Notes on group theory, v5 (S. Naculich, July 2024)

Pdf file generated on September 1, 2025.

1 Group theory

Definition of a group: A group G is a set of elements g that obeys:

- (Closure) If g, $g' \in G$, then $gg' \in G$.
- (Identity) There exists an element $e \in G$ such that $eg = ge = g$ for any $g \in G$..
- (Inverse) Given $g \in G$, there exists an element $g^{-1} \in G$ such that $g^{-1}g = gg^{-1} = e$.
- (Associativity) $(gg')g'' = g(g'g'')$.

If $gg' = g'g$ for all g, $g' \in G$, then the group is *abelian* (commutative), otherwise *nonabelian.*

Finite (infinite) groups have a finite (infinite) number of elements. A very important finite group is the symmetric group S_n, the group of permutations of n objects. The *order* (number of elements) of this group is $n!$. It is nonabelian for $n \geq 3$.

Infinite groups may be either discrete or continuous. An example of a continous group is U(1), the set of complex numbers z with modulus one, $|z| = 1$. An element of U(1) may be parametrized by $0 \leq \theta < 2\pi$, where $z = e^{i\theta}$. The identity element corresponds to $\theta = 0$, and the inverse of θ is $-\theta$. This group is abelian. The *dimension* of a continuous group is the number of parameters needed to specify an element. Thus, U(1) has dimension 1.

The complete set of simple Lie groups is given by [1,2]

- $A_n = \mathrm{SU}(n+1)$
- $B_n = \mathrm{SO}(2n+1)$
- $C_n = \mathrm{Sp}(2n)$
- $D_n = \mathrm{SO}(2n)$
- F_2
- G_4
- E_6
- E_7
- E_8

2 SO(N), the group of special orthogonal matrices

We denote by O(N) the group of real, $N \times N$, orthogonal matrices

$$R^T R = \mathbb{1}\,. \tag{2.1}$$

(R reminds us that these are rotation matrices in N dimensions.) Verify that such matrices satisfy the definition of a group. If in addition we require these matrices to have unit determinant

$$\det R = 1 \tag{2.2}$$

they are called "special" and the group is called SO(N). Now let r be a real, $N \times N$, antisymmetric (and therefore traceless) matrix

$$r^T = -r\,. \tag{2.3}$$

One may verify that $R = e^r$ is orthogonal. Tracelessness of r implies that R has unit determinant.

Since R is real, it may seem strange to introduce complex numbers, but it is conventional to write

$$R = e^{-iT}, \qquad \text{where } T \text{ is imaginary}, \qquad \text{and} \qquad T^T = -T\,. \tag{2.4}$$

(Note that T is therefore Hermitian. R is real and orthogonal, therefore unitary.)

Consider the set of imaginary, antisymmetric matrices T^{jk}, where $j,\ k \in \{1, \cdots, N\}$, whose matrix elements are given by

$$(T^{jk})^{mn} = -i(\delta^{jm}\delta^{kn} - \delta^{jn}\delta^{km}), \qquad \text{for example} \qquad T^{12} = \begin{pmatrix} 0 & -i & 0 & \cdots \\ i & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{2.5}$$

Using eq. (2.5), one may easily show that

$$\mathrm{Tr}(T^{ij}T^{kl}) = 2(\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk}) \tag{2.6}$$

and also that (be careful not to confuse i the index with $i = \sqrt{-1}$)

$$[T^{ij}, T^{kl}] = i(\delta^{ik}T^{jl} - \delta^{il}T^{jk} - \delta^{jk}T^{il} + \delta^{jl}T^{ik})\,. \tag{2.7}$$

We wish to choose a complete set of linearly independent matrices to be used as a basis. Since $T^{kj} = -T^{jk}$, we may restrict our attention to $j < k$. The number of such matrices is $N(N-1)/2$. Instead of using a double index (jk), let us relabel them in some way (possibly including minus signs) as T^a where $a \in \{1, \cdots, N(N-1)/2\}$. The matrices T^a are referred to as *generators* in the *fundamental representation* of the group. The most general imaginary hermitian matrix can then be written as a linear combination

$$T = \omega^a T^a \tag{2.8}$$

where ω^a are $N(N-1)/2$ arbitrary real parameters, and here and below we adopt the convention that repeated indices are implicitly summed over. An arbitrary element of SO(N) may be written

$$R(\omega) = e^{-i\omega^a T^a} \tag{2.9}$$

so that SO(N) has dimension $N(N-1)/2$.

If $i < j$ and $k < l$, the second term of eq. (2.6) vanishes so we have shown that the generators T^a are orthogonal with respect to the trace norm:

$$\mathrm{Tr}(T^a T^b) = 2\delta^{ab}\,. \tag{2.10}$$

Using eq. (2.5), one may easily show that (sum over a implied)

$$(T^a)^{mn}(T^a)^{pq} = \delta^{np}\delta^{mq} - \delta^{mp}\delta^{nq}\,, \tag{2.11}$$

$$(T^a T^a)^{mn} = (N-1)\delta^{mn} \tag{2.12}$$

which will be used below.

