Charged particles (Dec 2025)

This .pdf file generated January 16, 2026.

In these notes, we use the mostly minus metric $\eta_{00} = 1$. (Feynman, Jackson, Dirac, Gross notes)

1 Charged relativistic particles

1.1 Noncovariant Lagrangian approach

The action for a charged relativistic particle in an external electromagnetic field is
[Goldstein (1e) 19, 207; (2e) 23, 322; Jackson (2e), 574; Tong: Classical Dynamics, 36]

$$
\begin{aligned}
S &= \int \left(-m d\tau - qA_\mu dx^\mu\right) \\
&= \int \left(-m\sqrt{\eta_{\mu\nu} dx^\mu dx^\nu} - qA_\mu dx^\mu\right) \\
&= \int dt \left(-m\sqrt{1 - \left(\frac{d\mathbf{x}}{dt}\right)^2} - qA^0 + q\mathbf{A}\cdot\frac{d\mathbf{x}}{dt}\right) \\
&= \int dt\ L
\end{aligned} \tag{1.1}
$$

Under a gauge transformation $A_\mu \to A_\mu + \partial_\mu \chi$, one has $L \to L - q(d\chi/dt)$ so S is gauge invariant. The canonical momentum is

$$\mathbf{p} = \frac{\partial L}{\partial(d\mathbf{x}/dt)} = \boldsymbol{\pi} + q\mathbf{A} \tag{1.2}$$

where $\boldsymbol{\pi}$ is the (relativistic) mechanical momentum

$$\boldsymbol{\pi} = \frac{m(d\mathbf{x}/dt)}{\sqrt{1 - (d\mathbf{x}/dt)^2}} \tag{1.3}$$

While $\boldsymbol{\pi}$ is gauge-invariant, $\mathbf{p}$ is not. The Euler-Lagrange equation is

$$\frac{d\mathbf{p}}{dt} = \frac{d}{dt}\left(\boldsymbol{\pi} + q\mathbf{A}\right) = -q\nabla A^0 + q\nabla\left(\mathbf{A}\cdot\frac{d\mathbf{x}}{dt}\right) \tag{1.4}$$

that is

$$
\begin{aligned}
\frac{d\pi^i}{dt} + q\partial_0 A^i + q\partial_j A^i \frac{dx^j}{dt} &= -q\partial_i A^0 + q\partial_i A^j \frac{dx^j}{dt} \\
\frac{d\pi^i}{dt} &= q\left(-\partial_i A^0 - \partial_0 A^i\right) + q\left(\partial_i A^j - \partial_j A^i\right)\frac{dx^j}{dt}
\end{aligned} \tag{1.5}
$$

Defining

$$\mathbf{E} = -\nabla A^0 - \frac{\partial \mathbf{A}}{\partial t} \qquad\qquad \mathbf{B} = \nabla \times \mathbf{A} \tag{1.6}$$

we have

$$\epsilon^{ijk} B^k = \epsilon^{ijk} \epsilon^{klm} \partial_l A^m = \partial_i A^j - \partial_j A^i \tag{1.7}$$

so that

$$\frac{d\boldsymbol{\pi}}{dt} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \tag{1.8}$$

We define the "relativistic mass" (actually the kinetic plus rest energy) [Messiah, p. 883]

$$\pi^0 = \frac{m}{\sqrt{1 - \left(d\mathbf{x}/dt\right)^2}} \tag{1.9}$$

and the mechanical four-momentum

$$\pi^\mu = (\pi^0, \boldsymbol{\pi}) \qquad \Longrightarrow \qquad \pi^\mu \pi_\mu = m^2 \tag{1.10}$$

which is gauge invariant, but not conserved. Since

$$\frac{\boldsymbol{\pi}}{\pi^0} = \frac{d\mathbf{x}}{dt} \tag{1.11}$$

we have

$$\begin{aligned} \pi^0 \frac{d\pi^0}{dt} &= \boldsymbol{\pi} \cdot \frac{d\boldsymbol{\pi}}{dt} \\ \frac{d\pi^0}{dt} &= \mathbf{v} \cdot \frac{d\boldsymbol{\pi}}{dt} \\ &= q\mathbf{v} \cdot \mathbf{E} \\ &= q(-\partial_i A^0 - \partial_0 A^i)\frac{dx^i}{dt} \end{aligned} \tag{1.12}$$

We rewrite eqs. (**??**) and (**??**) as

$$\begin{aligned} \frac{d\pi^i}{d\tau} &= q\left(\partial^i A^0 - \partial^0 A^i\right)\frac{dx_0}{d\tau} + q\left(\partial^i A^j - \partial^j A^i\right)\frac{dx_j}{d\tau} \\ \frac{d\pi^0}{d\tau} &= q(-\partial^i A^0 + \partial^0 A^i)\frac{dx_i}{d\tau} \end{aligned} \tag{1.13}$$

or more compactly

$$\frac{d\pi^\mu}{d\tau} = qF^{\mu\nu}\frac{dx_\nu}{d\tau} \tag{1.14}$$

Since $\pi^\mu = m(dx^\mu/d\tau)$, this is just

$$m\frac{d^2x^\mu}{d\tau^2} = qF^{\mu\nu}\frac{dx_\nu}{d\tau} \tag{1.15}$$

1.2 Conservation of energy

The energy of the particle is defined as

$$\begin{aligned}
h &= \mathbf{p}\cdot\frac{d\mathbf{x}}{dt} - L \\
&= \boldsymbol{\pi}\cdot\frac{d\mathbf{x}}{dt} + q\mathbf{A}\cdot\frac{d\mathbf{x}}{dt} - \left(-m\sqrt{1-\left(\frac{d\mathbf{x}}{dt}\right)^2} - qA^0 + q\mathbf{A}\cdot\frac{d\mathbf{x}}{dt}\right) \\
&= \frac{m(d\mathbf{x}/dt)^2}{\sqrt{1-(d\mathbf{x}/dt)^2}} + m\sqrt{1-\left(\frac{d\mathbf{x}}{dt}\right)^2} + qA^0 \\
&= \frac{m}{\sqrt{1-(d\mathbf{x}/dt)^2}} + qA^0 \\
&= \pi^0 + qA^0
\end{aligned} \tag{1.16}$$

