}} \tag{1.25}$$

to obtain the scattering amplitude

$$iR(m)^2\Gamma_4 \tag{1.26}$$

2 Renormalization in four dimensions

2.1 Four-dimensional wavefunction and mass counterterms

Evaluate eq. (1.16) near four dimensions $d = 4 - 2\epsilon$, expanding in ϵ to write

$$\Sigma(p^2, m^2) = -\frac{\lambda m^2}{2(4\pi)^2}\left[\frac{1}{\epsilon} + 1 - \gamma + \log\left(\frac{4\pi\mu^2}{m^2}\right)\right] \tag{2.1}$$

Now we will determine the counterterms. The counterterm δZ is finite

$$\delta Z = -\frac{\lambda}{2(4\pi)^2} c_\phi \tag{2.2}$$

so that the field renormalization $R = 1 + \Sigma'(m^2) - \delta Z$ is

$$\boxed{R = 1 + \frac{\lambda}{2(4\pi)^2} c_\phi} \tag{2.3}$$

Moreover, $m_p^2 - m^2$ must be finite so eq. (1.14) implies that the mass counterterm must satisfy

$$\delta m^2|_{1/\epsilon} = -\Sigma(m^2)|_{1/\epsilon} = \frac{\lambda m^2}{2(4\pi)^2}\left[\frac{1}{\epsilon} + c_m\right] \tag{2.4}$$

Now we can evaluate the physical mass $m_p^2 = m^2 + \Sigma(m^2) + \delta m^2 - \delta Z\, m^2$ namely

$$\boxed{m_p^2 = m^2\left\{1 + \frac{\lambda}{2(4\pi)^2}\left[c_m + c_\phi - 1 + \gamma - \log\left(\frac{4\pi\mu^2}{m^2}\right)\right]\right\}} \tag{2.5}$$

2.2 Renormalized two-point function in four dimensions

Finally, we evaluate the renormalized two-point function in four dimensions

$$\langle T(\phi\phi)\rangle = \frac{i}{p^2 - m^2 - \Sigma - \delta m^2 + \delta Z\, p^2} = \frac{i}{\Gamma_2(p^2)} \tag{2.6}$$

where

$$\Gamma_2(p^2) = p^2 - m^2 + \frac{\lambda}{2(4\pi)^2}\left[\left(-c_m + 1 - \gamma + \log\left[\frac{4\pi\mu^2}{m^2}\right]\right) m^2 - c_\phi\, p^2\right] \tag{2.7}$$

In the $\overline{MS}$ scheme, we have $c_m = -\gamma + \log 4\pi$ and (?) $c_\phi = 0$, so

$$\Gamma_2(p^2) = p^2 - m^2 + \frac{\lambda m^2}{2(4\pi)^2}\left(1 + \log\left[\frac{\mu^2}{m^2}\right]\right) = p^2 - m_p^2 \tag{2.8}$$

On-shell renormalization implies $R = 1$ so $c_\phi = 0$. On-shell renormalization also implies $m_p^2 = m^2$ so

$$c_m + \gamma - \log\left[\frac{4\pi\mu^2}{m^2}\right] = 1 \tag{2.9}$$

so that the inverse propagator with on-shell renormalization is

$$\Gamma_2(p^2) = p^2 - m^2 \tag{2.10}$$

Since $\Sigma(p^2, 0) = 0$, no counterterms are necessary in the massless case and

$$\Gamma_2(p^2) = p^2 \tag{2.11}$$

2.3 Four-point function in four dimensions

Define

$$F \equiv 1 - x(1-x)\frac{p^2}{m^2} \tag{2.12}$$

Recall that

$$B(s) = \frac{\lambda^2 \mu^{4-d}\Gamma(\frac{4-d}{2})}{2(4\pi)^{d/2}}\left(\frac{m}{\mu}\right)^{d-4}\int_0^1 dx\, F(p_1+p_2)^{\frac{d-4}{2}} \tag{2.13}$$

Evaluate this near four dimensions $d = 4 - 2\epsilon$, expanding in ϵ to write

$$\begin{aligned} B(s) &= \frac{\lambda^2\mu^{2\epsilon}}{2(4\pi)^2}\frac{\Gamma(1+\epsilon)}{\epsilon}\left(\frac{4\pi\mu^2}{m^2}\right)^{\epsilon}\int_0^1 dx\ F^{-\epsilon} \\ &= \frac{\lambda^2\mu^{2\epsilon}}{2(4\pi)^2}\left(\frac{1}{\epsilon} - \gamma + \log\left[\frac{4\pi\mu^2}{m^2}\right]\right)\int_0^1 dx\ (1 - \epsilon\log F) + \mathcal{O}(\epsilon) \end{aligned} \tag{2.14}$$

Thus

$$\boxed{B(s) = \frac{\lambda^2\mu^{2\epsilon}}{2(4\pi)^2}\left(\frac{1}{\epsilon} - \gamma + \log\left[\frac{4\pi\mu^2}{m^2}\right] - \int_0^1 dx\ \log F\right) + \mathcal{O}(\epsilon)} \tag{2.15}$$

In the massless limit

$$B(s) = \frac{\lambda^2\mu^{2\epsilon}}{2(4\pi)^2}\left(\frac{1}{\epsilon} - \gamma + 2 + \log\left[\frac{4\pi\mu^2}{-s}\right]\right) + \mathcal{O}(\epsilon) \tag{2.16}$$

Now consider the 1PI four-point function

$$\Gamma_4 = -(\lambda + \delta\lambda)\mu^{2\epsilon} + B(s) + B(t) + B(u) \tag{2.17}$$

Clearly

$$\delta\lambda = \frac{3\lambda^2}{2(4\pi)^2}\left[\frac{1}{\epsilon} + c_\lambda\right] \tag{2.18}$$

Hence

$$\Gamma_4 = -\lambda\mu^{2\epsilon}\left[1+\frac{\lambda}{2(4\pi)^2}\left(3c_\lambda+3\gamma-3\log\left[\frac{4\pi\mu^2}{m^2}\right]+\int_0^1 dx\ \ [\log F(s)+\log F(t)+\log F(u)]\right)\right] \tag{2.19}$$

In the massless limit

$$\Gamma_4 = -\lambda\mu^{2\epsilon}\left[1+\frac{\lambda}{2(4\pi)^2}\left(3c_\lambda+3\gamma-\log\left[\frac{4\pi\mu^2}{-s}\right]-\log\left[\frac{4\pi\mu^2}{-t}\right]-\log\left[\frac{4\pi\mu^2}{-u}\right]\right)\right] \tag{2.20}$$

In the $\bar{M}S$ scheme, $c_\lambda+\gamma-\log 4\pi = 0$ so

$$\Gamma_4 = -\lambda\mu^{2\epsilon}\left[1+\frac{\lambda}{2(4\pi)^2}\left(-\log\left[\frac{\mu^2}{-s}\right]-\log\left[\frac{\mu^2}{-t}\right]-\log\left[\frac{\mu^2}{-u}\right]\right)\right] \tag{2.21}$$