We may freely rescale the fundamental generators with an arbitrary parameter L_f (called the *index* of the fundamental representation):

$$T^a_f = \sqrt{\frac{L_f}{2}}\, T^a \tag{2.13}$$

so that

$$\mathrm{Tr}(T^a_f T^b_f) = L_f \delta^{ab} \tag{2.14}$$

In general, as we will see later, the generators satisfy a set of commutation relations

$$[T^a_f, T^b_f] = ic^{abc} T^c_f \tag{2.15}$$

where c^{abc} is a set of real numbers called *structure constants*. For SO(N), this is evident from eq. (2.7), from which the values of c^{abc} may be inferred.

2.1 SO(3)

For SO(3), we choose our basis of generators to be (L reminds us of orbital angular momentum)

$$L^1 = T^{23}\,, \qquad L^2 = -T^{13}\,, \qquad L^3 = T^{12} \tag{2.16}$$

which can be expressed in terms of the Levi-Civita symbol ϵ^{abc} as

$$L^a = \tfrac{1}{2}\epsilon^{aij}T^{ij} \tag{2.17}$$

with sums over i and j (from 1 to 3) implied. Using eq. (2.5), this becomes

$$(L^a)^{mn} = i\epsilon^{man} \tag{2.18}$$

or explicitly

$$L^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}\,, \qquad L^2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}\,, \qquad L^3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,. \tag{2.19}$$

Using the identity

$$\epsilon^{ijm}\epsilon^{klm} = \delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk} \tag{2.20}$$

one can show that the SO(3) generators satisfy the commutation relations

$$[L^a, L^b] = i\epsilon^{abc}L^c\,. \tag{2.21}$$

Thus, writing $T^a_f = \sqrt{L_f/2}\; L^a$, we see that the structure constants for SO(3) are

$$c^{abc} = \sqrt{\frac{L_f}{2}}\;\epsilon^{abc}\,. \tag{2.22}$$

Consider a similarity transformation with the unitary matrix

$$L'^a = U^{-1}L^aU \qquad \text{where} \qquad U = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 0 & 1 \\ -i & 0 & -i \\ 0 & \sqrt{2} & 0 \end{pmatrix} \tag{2.23}$$

which preserves the commutation relations (2.21) as well as $\mathrm{tr}(L^aL^b) = 2\delta^{ab}$. In this basis, we have

$$L'^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\,, \qquad L'^2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\,, \qquad L'^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{2.24}$$

which are standard in quantum mechanics.

3 SU(N), the group of special unitary matrices

We denote by SU(N) the group of $N \times N$, unitary matrices with unit determinant:

$$U^\dagger U = \mathbb{1}, \qquad \det U = 1\,. \tag{3.1}$$

Such a matrix can be written as

$$U = e^{-iT}, \qquad \text{where} \quad T^\dagger = T \qquad \text{and} \quad \mathrm{Tr}(T) = 0 \tag{3.2}$$

that is, T is a traceless Hermitian matrix.

Consider the basis of traceless Hermitian matrices (e.g., see p. 148 of ref. [3]) with matrix elements

$$\begin{aligned}
(T_+^{jk})^{mn} &= \delta^{jm}\delta^{kn} + \delta^{jn}\delta^{km}\,, \qquad j < k \\
(T_-^{jk})^{mn} &= -i(\delta^{jm}\delta^{kn} - \delta^{jn}\delta^{km})\,, \qquad j < k \\
(T_D^{j})^{mn} &= \sqrt{\frac{2}{j(j+1)}}\delta^{mn} \times \begin{cases} 1, & \text{if } m \le j \\ -j, & \text{if } m = j+1 \\ 0, & \text{if } m > j+1 \end{cases}, \qquad 1 \le j \le N-1
\end{aligned} \tag{3.3}$$

Let us relabel these matrices as T^a where $a \in \{1, \cdots, N^2-1\}$. The most general traceless hermitian matrix can then be written as a linear combination of these generators

$$T = \omega^a T^a \tag{3.4}$$

where ω^a are $N^2 - 1$ arbitrary real parameters. An arbitary element of SU(N) may be written

$$U(\omega) = e^{-i\omega^a T^a} \tag{3.5}$$

so SU(N) has dimension $N^2 - 1$.

With a little bit of work one may show that the generators satisfy

$$\mathrm{Tr}(T^a T^b) = 2\delta^{ab}\,. \tag{3.6}$$

With rather more work one may show that

$$(T^a)^{mn}(T^a)^{pq} = 2\left(\delta^{np}\delta^{mq} - \frac{1}{N}\delta^{mn}\delta^{pq}\right)\,, \tag{3.7}$$

$$(T^a T^a)^{mn} = 2\left(N - \frac{1}{N}\right)\delta^{mn}\,. \tag{3.8}$$

As before, we may freely rescale the fundamental generators $T_f^a = \sqrt{\frac{L_f}{2}}\, T^a$ so that

$$\mathrm{Tr}(T_f^a T_f^b) = L_f \delta^{ab}\,, \tag{3.9}$$

$$[T_f^a, T_f^b] = i c^{abc} T_f^c \tag{3.10}$$

where c^{abc} are the structure constants of SU(N). It is conventional to choose $L_f = \frac{1}{2}$ for SU(N).