Thus h is the sum of the kinetic plus rest energy π^0 and the potential energy qA^0 of the particle. Observe that

$$\begin{aligned}
\frac{dh}{dt} &= \frac{d\pi^0}{dt} + q\frac{dA^0}{dt} \\
&= q(-\partial_i A^0 - \partial_0 A^i)\frac{dx^i}{dt} + q\left(\partial_0 A^0 + \partial_i A^0\frac{dx^i}{dt}\right) \\
&= q\partial_0\left(A^0 - A^i\frac{dx^i}{dt}\right) \\
&= -\frac{\partial L}{\partial t}
\end{aligned} \tag{1.17}$$

which is just Euler's second equation

$$\frac{dh}{dt} + \frac{\partial L}{\partial t} = 0 \tag{1.18}$$

Thus the energy h of the particle is conserved if L does not depend explicitly on time.

We can define h as the fourth component of the canonical four-momentum so that

$$\begin{aligned}
p^\mu &= (h, \mathbf{p}) \\
&= \pi^\mu + qA^\mu
\end{aligned} \tag{1.19}$$

which is not gauge invariant.

1.3 Hamiltonian approach

Inverting the equation for mechanical momentum gives

$$\frac{d\mathbf{x}}{dt} = \frac{\boldsymbol{\pi}}{\sqrt{\boldsymbol{\pi}^2 + m^2}} \tag{1.20}$$

Thus the Legendre transform is

$$\begin{aligned} H &= \frac{m}{\sqrt{1 - (d\mathbf{x}/dt)^2}} + qA^0 \\ &= \sqrt{\boldsymbol{\pi}^2 + m^2} + qA^0 \\ &= \sqrt{(\mathbf{p} - q\mathbf{A})^2 + m^2} + qA^0 \end{aligned} \tag{1.21}$$

Hamilton's equations give

$$\begin{aligned} \frac{dx^i}{dt} &= \frac{\partial H}{\partial p^i} = \frac{p^i - qA^i}{\sqrt{(\mathbf{p} - q\mathbf{A})^2 + m^2}} \\ \frac{dp^i}{dt} &= -\frac{\partial H}{\partial x^i} = q\left(\frac{(\mathbf{p} - q\mathbf{A}) \cdot \partial_i \mathbf{A}}{\sqrt{(\mathbf{p} - q\mathbf{A})^2 + m^2}} - \partial_i A^0\right) \\ &= q\partial_i(\mathbf{v} \cdot \mathbf{A} - A^0) \end{aligned} \tag{1.22}$$

which gives

$$\frac{d\pi^i}{dt} = -q(\partial_0 A^i + v^j \partial_j A^i) + q\partial_i(v^j A^j - A^0) \tag{1.23}$$

which of course is just eq. (**??**).

1.4 First covariant Lagrangian approach

The simplest covariant action is [Goldstein (1e) 209, (2e) 330; Coleman (2022) 50, 69]

$$\begin{aligned} S &= \int \left(-\tfrac{1}{2}m d\tau - qA_\mu dx^\mu\right) \\ &= \int d\tau \left(-\frac{1}{2}m\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} - qA_\mu\frac{dx^\mu}{d\tau}\right) \\ &= \int d\tau\ L_{\text{cov}} \end{aligned} \tag{1.24}$$

where $d\tau = \sqrt{\eta_{\mu\nu}dx^\mu dx^\nu}$. This action is not reparametrization invariant. Using

$$\begin{aligned} \frac{\partial L_{\text{cov}}}{\partial(dx^\mu/d\tau)} &= -m\frac{dx_\mu}{d\tau} - qA_\mu \\ \frac{\partial L_{\text{cov}}}{\partial x^\mu} &= -q\frac{\partial A_\nu}{\partial x^\mu}\frac{dx^\nu}{d\tau} \end{aligned} \tag{1.25}$$

we obtain the Euler-Lagrange equation

$$\begin{aligned} \frac{d}{d\tau}\left(m\frac{dx_\mu}{d\tau} + qA_\mu\right) &= q\partial_\mu A_\nu\frac{dx^\nu}{d\tau} \\ m\frac{d^2x_\mu}{d\tau^2} &= q\partial_\mu A_\nu\frac{dx^\nu}{d\tau} - q\partial_\nu A_\mu\frac{dx^\nu}{d\tau} \\ &= qF_{\mu\nu}\frac{dx^\nu}{d\tau} \end{aligned} \tag{1.26}$$

which is just eq. (**??**).

Because L_{cov} only depends on τ implicitly (through x^μ and $dx^\mu/d\tau$), Euler's second equation implies a conserved quantity

$$\begin{aligned} h &= \frac{\partial L_{\text{cov}}}{\partial(dx^\mu/d\tau)}\frac{dx^\mu}{d\tau} - L_{\text{cov}} \\ &= \left(-m\frac{dx_\mu}{d\tau} - qA_\mu\right)\frac{dx^\mu}{d\tau} + \frac{1}{2}m\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + qA_\mu\frac{dx^\mu}{d\tau} \\ &= -\frac{1}{2}m\frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau} \end{aligned} \tag{1.27}$$

By the equation of motion

$$\begin{aligned} \frac{dh}{d\tau} &= -m\frac{dx^\mu}{d\tau}\frac{d^2x_\mu}{d\tau^2} \\ &= -qF_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \\ &= 0 \end{aligned} \tag{1.28}$$

Of course, by the constraint $h = -m/2$ and so manifestly conserved.

1.5 Covariant Hamiltonian approach

The covariant Lagrangian

$$L_{\text{cov}} = -\frac{1}{2} m \eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} - q A_\mu \frac{dx^\mu}{d\tau} \tag{1.29}$$

yields the canonical four-momentum

$$\begin{aligned} p_\mu &= -\frac{\partial L_{\text{cov}}}{\partial (dx^\mu / ds)} \\ &= m \frac{dx_\mu}{d\tau} + q A_\mu \end{aligned} \tag{1.30}$$

the minus sign being necessary to produce agreement with the noncovariant approach.

