Notes on gauge theory (S. Naculich, July 2024)

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1 Electromagnetism

Reference: see chapter 10 of Griffiths [1]. Maxwell's equations are

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \times \mathbf{B} - \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{j}$$
$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{1.1}$$

where $\epsilon_0 \mu_0 = 1/c^2$. Show that these imply

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0 \tag{1.2}$$

The electric and magnetic fields may be expressed in terms of scalar (V) and vector ($\mathbf{A}$) potentials

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \qquad \mathbf{B} = \nabla \times \mathbf{A} \tag{1.3}$$

Show that these automatically satisfy the bottom two Maxwell equations. Henceforth, we will use Heaviside-Lorentz units, setting $\epsilon_0 = \mu_0 = 1$ and thus $c = 1$. Also, we will use relativistic notation, employing the mostly-minus, "West coast," particle physics convention, and not the mostly-plus, "East coast," relativity convention, for the Minkowski metric

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1.4}$$

The four-current

$$j^\mu = (\rho, \mathbf{j}) \tag{1.5}$$

obeys

$$\partial_\mu j^\mu = 0 \tag{1.6}$$

The Faraday tensor is a second-rank antisymmetric tensor defined by

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix} \tag{1.7}$$

We also define the dual Faraday tensor

$$\tilde{F}^{\mu\nu} = \tfrac{1}{2}\epsilon^{\mu\nu\kappa\lambda}F_{\kappa\lambda} \tag{1.8}$$

where $\epsilon^{\mu\nu\kappa\lambda}$ is the totally antisymmetric Levi-Civita tensor, and our convention is $\epsilon^{0123} = 1$. Show that

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -B^1 & -B^2 & -B^3 \\ B^1 & 0 & E^3 & -E^2 \\ B^2 & -E^3 & 0 & E^1 \\ B^3 & E^2 & -E^1 & 0 \end{pmatrix} \tag{1.9}$$

Show that the Maxwell equations may be expressed as

$$\partial_\mu F^{\mu\nu} = j^\nu \qquad\qquad \partial_\mu \tilde{F}^{\mu\nu} = 0 \tag{1.10}$$

Define the four-vector potential

$$A^\mu = (V, \mathbf{A}) \tag{1.11}$$

Show that the Faraday tensor is given by

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{1.12}$$

As before, show that this automatically satisfies $\partial_\mu \tilde{F}^{\mu\nu} = 0$.

Define a **gauge transformation** on A^μ as

$$A^\mu \to A'^\mu = A^\mu + \partial^\mu \chi \tag{1.13}$$

Show that the Faraday tensor is unchanged by eq. (1.13)

$$F_{\mu\nu} \to F'_{\mu\nu} = F_{\mu\nu} \tag{1.14}$$

The Lagrangian for electromagnetism

$$\mathcal{L}_{EM} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - j^\mu A_\mu \tag{1.15}$$

is gauge invariant provided eq. (1.6) holds. Show that Maxwell's equations may be derived using the Euler-Lagrange equations (see, e.g., Carroll [2]).

One may use gauge invariance to enforce the Lorenz gauge condition

$$\partial_\mu A^\mu = 0 \tag{1.16}$$

Show that Maxwell equations in free space ($j^\mu = 0$) become

$$\partial^2 A^\mu = 0 \tag{1.17}$$

with free wave solution

$$A^\mu = \epsilon^\mu e^{ik\cdot x}, \qquad k^2 = 0, \qquad \epsilon\cdot k = 0 \tag{1.18}$$

2 Minimal coupling prescription

The Lagrangian for a complex scalar field ϕ is

$$\mathcal{L}_{\partial\phi} = (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi - V(\phi^* \phi) \tag{2.1}$$

Consider a phase transformation

$$\phi \to \phi' = e^{-iq\chi} \phi \tag{2.2}$$

where q is the charge of the scalar field and χ is constant. This is called a **global** phase transformation because χ is the same everywhere. Observe that eq. (2.1) is invariant under a global phase transformation.

Next consider a phase transformation where χ depends on the spacetime coordinates: this is called a **local** phase transformation. Then

$$\partial_\mu \phi \to \partial_\mu \phi' = e^{-iq\chi} \left[\partial_\mu \phi - iq(\partial_\mu \chi)\phi\right] \tag{2.3}$$

and consequently eq. (2.1) is not invariant under a local phase transformation.