3.1 SU(2)

For SU(2), the generators in the defining representation are simply the Pauli spin matrices σ^a:

$$\begin{aligned} \sigma^1 &= T_+^{12}\,, \\ \sigma^2 &= T_-^{12}\,, \\ \sigma^3 &= T_D^1\,. \end{aligned} \tag{3.11}$$

The Pauli spin matrices are hermitian and traceless, and obey the nice relations

$$\sigma^a\sigma^b = \delta^{ab}\mathbb{1} + i\epsilon^{abc}\sigma^c\,. \tag{3.12}$$

From this we deduce

$$\mathrm{Tr}(\sigma^a\sigma^b) = 2\delta^{ab}\,, \tag{3.13}$$

$$[\sigma^a, \sigma^b] = 2i\epsilon^{abc}\sigma^c\,. \tag{3.14}$$

Thus, writing $T_f^a = \sqrt{L_f/2}\;\sigma^a$, we see that the structure constants for SU(2) are

$$c^{abc} = \sqrt{2L_f}\;\epsilon^{abc}\,. \tag{3.15}$$

It is conventional to choose $L_f = \frac{1}{2}$ and define $J^a = \frac{1}{2}\sigma^a$ for SU(2), so that

$$\mathrm{Tr}(J^aJ^b) = \tfrac{1}{2}\delta^{ab}\,, \tag{3.16}$$

$$[J^a, J^b] = i\epsilon^{abc}J^c\,. \tag{3.17}$$

Thus, if we choose $L_f = 2$ for SO(3) and $L_f = \frac{1}{2}$ for SU(2), the Lie algebras are the same, with structure constants $c^{abc} = \epsilon^{abc}$.

3.2 SU(3)

For SU(3), the generators in the defining representation are the Gell-Mann matrices λ^a:

$$\begin{array}{lll} \lambda^1 = T_+^{12} & \lambda^4 = T_+^{13} & \lambda^6 = T_+^{23} \\ \lambda^2 = T_-^{12} & \lambda^5 = T_-^{13} & \lambda^7 = T_-^{23} \\ \lambda^3 = T_D^1 & & \lambda^8 = T_D^2 \end{array} \tag{3.18}$$

which we recall obey $\mathrm{Tr}(\lambda^a\lambda^b) = 2\delta^{ab}$. The matrices obey

$$[\lambda^a, \lambda^b] = 2if^{abc}\lambda^c \tag{3.19}$$

where explicit values for f^{abc} may be found, e.g., on p. 517 of ref. [4]. Writing $T_f^a = \sqrt{L_f/2}\;\lambda^a$, we see that the structure constants for SU(3) are

$$c^{abc} = \sqrt{2L_f}\;f^{abc}\,. \tag{3.20}$$

Again, it is conventional to choose $L_f = \frac{1}{2}$ and define $t^a = \frac{1}{2}\lambda^a$ so that

$$\mathrm{Tr}(t^at^b) = \tfrac{1}{2}\delta^{ab}\,, \tag{3.21}$$

$$[t^a, t^b] = if^{abc}t^c\,. \tag{3.22}$$

4 Representations

4.1 Defining representation

In the previous sections, we have discussed the defining (or fundamental) representations of SO(N) and SU(N), which are $N \times N$ matrices U which act on an N-dimensional vector Φ as

$$\Phi' = U(\omega)\Phi \tag{4.1}$$

or in terms of components

$$\Phi'^m = U(\omega)^{mn}\Phi^n \tag{4.2}$$

where m, $n \in \{1, \cdots, N\}$. These (special, unitary) matrices can be expressed as

$$U(\omega) = e^{-i\omega^a T_f^a} \tag{4.3}$$

where T_f^a are the (Hermitian, traceless) generators in the fundamental representation, chosen to obey

$$\mathrm{Tr}(T_f^a T_f^b) = L_f \delta^{ab} \tag{4.4}$$

and ω^a are $\dim G$ real parameters. For group elements near the identity (that is, infinitesimal ω), we have

$$\delta\Phi^m \equiv \Phi'^m - \Phi^m = -i\omega^a (T_f^a)^{mn}\Phi^n\,. \tag{4.5}$$