If we define the "Hamiltonian" to be

$$\begin{aligned} H_{\text{cov}} &= -\left[(-p_\mu) \frac{dx^\mu}{d\tau} - L_{\text{cov}} \right] \\ &= \frac{1}{2m} (p - qA)^2 \end{aligned} \tag{1.31}$$

then Hamilton's equations yield

$$\begin{aligned} \frac{dx^\mu}{d\tau} &= \frac{\partial H_{\text{cov}}}{\partial p_\mu} = \frac{1}{m} (p^\mu - q A^\mu) \\ \frac{dp^\mu}{d\tau} &= -\frac{\partial H_{\text{cov}}}{\partial x_\mu} = \frac{q}{m} (p_\nu - q A_\nu) \partial^\mu A^\nu = q \frac{dx_\nu}{d\tau} \partial^\mu A^\nu \end{aligned} \tag{1.32}$$

from which we obtain

$$m \frac{d^2 x^\mu}{d\tau^2} = \frac{d}{d\tau} (p^\mu - q A^\mu) = q \frac{dx_\nu}{d\tau} \partial^\mu A^\nu - q \frac{\partial A^\mu}{\partial x_\nu} \frac{dx_\nu}{d\tau} = q F^{\mu\nu} \frac{dx_\nu}{d\tau} \tag{1.33}$$

which is again eq. (**??**).

1.6 Alternative covariant Lagrangian approach

An alternative covariant action is [Goldstein (2e) 326; Jackson (2e) 576; Coleman (2022) 50; Rohrlich (1990) 160].

$$\begin{aligned} S &= \int (-m d\tau - q A_\nu dx^\nu) \\ &= \int ds \left(-m\sqrt{\eta_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}} - qA_\nu \frac{dx^\nu}{ds} \right) \\ &= \int ds\ L_{\text{cov}} \end{aligned} \tag{1.34}$$

where now s denotes an arbitrary parametrization of the worldline. We have

$$\begin{aligned} \frac{\partial L_{\text{cov}}}{\partial(dx^\mu/ds)} &= -\ \frac{m(dx_\mu/ds)}{\sqrt{\eta_{\alpha\beta}(dx^\alpha/ds)(dx^\beta/ds)}} - qA_\mu \\ \frac{\partial L_{\text{cov}}}{\partial x^\mu} &= -q\frac{\partial A_\nu}{\partial x^\mu}\frac{dx^\nu}{ds} \end{aligned} \tag{1.35}$$

giving the Euler-Lagrange equation

$$\begin{aligned} \frac{d}{ds}\left(\frac{m(dx_\mu/ds)}{\sqrt{\eta_{\alpha\beta}(dx^\alpha/ds)(dx^\beta/ds)}} + qA_\mu \right) &= q\partial_\mu A_\nu \frac{dx^\nu}{ds} \\ \frac{d}{ds}\frac{m(dx_\mu/ds)}{\sqrt{\eta_{\alpha\beta}(dx^\alpha/ds)(dx^\beta/ds)}} &= q\partial_\mu A_\nu\frac{dx^\nu}{ds} - q\partial_\nu A_\mu \frac{dx^\nu}{ds} \\ \frac{d}{ds} m\frac{dx_\mu}{d\tau} &= qF_{\mu\nu}\frac{dx^\nu}{ds} \\ m\frac{d^2x_\mu}{d\tau^2} &= qF_{\mu\nu}\frac{dx^\nu}{d\tau} \end{aligned} \tag{1.36}$$

Observe that L_{cov} depends on s only through x^μ and (dx^μ/ds) so $\partial L_{\text{cov}}/\partial s = 0$ and therefore the second Euler equation says that $dh/ds = 0$. Indeed we find that h vanishes identically:

$$\begin{aligned} h &= \frac{\partial L_{\text{cov}}}{\partial(dx^\mu/ds)}\frac{dx^\mu}{ds} - L_{\text{cov}} \\ &= -\left(\frac{m(dx_\mu/ds)}{\sqrt{\eta_{\alpha\beta}(dx^\alpha/ds)(dx^\beta/ds)}} + qA_\mu \right)\frac{dx^\mu}{ds} + \left(m\sqrt{\eta_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}} + qA_\nu\frac{dx^\nu}{ds} \right) \\ &= 0 \end{aligned} \tag{1.37}$$

1.7 Alternative covariant Hamiltonian approach

The covariant Lagrangian

$$L_{\text{cov}} = -m\sqrt{\eta_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}} - qA_\nu\frac{dx^\nu}{ds} \tag{1.38}$$

yields the canonical four-momentum

$$\begin{aligned} p_\mu &= -\frac{\partial L_{\text{cov}}}{\partial(dx^\mu/ds)} \\ &= \frac{m(dx_\mu/ds)}{\sqrt{\eta_{\alpha\beta}(dx^\alpha/ds)(dx^\beta/ds)}} + qA_\mu \\ &= m\frac{dx_\mu}{d\tau} + qA_\mu \end{aligned} \tag{1.39}$$

the minus sign again being necessary to produce agreement with the noncovariant approach.

Jackson (2e) 577 then suggests defining the "Hamiltonian" as

$$\begin{aligned} H_{\text{cov}} &= -\tfrac{1}{2}\left[(-p_\mu)\frac{dx^\mu}{ds} - L_{\text{cov}}\right] \\ &= \tfrac{1}{2}\left[p_\mu\frac{dx^\mu}{ds} - m\sqrt{\eta_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}} - qA_\mu\frac{dx^\mu}{ds}\right] \\ &= \tfrac{1}{2}\left(p_\mu - qA_\mu\right)\frac{dx^\mu}{ds} - \tfrac{1}{2}m\sqrt{\eta_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}} \end{aligned} \tag{1.40}$$

We cannot obtain dx^μ/ds in terms of p^μ but if we let $s = \tau$, then eq. (**??**) allows us to write

$$\begin{aligned} H_{\text{cov}} &= \tfrac{1}{2}\left(p_\mu - qA_\mu\right)\frac{dx^\mu}{d\tau} - \tfrac{1}{2}m \\ &= \frac{1}{2m}(p-qA)^2 - \tfrac{1}{2}m \end{aligned} \tag{1.41}$$

(Of course, the constraint above would seem to imply $(p-qA)^2 = m^2$ so that $H_{\text{cov}} = 0$.)