Define the **gauge-covariant derivative**

$$D_\mu \phi = \partial_\mu \phi + iqA_\mu \phi \tag{2.4}$$

We now extend the definition of a gauge transformation (1.13) to also include a simultaneous local phase transformation on ϕ:

$$\begin{aligned} \phi &\to \phi' = e^{-iq\chi} \phi \\ A^\mu &\to A'^\mu = A^\mu + \partial^\mu \chi \end{aligned} \tag{2.5}$$

Show that eq. (2.5) implies

$$D_\mu \phi \to D'_\mu \phi' = e^{-iq\chi}\ D_\mu \phi \tag{2.6}$$

The **minimal coupling prescription** tells us to replace $\partial_\mu \phi$ with $D_\mu \phi$ in the Lagrangian (2.1) to obtain

$$\mathcal{L}_{D\phi} = (D_\mu \phi^*)(D^\mu \phi) - m^2 \phi^* \phi - V(\phi^* \phi) \tag{2.7}$$

Observe that eq. (2.7) is invariant under eq. (2.5), and therefore so is the total Lagrangian

$$\mathcal{L} = \mathcal{L}_{D\phi} + \mathcal{L}_{EM} = \mathcal{L}_{\partial\phi} + \mathcal{L}_{EM} + \mathcal{L}_{\text{int}}(\phi, A) \tag{2.8}$$

Show that the form of the interaction between the scalar field and the electromagnetic field is

$$\mathcal{L}_{\text{int}}(\phi, A) = iqA_\mu(\partial^\mu \phi^*)\phi - iqA_\mu \phi^*(\partial^\mu \phi) + q^2 A_\mu A^\mu \phi^* \phi \tag{2.9}$$

3 Nonabelian gauge theory

Reference: see chapter 46 of Coleman [3]. Let U be an $N \times N$ special unitary matrix

$$U = e^{-i\omega^a T_f^a}, \qquad (T_f^a)^\dagger = T_f^a \implies U^\dagger U = 1 \tag{3.1}$$

where ω^a are real parameters, and the generators in the fundamental representation T_f^a are hermitian and traceless. Recall that

$$\mathrm{Tr}(T_f^a T_f^b) = L_f \delta^{ab} \tag{3.2}$$

and the commutation relations are

$$[T_f^a, T_f^b] = i c^{abc} T_f^c \,. \tag{3.3}$$

Consider the Lagrangian for an N-component complex scalar field Φ

$$\mathcal{L}_{\partial\Phi} = (\partial_\mu \Phi)^\dagger (\partial^\mu \Phi) - m^2 \Phi^\dagger \Phi - V(\Phi^\dagger \Phi) \tag{3.4}$$

where Φ transforms under a unitary transformation U as

$$\Phi \to \Phi' = U\Phi \,. \tag{3.5}$$

Show that eq. (3.4) is invariant under a **global** transformation (where ω^a is constant). The Lagrangian (3.4) is not invariant, however, under a **local** transformation (where ω^a depend on location) because $\partial_\mu U \neq 0$. We now define the gauge-covariant derivative acting on Φ

$$D_\mu \Phi = \partial_\mu \Phi - i g A_\mu \Phi \tag{3.6}$$

where the gauge field $A_\mu = A_\mu^a T_f^a$ is an $N \times N$ hermitian matrix. Under a local transformation, the gauge-covariant derivative becomes

$$D_\mu \Phi \to D'_\mu \Phi' = \partial_\mu \Phi' - i g A'_\mu \Phi' \tag{3.7}$$

where the expression for A'_μ is determined below by the requirement that

$$D'_\mu \Phi' = U(D_\mu \Phi) \,. \tag{3.8}$$

In turn, eq. (3.8) ensures that the **gauged** Lagrangian

$$\mathcal{L}_{D\Phi} = (D_\mu \Phi)^\dagger (D^\mu \Phi) - m^2 \Phi^\dagger \Phi - V(\Phi^\dagger \Phi) \tag{3.9}$$