Now since the T_f^a are Hermitian, traceless matrices (and, for SO(N), imaginary), it follows that the commutator $i[T_f^a, T_f^b]$ is also Hermitian and traceless (and, for SO(N), imaginary), and so can be expressed as a real linear combination of all of the generators: $i[T_f^a, T_f^b] = \omega^c T_f^c$. We express this by writing

$$[T_f^a, T_f^b] = i c^{abc} T_f^c \tag{4.6}$$

for some set of real constants c^{abc}, called structure constants. We describe this by saying that the generators satisfy a Lie algebra. Note that c^{abc} is manifestly antisymmetric in the first two indices. If in addition we use eq. (4.4), one finds

$$c^{abc} = -\frac{i}{L_f}\,\mathrm{Tr}([T_f^a, T_f^b]T_f^c) \tag{4.7}$$

from which one easily verifies that the structure constants c^{abc} are antisymmetric in all indices. As we saw in the previous sections

$$c^{abc} = \begin{cases} \sqrt{2L_f}\, f^{abc}, & \text{for SU}(N) \text{ and Sp}(N) \\ \sqrt{\frac{L_f}{2}}\, f^{abc}, & \text{for SO}(N) \end{cases} \tag{4.8}$$

where $f^{abc} = \epsilon^{abc}$ for SO(3), SU(2), and Sp(2), and the Gell-Mann structure constants for SU(3).

Using the Jacobi identify together with eq. (4.6), one obtains

$$\begin{aligned} 0 &= [[T_f^a, T_f^b], T_f^d] + [[T_f^b, T_f^d], T_f^a] + [[T_f^d, T_f^a], T_f^b] \\ &= -\left(c^{abc}c^{cde} + c^{bdc}c^{cae} + c^{dac}c^{cbe}\right) T_f^e \end{aligned} \tag{4.9}$$

so that the structure constants obey

$$c^{abc}c^{cde} + c^{bdc}c^{cae} + c^{dac}c^{cbe} = 0\,. \tag{4.10}$$

Now let's see how the commutator arises through group multiplication. Let $\omega_1^a = \epsilon\delta^{aa_1}$ and $\omega_2^a = \epsilon\delta^{aa_2}$. Observe that

$$\begin{aligned} U^{-1}(\omega_1)U^{-1}(\omega_2)U(\omega_1)U(\omega_2) &= 1 - \epsilon^2[T_f^{a_1}, T_f^{a_2}] + \mathcal{O}(\epsilon^3) \\ &= 1 - \epsilon^2 i c^{a_1a_2a}T_f^a + \mathcal{O}(\epsilon^3) \end{aligned} \tag{4.11}$$

But a product of group elements is another group element

$$U^{-1}(\omega_1)U^{-1}(\omega_2)U(\omega_1)U(\omega_2) = U(\omega) = e^{-i\omega^a T_f^a} \tag{4.12}$$

from which we see that

$$\omega^a = \epsilon^2 c^{a_1a_2a} + \mathcal{O}(\epsilon^3)\,. \tag{4.13}$$

4.2 Other representations

Lie groups also have other (unitary) representations R, which are $r \times r$ matrices $D_R(\omega)$ (where $r = \dim R$) that obey the same group multiplication laws as the defining representation

$$U(\omega_1)U(\omega_2) = U(\omega_3) \longrightarrow D_R(\omega_1)D_R(\omega_2) = D_R(\omega_3) \tag{4.14}$$

These matrices act on an r-dimensional vector V as

$$V' = D_R(\omega)V \tag{4.15}$$

or in terms of components

$$V'^i = D_R(\omega)^{ij}V^j \tag{4.16}$$

where i, $j \in \{1, \cdots, r\}$. These (unitary) representations are parametrized as

$$D_R(\omega) = e^{-i\omega^a T_R^a} \tag{4.17}$$

where T_R^a are the (Hermitian) generators in the R representation. For group elements near the identity, we have

$$\delta V^i \equiv V'^i - V^i = -i\omega^a (T_R^a)^{ij} V^j \tag{4.18}$$

Repeating the calculation above, we have

$$D_R^{-1}(\omega_1)D_R^{-1}(\omega_2)D_R(\omega_1)D_R(\omega_2) = 1 - \epsilon^2[T_R^{a_1}, T_R^{a_2}] + \mathcal{O}(\epsilon^3) \tag{4.19}$$

But using eq. (4.13), we have

$$\begin{aligned} D_R^{-1}(\omega_1)D_R^{-1}(\omega_2)D_R(\omega_1)D_R(\omega_2) &= D_R(\omega) = 1 - i\omega^a T_R^a + \mathcal{O}(\omega^2) \\ &= 1 - i\epsilon^2 c^{a_1 a_2 a} T_R^a + \mathcal{O}(\epsilon^3) \end{aligned} \tag{4.20}$$

Comparing these two equations, it follows that

$$[T_R^a, T_R^b] = ic^{abc} T_R^c \tag{4.21}$$

with exactly the same structure constants as T_f^a.