Hamilton's equations applied to eq. (**??**) give

$$\begin{aligned} \frac{dx^\mu}{d\tau} &= \frac{\partial H_{\text{cov}}}{\partial p_\mu} = \frac{1}{m}\left(p^\mu - qA^\mu\right) \\ \frac{dp^\mu}{d\tau} &= -\frac{\partial H_{\text{cov}}}{\partial x_\mu} = \frac{q}{m}\left(p_\nu - qA_\nu\right)\partial^\mu A^\nu = q\frac{dx_\nu}{d\tau}\partial^\mu A^\nu \end{aligned} \tag{1.42}$$

from which we obtain eq. (**??**) once again

$$m\frac{d^2x^\mu}{d\tau^2} = \frac{d}{d\tau}\left(p^\mu - qA^\mu\right) = q\frac{dx_\nu}{d\tau}\partial^\mu A^\nu - q\frac{\partial A^\mu}{\partial x_\nu}\frac{dx_\nu}{d\tau} = qF^{\mu\nu}\frac{dx_\nu}{d\tau} \tag{1.43}$$

1.8 Lagrangian with Lagrange multipliers

A Lagrange multiplier approach [Barut (1964) 65] starts with

$$\overline{L} = L + \frac{\lambda}{2}\left(\frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau} - 1\right) \tag{1.44}$$

The Euler-Lagrange equation for λ yields

$$\frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau} = 1 \tag{1.45}$$

The Euler-Lagrange equation for x^μ yields

$$0 = \frac{d}{d\tau}\left(\lambda\frac{dx_\mu}{d\tau} + \frac{\partial L}{\partial(dx^\mu/d\tau)}\right) - \frac{\partial L}{\partial x^\mu} \tag{1.46}$$

Multiplying by $(dx^\mu/d\tau)$ we have

$$0 = \frac{d\lambda}{d\tau}\left(\frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau}\right) + \lambda\frac{dx^\mu}{d\tau}\frac{d^2x_\mu}{d\tau^2} + \frac{dx^\mu}{d\tau}\frac{d}{d\tau}\left(\frac{\partial L}{\partial(dx^\mu/d\tau)}\right) - \frac{dx^\mu}{d\tau}\frac{\partial L}{\partial x^\mu} \tag{1.47}$$

or

$$\frac{d\lambda}{d\tau} = \frac{dx^\mu}{d\tau}\frac{\partial L}{\partial x^\mu} - \frac{dx^\mu}{d\tau}\frac{d}{d\tau}\left(\frac{\partial L}{\partial(dx^\mu/d\tau)}\right) \tag{1.48}$$

which we can integrate (cf Barut, or just verify by differentiating) to obtain

$$\lambda = L - \frac{dx^\nu}{d\tau}\left(\frac{\partial L}{\partial(dx^\nu/d\tau)}\right) + \text{const} \tag{1.49}$$

Substitute into eq. (**??**) to obtain the equation of motion

$$0 = \frac{d}{d\tau}\left(\left[L - \frac{dx^\nu}{d\tau}\left(\frac{\partial L}{\partial(dx^\nu/d\tau)}\right) + \text{const}\right]\frac{dx_\mu}{d\tau} + \frac{\partial L}{\partial(dx^\mu/d\tau)}\right) - \frac{\partial L}{\partial x^\mu} \tag{1.50}$$

If the Lagrangian is

$$L = -m - qA_\mu\frac{dx^\mu}{d\tau} \tag{1.51}$$

then (setting the constant to zero) we have $\lambda = -m$ and so

$$\frac{d}{d\tau}\left(m\frac{dx_\mu}{d\tau} + qA_\mu\right) = q\partial_\mu A_\nu\frac{dx^\nu}{d\tau} \tag{1.52}$$

as found before.

2 Particle-field interaction

We follow Coleman's Lectures on Relativity, ch. 3. [See also Jackson, ch 14; Rohrlich, ch 4.7-4.8 (1990) 77] The action for the electromagnetic field

$$S_{EM} = \int d^4x \left(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - J^\mu A_\mu\right) \tag{2.1}$$

yields the equation of motion (in Lorenz gauge)

$$\partial^2 A^\mu = J^\mu \tag{2.2}$$

whose solution is

$$A^\mu(x) = A^\mu_{\text{free}}(x) + \int d^4y\ D(x-y)J^\mu(y) \tag{2.3}$$

where $A^\mu_{\text{free}}(x)$ is a complementary solution and $D(x)$ is the Green function

$$\partial^2 D(x) = \delta^{(4)}(x), \qquad D(x) = \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot x}\frac{(-1)}{k^2} \tag{2.4}$$

Choosing retarded boundary conditions, we have

$$A^\mu(x) = A^\mu_{\text{in}}(x) + \int d^4y\ D_R(x-y)J^\mu(y) \tag{2.5}$$

Evaluating the contour integral by passing below the poles we obtain

$$D_R(x) = \frac{1}{4\pi r}\delta(r-t) = \frac{1}{2\pi}\delta(x^2)\theta(x^0) \tag{2.6}$$

See also Jackson (2e), sec. 12.11.