is invariant under a local unitary transformation. Combining eqs. (3.7) and (3.8), we find

$$\begin{aligned}
\partial_\mu \Phi' - i g A'_\mu \Phi' &= U \; D_\mu \Phi \\
&= U \left(\partial_\mu - i g A_\mu\right) \left(U^{-1} \Phi'\right) \\
&= \partial_\mu \Phi' + \left[U(\partial_\mu U^{-1}) - i g U A_\mu U^{-1}\right] \Phi' \\
&= \partial_\mu \Phi' + \left[-(\partial_\mu U) U^{-1} - i g U A_\mu U^{-1}\right] \Phi'
\end{aligned} \tag{3.10}$$

where we used $0 = \partial_\mu(UU^{-1}) = (\partial_\mu U)U^{-1} + U(\partial_\mu U^{-1})$. That is, under a local transformation the gauge field transforms as

$$A'_\mu = UA_\mu U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1} \,. \tag{3.11}$$

For infinitesimal gauge transformations $U = 1 - i\omega^a T^a_f + \mathcal{O}(\omega^2)$, one has

$$\begin{aligned} \Phi'^{\,m} &= \Phi^m - i\omega^a (T^a_f)^{mn}\Phi^n + \mathcal{O}(\omega^2)\,, \\ A'^{\,a}_\mu &= A^a_\mu + c^{abc}\omega^b A^c_\mu - \frac{1}{g}\partial_\mu\omega^a + \mathcal{O}(\omega^2)\,. \end{aligned} \tag{3.12}$$

Next, consider a field ϕ in the adjoint representation, whose components transform under infinitesimal transformations as

$$\begin{aligned} \phi'^{\,a} &= \phi^a - i\omega^b (T^b_{\rm adj})^{ac}\phi^c + \mathcal{O}(\omega^2) \\ &= \phi^a + c^{abc}\omega^b\phi^c + \mathcal{O}(\omega^2) \end{aligned} \tag{3.13}$$

where we recall that $(T^b_{\rm adj})^{ac} = ic^{abc}$. The adjoint field may be expressed as an $N \times N$ hermitian matrix

$$\phi = \phi^a T^a_f \tag{3.14}$$

which transforms as

$$\phi \to \phi' = U\phi U^{-1} \tag{3.15}$$

yielding eq. (3.13). The ungauged Lagrangian for the adjoint field

$$\mathcal{L}_{\partial\phi} = \frac{1}{2L_f}\,\mathrm{tr}\left[(\partial_\mu\phi)(\partial^\mu\phi) - m^2\phi^2\right] = \tfrac{1}{2}(\partial_\mu\phi^a)(\partial^\mu\phi^a) - \tfrac{1}{2}m^2\phi^a\phi^a \tag{3.16}$$

is invariant under a global, but not a local, transformation. Recalling that the covariant derivative acts on the components of Φ, a field in the fundamental representation, as

$$(D_\mu\Phi)^m = \partial_\mu\Phi^m - igA^a_\mu(T^a_f)^{mn}\Phi^n \tag{3.17}$$

we expect that the covariant derivative acts on ϕ^a, the components of the adjoint field, as

$$\begin{aligned} (D_\mu\phi)^a &= \partial_\mu\phi^a - igA^b_\mu(T^b_{\rm adj})^{ac}\phi^c \\ &= \partial_\mu\phi^a + gc^{abc}A^b_\mu\phi^c\,. \end{aligned} \tag{3.18}$$

One may easily verify that this is equivalent to writing

$$D_\mu\phi = \partial_\mu\phi - ig[A_\mu, \phi]\,. \tag{3.19}$$

One can show that eqs. (3.15) and (3.19) together with eq. (3.11) implies

$$D_\mu\phi \to D'_\mu\phi' = U(D_\mu\phi)U^{-1}\,. \tag{3.20}$$

Consequently, the gauged Lagrangian

$$\mathcal{L}_{D\phi} = \frac{1}{2L_f}\,\mathrm{tr}\left[(D_\mu\phi)(D^\mu\phi) - m^2\phi^2\right] = \tfrac{1}{2}(D_\mu\phi^a)(D^\mu\phi^a) - \tfrac{1}{2}m^2\phi^a\phi^a \tag{3.21}$$

is invariant under a local unitary transformation.