Furthermore, if T_f^a are chosen to be orthonormal (up to a constant L_f), then so will be T_R^a (cf. p. 498 of Peskin and Schroeder [5])

$$\mathrm{Tr}(T_R^a T_R^b) = L_R \delta^{ab} \tag{4.22}$$

but with an index L_R that depends on the representation and is some multiple of L_f. Combining eqs. (4.21) and (4.22), we obtain

$$c^{abc} = -\frac{i}{L_R}\,\mathrm{Tr}([T_R^a, T_R^b]T_R^c) \tag{4.23}$$

4.3 Adjoint representation

We now define the generators in the *adjoint* representation by

$$\left(T_{\mathrm{adj}}^a\right)^{de} = ic^{dae}\,. \tag{4.24}$$

where a, d, $e \in \{1, \cdots, \dim G\}$. Note that these are purely imaginary, antisymmetric matrices (and therefore Hermitian, of course). One verifies using eq. (4.10) and the antisymmetry of c^{abc} in the first two indices that

$$\left([T_{\mathrm{adj}}^a, T_{\mathrm{adj}}^b]\right)^{de} = -c^{dac}c^{cbe} + c^{dbc}c^{cae} = -c^{abc}c^{dce} = ic^{abc}\left(T_{\mathrm{adj}}^c\right)^{de} \tag{4.25}$$

verifying that the matrices (4.24) obey eq. (4.21). The index of the adjoint representation is

$$L_{\mathrm{adj}}\delta^{ab} = \mathrm{Tr}(T_{\mathrm{adj}}^a T_{\mathrm{adj}}^b) = c^{acd}c^{bcd}\,. \tag{4.26}$$

The adjoint representation matrices

$$D_{\mathrm{adj}}(\omega) = e^{-i\omega^a T_{\mathrm{adj}}^a} \tag{4.27}$$

act on a $\dim G$-dimensional vector with components ϕ^b as

$$\phi'^b = D_{\rm adj}(\omega)^{bc}\phi^c \tag{4.28}$$

For group elements near the identity, we have

$$\delta\phi^b = -i\omega^a (T^a_{\rm adj})^{bc}\phi^c = c^{bac}\omega^a\phi^c \tag{4.29}$$

Instead of regarding ϕ^b as components of a $\dim G$-dimensional vector, we can use them to construct the $\dim G \times \dim G$ hermitian matrix

$$\phi \equiv \phi^a T^a_f \tag{4.30}$$

We will demonstrate that ϕ transforms as

$$\phi' = U(\omega)\,\phi\, U^{-1}(\omega) \tag{4.31}$$

Under a infinitesimal group element

$$\begin{aligned} \phi' &= e^{-i\omega^a T^a_f}\, \phi^c T^c_f\, e^{i\omega^b T^b_f} \\ &= \phi - i\omega^a\phi^c [T^a_f, T^c_f] \\ &= \phi + \omega^a \phi^c c^{bac} T^b_f \end{aligned} \tag{4.32}$$

that is

$$\delta\phi = \phi' - \phi = c^{bac}\omega^a\phi^c T^b_f \tag{4.33}$$

precisely in agreement with eq. (4.29).

4.4 Representations of SU(2)

In quantum mechanics, we derive the spin-j representation matrices of SU(2) in a basis in which J^3 is diagonal:

$$(J^3)^{mm'} = m\delta^{mm'}, \qquad m, m' = j, \cdots, -j \tag{4.34}$$

The index of the spin$-j$ representation is

$$L_j = {\rm tr}(J^3 J^3) = \frac{j(j+1)(2j+1)}{3} \tag{4.35}$$

so $L_{1/2} = \frac{1}{2}$, $L_1 = 2$, $L_{3/2} = 5$, $L_2 = 10$, etc. We also have

$$(J^1)^2 + (J^2)^2 + (J^3)^2 = j(j+1)\mathbb{1} \tag{4.36}$$

so that the quadratic Casimir of the spin-j representation is

$$C_j = 2j(j+1) \tag{4.37}$$

that is $C_{1/2} = \frac{3}{2}$, $C_1 = 4$, $C_{3/2} = \frac{15}{2}$, $C_2 = 6$, etc.

4.5 Complex conjugate representations

Given a representation R with matrices $D_R(\omega)$, define the conjugate representation $\overline{R}$ with matrices

$$D_{\overline{R}}(\omega) = D_R^*(\omega) \tag{4.38}$$

which is also a representation since

$$D_R(\omega_1)D_R(\omega_2) = D_R(\omega_3) \qquad \Longrightarrow \qquad D_R^*(\omega_1)D_R^*(\omega_2) = D_R^*(\omega_3) \tag{4.39}$$

The generators of the conjugate representation

$$D_{\overline{R}}(\omega) = e^{-i\omega^a T^a_{\overline{R}}} \tag{4.40}$$

are given by

$$T^a_{\overline{R}} = -(T^a_R)^* \tag{4.41}$$

In general, the representation $\overline{R}$ is distinct from R, unless it is a real or pseudo-real representation.

If the generators T^a_R are all imaginary,

$$(T^a_R)^* = -T^a_R \tag{4.42}$$

then

$$T^a_{\overline{R}} = T^a_R \qquad \Longrightarrow \qquad D_{\overline{R}}(\omega) = D_R(\omega) \tag{4.43}$$

and so R is self-conjugate, or real.