The covariant action for a charged particle

$$S_{\text{pcl}} = \int d\tau \left(-\frac{1}{2}m\eta_{\mu\nu}\frac{dy^\mu}{d\tau}\frac{dy^\nu}{d\tau} - qA_\mu\frac{dy^\mu}{d\tau}\right) \tag{2.7}$$

yields the equation of motion

$$m\frac{d^2y^\mu}{d\tau^2} = qF^{\mu\nu}\frac{dy_\nu}{d\tau} \tag{2.8}$$

Comparing eqs. (**??**) and (**??**), the charged particle produces the current

$$J^\mu(x) = q\int d\tau\ \delta^{(4)}(x-y(\tau))\frac{dy^\mu}{d\tau} \tag{2.9}$$

which generates the field

$$A^\mu(x) = A^\mu_{\rm in}(x) + \frac{q}{2\pi}\int d\tau\ \delta(z^2)\theta(z^0)\frac{dy^\mu}{d\tau}, \qquad z^\mu = x^\mu - y^\mu(\tau) \tag{2.10}$$

Let $\tau = \tau_0$ be the point of the trajectory $y^\mu(\tau)$ satisfying $z^2 = 0$, i.e. the unique event where the past light-cone centered at x^μ intersects the particle's worldline

$$\begin{aligned} z^0 &= |\mathbf{z}| \\ x^0 - y^0(\tau_0) &= |\mathbf{x} - \mathbf{y}(\tau_0)| \end{aligned} \tag{2.11}$$

Observing that z^2 decreases as τ increases, one has

$$\delta(z^2) = \left|\frac{\partial z^2}{\partial \tau}\right|^{-1}\delta(\tau - \tau_0), \qquad \left|\frac{\partial z^2}{\partial \tau}\right| = -\frac{\partial z^2}{\partial \tau} = 2z_\nu\frac{dy^\nu}{d\tau} \tag{2.12}$$

Thus one obtains the Liénard-Wiechert potential (see also Jackson (3e), eq. 14.6)

$$A^\mu(x) = \left(\frac{q}{4\pi}\right)\frac{v^\mu}{z\cdot v}\bigg|_{\tau=\tau_0}, \qquad v^\mu \equiv \frac{dy^\mu}{d\tau} = (\gamma, \gamma\mathbf{v}) \tag{2.13}$$

so $v^2 = 1$. Define $R \equiv z^0$ and $\mathbf{n} = \mathbf{z}/R$ with all quantities evaluated at $\tau = \tau_0$. Then

$$z\cdot v = R\gamma(1 - \mathbf{n}\cdot\mathbf{v}) \tag{2.14}$$

Thus (Jackson (2e), eq. 14.8)

$$A^0 = \left(\frac{q}{4\pi}\right)\frac{1}{R(1-\mathbf{n}\cdot\mathbf{v})}, \qquad \mathbf{A} = \left(\frac{q}{4\pi}\right)\frac{\mathbf{v}}{R(1-\mathbf{n}\cdot\mathbf{v})} \tag{2.15}$$

Consider a particle moving at constant speed in the $+x$ direction and passing through the origin:

$$y^\mu(\tau) = (\gamma\tau, \gamma v\tau, 0, 0), \qquad v^\mu = (\gamma, \gamma v, 0, 0) \tag{2.16}$$

Let's evaluate the potential at $x^\mu = (t, x, 0, 0)$. Then $z^2 = 0$ implies $t - \gamma\tau_0 = x - \gamma v\tau_0$ so that

$$\tau_0 = \frac{t-x}{\gamma(1-v)}, \qquad z^\mu = \frac{x - vt}{1-v}(1,1,0,0) \qquad z\cdot v = \gamma(x - vt) \tag{2.17}$$

yielding

$$A^\mu(t,x,0,0) = \frac{q}{4\pi(x-vt)}(1, v, 0, 0) \tag{2.18}$$

We'll see from results below that this gives

$$E_x(t,x,0,0) = F_{01}(t,x,0,0) = \frac{q}{4\pi}\frac{1}{\gamma^2(x-vt)^2} \tag{2.19}$$

Coleman shows that the field strength tensor is given by

$$F_{\mu\nu}(x) = \frac{q}{4\pi}\left[\frac{1}{(z\cdot v)^2}z_\mu a_\nu + \frac{(1-z\cdot a)}{(z\cdot v)^3}z_\mu v_\nu\right]\Bigg|_{\tau=\tau_0} - (\mu\leftrightarrow\nu), \qquad a^\mu \equiv \frac{d^2y^\mu}{d\tau^2} \tag{2.20}$$

Alternatively, Jackson (3e), eq. 14.11 gives the equivalent expression (easily verified)

$$F_{\mu\nu}(x) = \frac{q}{4\pi}\frac{1}{(z\cdot v)}\frac{d}{d\tau}\left[\frac{z_\mu v_\nu}{z\cdot v}\right]\Bigg|_{\tau=\tau_0} - (\mu\leftrightarrow\nu) \tag{2.21}$$

Jackson then derives (eqs. 14.13, 14.14) explicit results

$$\begin{aligned}\mathbf{E} &= \frac{q}{4\pi}\left[\frac{\mathbf{n}-\mathbf{v}}{\gamma^2R^2(1-\mathbf{n}\cdot\mathbf{v})^3} + \frac{\mathbf{n}\times\{(\mathbf{n}-\mathbf{v})\times\dot{\mathbf{v}}\}}{R(1-\mathbf{n}\cdot\mathbf{v})^3}\right]\\ \mathbf{B} &= \mathbf{n}\times\mathbf{E}\end{aligned} \tag{2.22}$$

which reduce for non-relativistic motion to

$$\begin{aligned}\mathbf{E} &= \frac{q}{4\pi}\left[\frac{\mathbf{n}}{R^2} + \frac{\mathbf{n}\times\{\mathbf{n}\times\dot{\mathbf{v}}\}}{R}\right]\\ \mathbf{B} &= \mathbf{n}\times\mathbf{E}\end{aligned} \tag{2.23}$$

Feynman, vol. 1, eq. 28.3 gives

$$\begin{aligned}\mathbf{E} &= \frac{q}{4\pi\epsilon_0}\left[\frac{\mathbf{n}}{R^2} + R\frac{d}{dt}\left(\frac{\mathbf{n}}{R^2}\right) + \frac{d^2\mathbf{n}}{dt^2}\right]\\ \mathbf{B} &= \mathbf{n}\times\mathbf{E}\end{aligned} \tag{2.24}$$

It is remarkable that the fields are completely determined only by the behavior of the charge at the retarded time.