Next we define the nonabelian field strength tensor $F_{\mu\nu} = F^a_{\mu\nu} T^a_f$ via

$$F_{\mu\nu}\Phi = \frac{i}{g}(D_\mu D_\nu - D_\nu D_\mu)\Phi\,. \tag{3.22}$$

We may determine how $F_{\mu\nu}$ transforms by examining how the right hand side of this equation transforms. Temporarily let $\Psi_\nu = D_\nu\Phi$. We know from eq. (3.8) that Ψ_ν transforms as $\Psi'_\nu = U\Psi_\nu$, that is, as a field in the fundamental representation. This means, again using eq. (3.8), that the covariant derivative acting on Ψ_ν transforms as $D'_\mu\Psi'_\nu = UD_\mu\Psi_\nu$, or in other words $D'_\mu D'_\nu\Phi' = UD_\mu D_\nu\Phi$. Thus, eq. (3.22) inplies

$$\begin{aligned} F'_{\mu\nu}\Phi' &= \frac{i}{g}(D'_\mu D'_\nu - D'_\nu D'_\mu)\Phi' \\ &= \frac{i}{g}U(D_\mu D_\nu - D_\nu D_\mu)\Phi \\ &= UF_{\mu\nu}\Phi \\ &= UF_{\mu\nu}U^{-1}\Phi' \end{aligned} \tag{3.23}$$

which is to say that $F_{\mu\nu}$ transforms in the adjoint representation (see eq. (3.15))

$$F_{\mu\nu} \to F'_{\mu\nu} = UF_{\mu\nu}U^{-1}\,. \tag{3.24}$$

For infinitesimal gauge transformations $U \approx 1 - i\omega^a T^a$, one has

$$F^a_{\mu\nu} \to F'^{\,a}_{\mu\nu} = F^a_{\mu\nu} + c^{abc}\omega^b F^c_{\mu\nu} + \mathcal{O}(\omega^2)\,. \tag{3.25}$$

We can now write a gauge-invariant Lagrangian for the gauge field, namely

$$\mathcal{L}_{\rm YM} = -\tfrac{1}{4L_f}\,\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) = -\tfrac{1}{4L_f}F^a_{\mu\nu}F^{b\mu\nu}\,\mathrm{Tr}(T^a_f T^b_f) = -\tfrac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}\,. \tag{3.26}$$

Finally, we can derive the relation between the field strength $F_{\mu\nu}$ and the vector potential A_μ using eq. (3.22)

$$\begin{aligned} F_{\mu\nu}\Phi &= \frac{i}{g}(\partial_\mu - igA_\mu)(\partial_\nu - igA_\nu)\Phi - (\mu \leftrightarrow \nu) \\ &= (\partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu])\Phi \\ &= (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gc^{abc}A^b_\mu A^c_\nu)T^a_f\Phi \end{aligned} \tag{3.27}$$

thus

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gc^{abc}A^b_\mu A^c_\nu\,. \tag{3.28}$$

Since the field strength tensor transforms in the adjoint representation, the covariant derivative acts on it as (see eq. (3.19))

$$D_\lambda F_{\mu\nu} = \partial_\lambda F_{\mu\nu} - ig[A_\lambda, F_{\mu\nu}] \tag{3.29}$$

or equivalently

$$D_\lambda F^a_{\mu\nu} = \partial_\lambda F^a_{\mu\nu} + gc^{abc} A^b_\lambda F^c_{\mu\nu} \,. \tag{3.30}$$

Show that the Euler-Lagrange equations for the Yang-Mills Lagrangian (3.26) can be written

$$D_\mu F^{a\mu\nu} = 0 \tag{3.31}$$

One may verify that the Bianchi identity holds automatically

$$D_\mu \tilde{F}^{a\mu\nu} = 0 \tag{3.32}$$

by using the Jacobi identity $c^{abe}c^{ecd} + c^{bce}c^{ead} + c^{cae}c^{ebd} = 0$ for the structure constants. An alternative proof of the Bianchi identity uses

$$[D_\mu, [D_\nu, D_\lambda]]\Phi + \text{cyclic permutations} = 0 \tag{3.33}$$

References

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