If instead

$$(T^a_R)^* = -S^{-1}T^a_R S \tag{4.44}$$

for some matrix S, then

$$T^a_{\overline{R}} = S^{-1}T^a_R S \qquad \Longrightarrow \qquad D_{\overline{R}}(\omega) = S^{-1}D_R(\omega)S \tag{4.45}$$

meaning that $\overline{R}$ is equivalent (by similarity transformation) to R. In that case, R is called pseudo-real.

The adjoint representation is always real. The fundamental representation of SO(N) is real. The fundamental representation of SU(N) is pseudo-real only for $N = 2$, otherwise neither real nor pseudo-real.

5 Quadratic Casimir

The Casimir C_R of the representation R is defined as (implied sum over a)

$$(T^a_R T^a_R)^{ij} = L_f C_R \delta^{ij} \tag{5.1}$$

Combining eqs. (4.22) and (5.1) we obtain

$$C_R = \frac{L_R}{L_f}\frac{\dim G}{\dim R} \qquad C_f = \frac{\dim G}{\dim f} \qquad C_{\rm adj} = \frac{L_{\rm adj}}{L_f} \tag{5.2}$$

The Casimir of the adjoint representation is

$$L_f C_{\rm adj} \delta^{ab} = (T^c_{\rm adj} T^c_{\rm adj})^{ab} = c^{acd} c^{bcd} \tag{5.3}$$

This together with eq. (4.26) is consistent with eq. (5.2). Using eqs. (4.6) and (4.7) we have

$$\begin{aligned} c^{acd}c^{bcd} &= -\frac{i}{L_f}\,\mathrm{Tr}([T^a_f, T^c_f]T^d_f)c^{bcd} = \frac{1}{L_f}\,\mathrm{Tr}([T^a_f,[T^b_f,T^d_f]T^d_f) \\ &= \frac{1}{L_f}\Big[\mathrm{Tr}(T^a_f T^b_f T^d_f T^d_f) - \mathrm{Tr}(T^a_f T^d_f T^b_f T^d_f) - \mathrm{Tr}(T^b_f T^d_f T^a_f T^d_f) + \mathrm{Tr}(T^d_f T^b_f T^a_f T^d_f)\Big] \\ &= 2L_f C_f \delta^{ab} - \frac{2}{L_f}\,\mathrm{Tr}(T^a_f T^d_f T^b_f T^d_f) \end{aligned} \tag{5.4}$$

We will use this below to evaluate $C_{\rm adj}$ for various groups. Let's also evaluate

$$c^{dae}c^{ebf}c^{fcd} = -c^{dca}c^{ebf}c^{fed} - c^{dec}c^{ebf}c^{fad} = -L_f C_{\rm adj} c^{dca}\delta^{bd} - c^{ecd}c^{fbe}c^{daf} \tag{5.5}$$

which implies

$$c^{dae}c^{ebf}c^{fcd} = -\tfrac{1}{2}L_f C_{\rm adj} c^{abc} \tag{5.6}$$

Alternatively we have

$$\begin{aligned} c^{dae}c^{ebf}c^{fcd} &= -\frac{i}{L_f}\,\mathrm{Tr}([T^d_f,T^a_f]T^e_f)c^{ebf}c^{fcd} = -\frac{1}{L_f}\,\mathrm{Tr}([[T^f_f,T^c_f],T^a_f]T^e_f)c^{ebf} \\ &= \frac{i}{L_f}\,\mathrm{Tr}([[[T^e_f,T^b_f],T^c_f],T^a_f]T^e_f) \\ &= \frac{i}{L_f}\Big[\mathrm{Tr}(ebcae) - \mathrm{Tr}(becae) - \mathrm{Tr}(cebae) + \mathrm{Tr}(cbeae) \\ &\qquad - \mathrm{Tr}(aebce) + \mathrm{Tr}(abece) + \mathrm{Tr}(acebe) - \mathrm{Tr}(acbee)\Big] \end{aligned} \tag{5.7}$$

Consider the tensor product of representations $T^a_{R_1\otimes R_2} = T^a_{R_1}\otimes \mathbb{1} + \mathbb{1}\otimes T^a_{R_2}$. The Casimir is then

$$T^a_{R_1\otimes R_2}T^a_{R_1\otimes R_2} = T^a_{R_1}T^a_{R_1}\otimes\mathbb{1} + 2T^a_{R_1}\otimes T^a_{R_2} + \mathbb{1}\otimes T^a_{R_2}T^a_{R_2} \tag{5.8}$$

If the generators are traceless, we have

$$\begin{aligned}\operatorname{tr}\left(T^a_{R_1\otimes R_2}T^a_{R_1\otimes R_2}\right) &= \operatorname{tr}\left(T^a_{R_1}T^a_{R_1}\right)\dim R_2 + \dim R_1 \operatorname{tr}\left(T^a_{R_2}T^a_{R_2}\right) \\ &= L_f(C_{R_1}+C_{R_2})\dim R_1 \dim R_2\end{aligned} \tag{5.9}$$