After changing changing his metric to ours, Rohrlich defines $\rho = R^\mu v_\mu$ where R^μ is what we are calling z^μ so ρ is what we are calling $z \cdot v$. Then Rohrlich defines u^μ via

$$R^\mu = \rho(u^\mu + v^\mu), \qquad u^2 = -1, \qquad u^\mu v_\mu = 0 \tag{2.25}$$

Using $v^2 = 1$, one easily verifies $R^2 = 0$ and $R^\mu v_\mu = \rho$ and $R^\mu u_\mu = -\rho$. Thus

$$u^\mu = \frac{R^\mu}{\rho} - v^\mu \tag{2.26}$$

Also using $a^\mu v_\mu = 0$, one has

$$a_u \equiv -a^\mu u_\mu = -\frac{a^\mu R_\mu}{\rho} \tag{2.27}$$

Then Rohrlich writes (after correcting for -4π due to his conventions)

$$F^{\mu\nu} = -\frac{q}{4\pi}\left[\frac{v^\mu u^\nu}{\rho^2} + \frac{1}{\rho}\left(a^\mu v^\nu - a_u u^\mu u^\nu - u^\mu a^\nu\right)\right] - (\mu \leftrightarrow \nu) \tag{2.28}$$

which can easily be shown to agree with Coleman's expression. Rohrlich also writes

$$F^{\mu\nu} = -\frac{q}{4\pi\rho}\frac{d}{d\tau}\left(\frac{v^\mu R^\nu}{\rho}\right) \tag{2.29}$$

which agrees with Jackson's expression.

2.1 Radiation

Spacetime symmetry leads to a conserved energy-momentum tensor

$$T_c^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\alpha)}\partial^\nu A_\alpha - \eta^{\mu\nu}\mathcal{L} = -F^{\mu\alpha}\partial^\nu A_\alpha + \tfrac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \tag{2.30}$$

which is neither symmetric nor gauge invariant. By adding a (conserved) total derivative, we obtain the symmetric (Belinfante) energy-momentum tensor

$$T_b^{\mu\nu} = T_c^{\mu\nu} + \partial_\alpha\left(F^{\mu\alpha}A^\nu\right) = F^{\mu\alpha}F_\alpha{}^\nu + \tfrac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \tag{2.31}$$

One observes that

$$\partial_\mu T_b^{\mu\nu} = F^{\nu\lambda}j_\lambda \tag{2.32}$$

Plugging in the field strength above, one obtains

$$\begin{aligned} T_b^{\mu\nu} &= \frac{q^2}{4\pi\rho^4}\left(u^\mu u^\nu - v^\mu v^\nu - \tfrac{1}{2}\eta^{\mu\nu}\right) \\ &+ \frac{q^2}{4\pi\rho^3}\left(2a_u\frac{R^\mu R^\nu}{\rho^2} - a_u\frac{v^\mu R^\nu + v^\nu R^\mu}{\rho} + \frac{a^\mu R^\nu + a^\nu R^\mu}{\rho}\right) \\ &+ \frac{q^2}{4\pi\rho^2}(a_u^2 - a^\lambda a_\lambda)\frac{R^\mu R^\nu}{\rho^2} \end{aligned} \tag{2.33}$$

The energy flux is given by

$$\mathbf{S} = \mathbf{E}\times\mathbf{B} = \left(\frac{q}{4\pi}\right)^2\left|\frac{\mathbf{n}\times\{\mathbf{n}\times d\mathbf{v}/dt\}}{R}\right|^2\mathbf{n} \tag{2.34}$$

and integrating over a sphere gives the Larmor formula for power radiated

$$P = \frac{2}{3}\left(\frac{q^2}{4\pi}\right)\left(\frac{d\mathbf{v}}{dt}\right)^2 \tag{2.35}$$

agreeing with Coleman (eq. 3.172). Purcell and Morin write (eq. H.7)

$$P = \frac{2}{3}\left(\frac{q^2}{4\pi\epsilon_0 c^3}\right)\left(\frac{d\mathbf{v}}{dt}\right)^2 \tag{2.36}$$

The relativistic generalization given by Liénard is (Jackson (2e), eq. 14.26)

$$P = \frac{2}{3}\left(\frac{q^2}{4\pi}\right)\gamma^6\left[\left(\frac{d\mathbf{v}}{dt}\right)^2 - \left(\mathbf{v}\times\frac{d\mathbf{v}}{dt}\right)^2\right] \tag{2.37}$$

2.2 Regularization

Let's alter the action to take the form

$$S = \int d^4x \left(-\tfrac{1}{4}\overline{F}_{\mu\nu}\overline{F}^{\mu\nu}\right) + \int d\tau \left(-\frac{1}{2}m_0\eta_{\mu\nu}\frac{dy^\mu}{d\tau}\frac{dy^\nu}{d\tau} - q\int d^4x\ \overline{A}_\mu(x)f(x-y(\tau))\frac{dy^\mu}{d\tau}\right) \quad (2.38)$$

where $f(x) = \lambda^4 F(\lambda x)$ for any $F(x)$ obeying $\int d^4x\ F(x) = 1$. We can write the electromagetic action as

$$S_{\text{EM}} = \int d^4x \left(-\tfrac{1}{4}\overline{F}_{\mu\nu}\overline{F}^{\mu\nu} - \overline{J}^\mu\overline{A}_\mu\right) \quad (2.39)$$

where

$$\begin{aligned}\overline{J}^\mu(x) &= q\int d\tau\ f(x-y(\tau))\frac{dy^\mu}{d\tau}\\ &= q\int d\tau\ \int d^4x'\ f(x-x')\delta^{(4)}(x'-y(\tau))\frac{dy^\mu}{d\tau}\\ &= \int d^4x'\ f(x-x')J^\mu(x')\end{aligned} \quad (2.40)$$

Because Maxwell's equation is linear

$$\partial_\mu\overline{F}^{\mu\nu} = \overline{J}^\nu \quad (2.41)$$

the solution is

$$\overline{F}^{\mu\nu}(x) = \int d^4x'\ f(x-x')F^{\mu\nu}(x') \quad (2.42)$$

where $F^{\mu\nu}$ is the solution of

$$\partial_\mu F^{\mu\nu} = J^\nu \quad (2.43)$$

Now using $F_{\mu\nu}$ from above, and expanding in λ^{-1}, Coleman obtains

$$\overline{F}^{\mu\nu} = \lambda\frac{q}{8\pi}\left(\frac{dy^\mu}{d\tau}\frac{d^2y^\nu}{d\tau^2} - \frac{d^2y^\mu}{d\tau^2}\frac{dy^\nu}{d\tau}\right) + \frac{q}{6\pi}\left(\frac{d^3y^\mu}{d\tau^3}\frac{dy^\nu}{d\tau} - \frac{dy^\mu}{d\tau}\frac{d^3y^\nu}{d\tau^3}\right) + \mathcal{O}(1/\lambda^2) \quad (2.44)$$

Now plug this into

$$m_0\frac{d^2y^\mu}{d\tau^2} = q\overline{F}^{\mu\nu}\frac{dy_\nu}{d\tau} + F^\mu_{\text{ext}} \quad (2.45)$$

(I am not sure about this! shouldn't there be dependence on $f(x)$ due to the altered action eq. (**??**)?) to obtain

$$m_0\frac{d^2y^\mu}{d\tau^2} = \frac{\lambda q^2}{8\pi}\left(-\frac{d^2y^\mu}{d\tau^2}\right) + \frac{q^2}{6\pi}\left(\frac{d^3y^\mu}{d\tau^3} + \frac{dy^\mu}{d\tau}\frac{d^2y^\nu}{d\tau^2}\frac{d^2y_\nu}{d\tau^2}\right) \quad (2.46)$$

where we use

$$\begin{aligned}(dy^\nu/d\tau)(dy_\nu/d\tau) &= 1 \\ (d^2y^\nu/d\tau^2)(dy_\nu/d\tau) &= 0 \\ (d^3y^\nu/d\tau^3)(dy_\nu/d\tau) &= -(d^2y^\nu/d\tau^2)(d^2y_\nu/d\tau^2)\end{aligned} \tag{2.47}$$

Defining the renormalized mass as

$$m = m_0 + \frac{\lambda q^2}{8\pi} \tag{2.48}$$

we can write

$$\begin{aligned} m\frac{d^2y^\mu}{d\tau^2} &= F^\mu_{\text{rad}} + F^\mu_{\text{ext}} \\ F^\mu_{\text{rad}} &= \frac{2}{3}\frac{q^2}{4\pi}\left(\frac{d^3y^\mu}{d\tau^3} + \frac{dy^\mu}{d\tau}\frac{d^2y^\nu}{d\tau^2}\frac{d^2y_\nu}{d\tau^2}\right)\end{aligned} \tag{2.49}$$

In the nonrelativistic limit (and restoring units) we have

$$m\frac{d^2\mathbf{y}}{dt^2} = \frac{2}{3}\frac{q^2}{4\pi\epsilon_0 c^3}\frac{d^3\mathbf{y}}{dt^3} + \mathbf{F}_{\text{ext}} \tag{2.50}$$

For a free particle this has a runaway solution

$$\mathbf{y} = \mathbf{a} + \mathbf{b}t + \mathbf{c}e^{t/\lambda}, \tag{2.51}$$

where[1]

$$\lambda = \frac{2}{3}\frac{q^2}{4\pi\epsilon_0}\frac{1}{mc^3} = \frac{2}{3}\alpha\frac{\hbar c}{mc^3} = 6\times 10^{-24}\ \text{s} \tag{2.52}$$

For a linear restoring force, the equation of motion is

$$\frac{d^2\mathbf{y}}{dt^2} = \lambda\frac{d^3\mathbf{y}}{dt^3} - \omega_0^2\mathbf{y} \tag{2.53}$$

The ansatz $\mathbf{y} = \mathbf{y}_0 e^{i\omega t}$ yields the third order equation

$$\omega^2 = i\lambda\omega^3 + \omega_0^2 \tag{2.54}$$

two of the solutions of which are approximately (for small $\lambda\omega_0$)

$$\omega = \pm\omega_0 + \tfrac{1}{2}i\lambda\omega_0^2 \tag{2.55}$$

[1] Coleman, eq. 3.158 says 5×10^{-25} s, but e is written in Gaussian units, so answer is off by 4π. Footnote 8 on p. 66 clarifies that Coleman is using Heaviside-Lorentz units, $\epsilon_0 = 1 = c$.

and thus

$$\mathbf{y} = \mathbf{y}_0 e^{-\frac{1}{2}\lambda\omega_0^2 t} e^{\pm i\omega_0 t} \tag{2.56}$$

A real solution is of the form

$$\mathbf{y} = \mathbf{y}_0 e^{-\frac{1}{2}\lambda\omega_0^2 t} \cos(\omega_0 t + \phi) \tag{2.57}$$

The time-averaged mechanical energy of the oscillator is

$$E_{\text{mech}} = K + U = \tfrac{1}{2} m\omega_0^2 y_0^2 e^{-\lambda\omega_0^2 t} \tag{2.58}$$

and the rate of energy loss

$$\begin{aligned} \frac{dE_{\text{mech}}}{dt} &= -\tfrac{1}{2}\lambda m\omega_0^4 y_0^2 e^{-\lambda\omega_0^2 t} \\ &= -\frac{1}{3}\left(\frac{q^2}{4\pi c^3}\right)\omega_0^4 y_0^2 e^{-\lambda\omega_0^2 t} \end{aligned} \tag{2.59}$$

The time average of the square of the acceleration is

$$\left|\frac{d\mathbf{v}}{dt}\right|^2 = \tfrac{1}{2}\omega_0^4 y_0^2 e^{-\lambda\omega_0^2 t} \tag{2.60}$$

and thus the energy loss is precisely consistent with the Larmor formula

$$\frac{dE_{\text{mech}}}{dt} = -P = -\frac{2}{3}\left(\frac{q^2}{4\pi c^3}\right)\left(\frac{d\mathbf{v}}{dt}\right)^2 \tag{2.61}$$

3 Gravitational scattering

A relativistic particle in a gravitational field has action

$$S = -m\int d\tau = -m\int d\sigma \left(\frac{d\tau}{d\sigma}\right) = -m\int d\sigma \sqrt{g_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}}\,. \tag{3.1}$$