But since $T^a_{R_1\otimes R_2} = \oplus_i T^a_{R_i}$ we also have

$$\operatorname{tr}\left(T^a_{R_1\otimes R_2}T^a_{R_1\otimes R_2}\right) = \sum_i L_f C_{R_i}\dim R_i \tag{5.10}$$

Consequently

$$(C_{R_1}+C_{R_2})\dim R_1 \dim R_2 = \sum_i C_{R_i}\dim R_i \tag{5.11}$$

5.1 SU(N)

For SU(N), we have $\dim f = N$ and $\dim G = N^2 - 1$, so

$$C_f = \frac{\dim G}{\dim f} = N - \frac{1}{N}, \tag{5.12}$$

Moreover, the fundamental generators obey eq. (3.7)

$$(T_f^a)^{mn}(T_f^a)^{pq} = L_f\left(\delta^{np}\delta^{mq} - \frac{1}{N}\delta^{mn}\delta^{pq}\right) \qquad \Longrightarrow \qquad C_f = N - \frac{1}{N} \tag{5.13}$$

For P and Q any product of generators, this implies

$$\begin{aligned} \mathrm{Tr}(PT^a)\,\mathrm{Tr}(QT^a) &= L_f\left[\mathrm{Tr}(PQ) - \frac{1}{N}\,\mathrm{Tr}(P)\,\mathrm{Tr}(Q)\right] \\ \mathrm{Tr}(PT^aQT^a) &= L_f\left[\mathrm{Tr}(P)\,\mathrm{Tr}(Q) - \frac{1}{N}\,\mathrm{Tr}(PQ)\right] \end{aligned} \tag{5.14}$$

Since $f \otimes \bar{f} = 1 \oplus (\mathrm{adj})$, by eq. (5.11) we have $C_{\mathrm{adj}} \dim G = 2C_f(\dim f)^2 = 2N(N^2-1)$ hence

$$C_{\mathrm{adj}} = 2N \tag{5.15}$$

and

$$c^{acd}c^{bcd} = L_f C_{\mathrm{adj}}\delta^{ab} = 2L_f N\delta^{ab} \tag{5.16}$$

Alternatively, we can use eqs. (5.4) and (5.14) to show

$$c^{acd}c^{bcd} = 2L_fC_f\delta^{ab} - \frac{2}{L_f}\,\mathrm{Tr}(T_f^aT_f^dT_f^bT_f^d) = 2L_f\left(N - \frac{1}{N}\right)\delta^{ab} + \frac{2}{N}L_f\delta^{ab} = 2L_fN\delta^{ab} \tag{5.17}$$

Since for SU(N), we have $c^{abc} = \sqrt{2L_f}f^{abc}$, this implies

$$f^{acd}f^{bcd} = N\delta^{ab} \tag{5.18}$$

Let's use eqs. (5.7) and (5.14) to evaluate

$$c^{dae}c^{ebf}c^{fcd} = iN\Big[\,\mathrm{Tr}(abc) - \mathrm{Tr}(cba)\Big] = -L_fNc^{abc} \tag{5.19}$$

which is also consistent with eq. (5.6).

5.2 SO(N)

For SO(N), we have $\dim G = N(N-1)/2$ and $\dim f = N$, so

$$C_f = \frac{\dim G}{\dim f} = \tfrac{1}{2}(N-1). \tag{5.20}$$

Generators in the defining represention of SO(N) are hermitian $N \times N$ matrices that satisfy

$$(T^a)^T = -T^a \tag{5.21}$$

This also implies that they are imaginary and traceless. They obey eq. (2.11) (see also ref. [6])

$$(T_f^a)^{mn}(T_f^a)^{pq} = \frac{L_f}{2}\left(\delta^{np}\delta^{mq} - \delta^{mp}\delta^{pq}\right) \qquad \Longrightarrow \qquad C_f = \frac{N-1}{2} \tag{5.22}$$

For P and Q any product of generators, this implies

$$\begin{aligned} \mathrm{Tr}(PT^a)\,\mathrm{Tr}(QT^a) &= \frac{L_f}{2}\Big[\mathrm{Tr}(PQ) - \mathrm{Tr}(PQ^T)\Big] \\ \mathrm{Tr}(PT^aQT^a) &= \frac{L_f}{2}\Big[\mathrm{Tr}(P)\,\mathrm{Tr}(Q) - \mathrm{Tr}(PQ^T)\Big] \end{aligned} \tag{5.23}$$

From eq. (5.21), we have

$$Q^T = (-1)^{n_Q} Q^R \tag{5.24}$$

where Q^R denotes the product of generators Q in reverse order, and n_Q denotes the number of factors in Q, so we can recast eq. (5.23) as [7]

$$\begin{aligned} \mathrm{Tr}(PT^a)\,\mathrm{Tr}(QT^a) &= \frac{L_f}{2}\Big[\mathrm{Tr}(PQ) - (-1)^{n_Q}\,\mathrm{Tr}(PQ^R)\Big] \\ \mathrm{Tr}(PT^aQT^a) &= \frac{L_f}{2}\Big[\mathrm{Tr}(P)\,\mathrm{Tr}(Q) - (-1)^{n_Q}\,\mathrm{Tr}(PQ^R)\Big] \end{aligned} \tag{5.25}$$

We can use eqs. (5.4) and (5.25) to show that

$$\begin{aligned} c^{acd}c^{bcd} &= 2L_f C_f \delta^{ab} - \frac{2}{L_f}\,\mathrm{Tr}(T_f^a T_f^d T_f^b T_f^d) = 2L_f\left(\frac{N-1}{2}\right)\delta^{ab} - L_f\delta^{ab} \\ &= L_f(N-2)\delta^{ab} \end{aligned} \tag{5.26}$$

Since $c^{acd}c^{bcd} = L_f C_{\mathrm{adj}}\delta^{ab}$, we have

$$C_{\mathrm{adj}} = N-2 \tag{5.27}$$

Since for SO(N), we have $c^{abc} = \sqrt{\frac{L_f}{2}}\,f^{abc}$, this implies

$$f^{acd}f^{bcd} = 2(N-2)\delta^{ab} \tag{5.28}$$

Let's use eqs. (5.7) and (5.25) to evaluate

$$c^{dae}c^{ebf}c^{fcd} = \frac{i}{2}(N-2)\Big[\mathrm{Tr}(abc) - \mathrm{Tr}(cba)\Big] = -L_f\frac{N-2}{2}c^{abc} \tag{5.29}$$

which is also consistent with eq. (5.6).

5.3 Sp(N)

For Sp(N), with N even, we have $\dim G = N(N+1)/2$ and $\dim f = N$, so

$$C_f = \frac{\dim G}{\dim f} = \tfrac{1}{2}(N+1). \tag{5.30}$$

Generators in the defining represention of Sp(N) are hermitian $N \times N$ matrices that satisfy

$$(T^a)^T = JT^aJ \tag{5.31}$$

where $J^2 = -1$ and $J^T = -J$. This implies that they are traceless. They also obey [6]

$$(T_f^a)^{mn}(T_f^a)^{pq} = \frac{L_f}{2}\left(\delta^{np}\delta^{mq} - J^{mp}J^{pq}\right) \qquad \Longrightarrow \qquad C_f = \frac{N+1}{2} \tag{5.32}$$

For P and Q any product of generators, this implies

$$\begin{aligned} \mathrm{Tr}(PT^a)\,\mathrm{Tr}(QT^a) &= \frac{L_f}{2}\Big[\mathrm{Tr}(PQ) + \mathrm{Tr}(PJQ^TJ)\Big] \\ \mathrm{Tr}(PT^aQT^a) &= \frac{L_f}{2}\Big[\mathrm{Tr}(P)\,\mathrm{Tr}(Q) - \mathrm{Tr}(PJQ^TJ)\Big] \end{aligned} \tag{5.33}$$

From eq. (5.31), we have

$$Q^T = (-1)^{n_Q-1}JQ^RJ \tag{5.34}$$

so we can recast eq. (5.33) as [7]

$$\begin{aligned} \mathrm{Tr}(PT^a)\,\mathrm{Tr}(QT^a) &= \frac{L_f}{2}\Big[\mathrm{Tr}(PQ) - (-1)^{n_Q}\,\mathrm{Tr}(PQ^R)\Big] \\ \mathrm{Tr}(PT^aQT^a) &= \frac{L_f}{2}\Big[\mathrm{Tr}(P)\,\mathrm{Tr}(Q) + (-1)^{n_Q}\,\mathrm{Tr}(PQ^R)\Big] \end{aligned} \tag{5.35}$$

We can use eqs. (5.4) and (5.35) to show that

$$\begin{aligned} c^{acd}c^{bcd} &= 2L_fC_f\delta^{ab} - \frac{2}{L_f}\,\mathrm{Tr}(T_f^aT_f^dT_f^bT_f^d) = 2L_f\left(\frac{N+1}{2}\right)\delta^{ab} + L_f\delta^{ab} \\ &= L_f(N+2)\delta^{ab} \end{aligned} \tag{5.36}$$

Since $c^{acd}c^{bcd} = L_fC_{\mathrm{adj}}\delta^{ab}$, we have

$$C_{\mathrm{adj}} = N+2 \tag{5.37}$$

Since for Sp(N), we have $c^{abc} = \sqrt{2L_f}f^{abc}$, this implies

$$f^{acd}f^{bcd} = \tfrac{1}{2}(N+2)\delta^{ab} \tag{5.38}$$

Let's use eqs. (5.7) and (5.35) to evaluate

$$c^{dae}c^{ebf}c^{fcd} = \frac{i}{2}(N+2)\Big[\mathrm{Tr}(abc) - \mathrm{Tr}(cba)\Big] = -L_f\frac{N+2}{2}c^{abc} \tag{5.39}$$

which is also consistent with eq. (5.6).

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