Euler's equations then yield

$$\frac{d}{d\sigma}\left(\frac{g_{\mu\nu}(dx^\nu/d\sigma)}{(d\tau/d\sigma)}\right) = \tfrac{1}{2}\partial_\mu g_{\kappa\lambda}\frac{(dx^\kappa/d\sigma)(dx^\lambda/d\sigma)}{(d\tau/d\sigma)}\,. \tag{3.2}$$

Setting $\sigma = \tau$, we have

$$\frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\nu}{d\tau}\right) = \tfrac{1}{2}\partial_\mu g_{\kappa\lambda}\frac{dx^\kappa}{d\tau}\frac{dx^\lambda}{d\tau}\,. \tag{3.3}$$

We can rewrite this as

$$\begin{aligned} g_{\mu\nu}\frac{d^2x^\nu}{d\tau^2} &= \tfrac{1}{2}\partial_\mu g_{\kappa\lambda}\frac{dx^\kappa}{d\tau}\frac{dx^\lambda}{d\tau} - \partial_\lambda g_{\mu\nu}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} \\ &= \tfrac{1}{2}\left[\partial_\mu g_{\kappa\lambda}\frac{dx^\kappa}{d\tau}\frac{dx^\lambda}{d\tau} - \partial_\lambda g_{\mu\kappa}\frac{dx^\kappa}{d\tau}\frac{dx^\lambda}{d\tau} - \partial_\kappa g_{\mu\lambda}\frac{dx^\kappa}{d\tau}\frac{dx^\lambda}{d\tau}\right] \end{aligned} \tag{3.4}$$

which gives the usual geodesic equation:

$$\frac{d^2x^\nu}{d\tau^2} = -\tfrac{1}{2}g^{\nu\mu}\left[\partial_\lambda g_{\mu\kappa} + \partial_\kappa g_{\mu\lambda} - \partial_\mu g_{\kappa\lambda}\right]\frac{dx^\kappa}{d\tau}\frac{dx^\lambda}{d\tau} = \quad - \;\Gamma^\nu_{\kappa\lambda}\frac{dx^\kappa}{d\tau}\frac{dx^\lambda}{d\tau}\,. \tag{3.5}$$

Returning to eq. (**??**), we can rewrite it as

$$\frac{dp_\mu}{d\tau} = \tfrac{1}{2}\partial_\mu g_{\kappa\lambda}\frac{dx^\kappa}{d\tau}p^\lambda \qquad \text{where} \qquad p^\mu \equiv m\frac{dx^\mu}{d\tau}\,. \tag{3.6}$$

Since the Schwarzchild metric in isotropic coordinates

$$ds^2 = A(r)(dx^0)^2 - B(r)(d\mathbf{x})^2, \qquad\qquad r = \mathbf{x}^2 \tag{3.7}$$

is independent of x^0, i.e., $\partial_0 g_{\mu\nu} = 0$, we can obtain a conserved quantity from eq. (**??**)

$$p_0 = mA\frac{dx^0}{d\tau} = \text{const} = E \tag{3.8}$$

where we have evaluated the constant at $r \to \infty$ where $A \to 1$. Then

$$d\tau^2 = A(dx^0)^2 - B(d\mathbf{x})^2 \tag{3.9}$$

together with

$$dx^0 = \frac{E}{mA} d\tau \tag{3.10}$$

implies

$$\begin{aligned} B(d\mathbf{x})^2 &= \left[\frac{E^2}{m^2 A} - 1\right] (d\tau)^2 \\ |d\mathbf{x}| &= \frac{1}{m}\sqrt{\frac{E^2 - m^2 A}{AB}} d\tau \end{aligned} \tag{3.11}$$

from which we have

$$\frac{dx^0}{d\tau} = \frac{E}{mA}, \qquad \frac{|d\mathbf{x}|}{dx^0} = \frac{A}{E}\sqrt{\frac{E^2 - m^2 A}{AB}} \qquad \frac{|d\mathbf{x}|}{d\tau} = \frac{1}{m}\sqrt{\frac{E^2 - m^2 A}{AB}}\,. \tag{3.12}$$

In isotropic coordinates, the $\mu = i$ Euler equation (**??**) is

$$\begin{aligned} \frac{d}{d\tau}\left(-B\frac{d\mathbf{x}}{d\tau}\right) &= \frac{1}{2}\left[\left(\frac{dx^0}{d\tau}\right)^2 \nabla A - \left(\frac{d\mathbf{x}}{d\tau}\right)^2 \nabla B\right] \\ &= \frac{1}{2}\left[\left(\frac{E}{mA}\right)^2 \nabla A - \left(\frac{E^2 - m^2 A}{m^2 AB}\right) \nabla B\right]. \end{aligned} \tag{3.13}$$

Can we also start from ?

$$S = \int dt \sqrt{A - B\left(\frac{d\mathbf{x}}{dt}\right)^2} \tag{3.14}$$

To first order in the gravitational field we have

$$A \approx 1 + 2\phi, \qquad B \approx 1 - 2\phi \tag{3.15}$$

so that eq. (**??**) becomes

$$\begin{aligned} \frac{d^2\mathbf{x}}{d\tau^2} &= -\ \gamma^2(1+\beta^2)\nabla\phi \\ \frac{d}{d\tau}\left(m\frac{d\mathbf{x}}{d\tau}\right) &= -\ m\gamma^2(1+\beta^2)\nabla\phi \\ \frac{d\mathbf{p}}{dt} &= -\ m\gamma(1+\beta^2)\nabla\phi\,. \end{aligned} \tag{3.16}$$

Equivalently, eq. (**??**) becomes

$$dp_i = \tfrac{1}{2}\left[\partial_i A\; p^0 dx^0 - \partial_i B\; \mathbf{p}\cdot d\mathbf{x}\right] = \partial_i\phi\left[p^0 dx^0 + \mathbf{p}\cdot d\mathbf{x}\right] \tag{3.17}$$

where the last equality holds to first order in the gravitational field. If we let then $p^0 = m\gamma$, $p^i = m\gamma\beta^i$, and $dx^i = \beta^i dx^0$, then

$$dp^i = -m\gamma(1+\beta^2)\partial_i\phi\; dx^0 = m\gamma(1+\beta^2) g^i\; dx^0 \tag{3.18}$$

the same as in eq. (**??**).