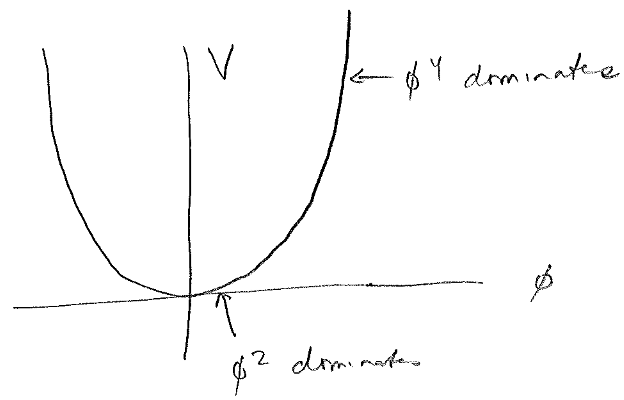
Spontaneous symmetry breaking

Classical real scalar field ϕ

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - V(\phi) \quad \Rightarrow \quad \partial^2\phi + \frac{\partial V}{\partial\phi} = 0$$

Consider

$$V(\phi) = \frac{1}{2}\mu^2\phi^2 + \frac{1}{4}\lambda\phi^4 \quad \Rightarrow \quad \underbrace{(\partial^2+\mu^2)\phi = \lambda\phi^3}_{\text{nonlinear eqn}}$$



symmetric under $\phi \to -\phi$

$\phi = 0$ is a stable solution of the e.o.m.

For ϕ small, e.o.m. is $(\partial^2+\mu^2)\phi \approx 0$ (free field)

Approximate solution

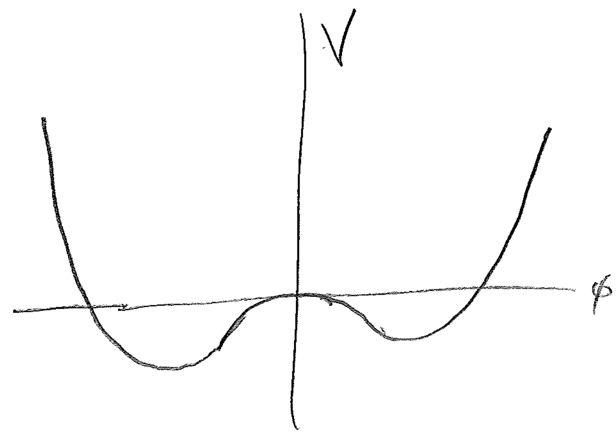
$$\phi(x,t) \approx \int \widetilde{dk}\,\left[a(k)e^{-ik\cdot x} + a^*(k)e^{ik\cdot x}\right]\Big|_{k^0=\omega_k}$$

This is valid provided coefficients $a(k)$ are small enough that cubic term is negligible

Quantize in usual way.

Instead consider

$$V(\phi) = -\tfrac{1}{2}\mu^2\phi^2 + \tfrac{1}{4}\lambda\phi^4 \Rightarrow (\partial^2 - \mu^2)\phi = \lambda\phi^3$$



symmetric under $\phi \to -\phi$

$\phi = 0$ is an unstable solution to e. om.

e.g. if $\phi(x,t) = f(t)$ (spatially uniform)

$$\ddot{\phi} - \mu^2\phi \approx 0 \quad \text{for small } \phi$$

$$\phi \simeq C e^{\pm\mu t}$$

Try ansatz $\phi = e^{-ik\cdot x}$

$$(\partial^2 - \mu^2)\phi = (-k^2 - \mu^2)\phi \approx 0$$

$$k^2 = -\mu^2$$

$$k^0 = \pm\sqrt{-\mu^2 + \vec{k}^2}$$

If $\vec{k}^2 < \mu^2$ then $k^0 = \pm i\sqrt{\mu^2 - \vec{k}^2}$

$$\phi = e^{\pm\sqrt{\mu^2 - \vec{k}^2}\, t + i\vec{k}\cdot\vec{x}}$$

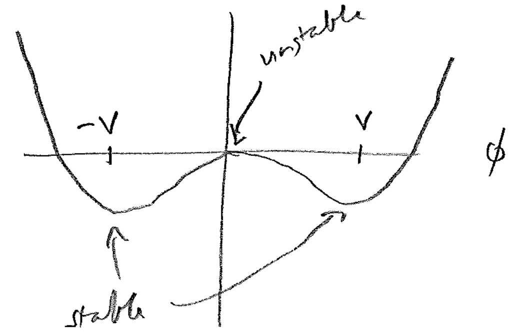
If interpret as particles

$$E^2 = \vec{p}^2 - \hbar^2\mu^2 \Rightarrow m = i\hbar\mu$$

$$E \leq |\vec{p}|$$

$$v = \frac{|p|}{E} > 1 \quad \text{tachyons}$$

Problem: we're expanding around the wrong "vacuum state"



Find stable minima of potential. $\frac{\partial V}{\partial \phi} = 0$

$$V = -\frac{1}{2}\mu^2\phi^2 + \frac{1}{4}\lambda\phi^4$$

$$\frac{\partial V}{\partial \phi} = -\mu^2\phi + \lambda\phi^3$$

$$= \lambda\phi\left(\phi^2 - \frac{\mu^2}{\lambda}\right) \qquad \text{Define } v^2 = \frac{\mu^2}{\lambda}$$

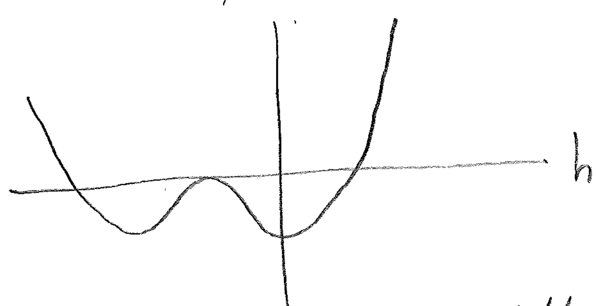
$$= \lambda\phi(\phi^2 - v^2)$$

$$\Rightarrow \phi = 0, \pm v$$

Define a new field by shifting ϕ

$$\phi = v + h$$

(could alternatively shift $\phi = -v + h$)



$\phi \to -\phi$ symmetry is hidden ("broken")

Now $h = 0$ is a stable solution

Recall from

$$0 = \partial^2\phi + \frac{\partial V}{\partial \phi}$$

$$= \partial^2\phi + \lambda\phi(\phi+v)(\phi-v)$$

$$= \partial^2 h + \lambda(v+h)(2v+h)h$$

$$= \partial^2 h + 2\lambda v^2 h + 3\lambda v h^2 + \lambda h^3$$

For small h:

$$\partial^2 h + 2\lambda v^2 h \approx 0$$

Ansatz $h = e^{-ik\cdot x}$

$$\Rightarrow k^2 = 2\lambda v^2 = 2\mu^2 \qquad \Rightarrow k^0 = \pm\sqrt{\vec{k}^2 + 2\mu^2}$$

$$h(x,t) \approx \int \tilde{dk}\, \left[a(k)\, e^{-ik\cdot x} + a^*(k)\, e^{ik\cdot x}\right]\Big|_{k^0 = \sqrt{\vec{k}^2 + 2\mu^2}}$$

When we quantize, $a, a^\dagger$ = annihilation & creation ops.

$$p^\mu = \hbar k^\mu$$

$$p^2 = \hbar^2 k^2 = 2\hbar^2\mu^2 \equiv m^2$$

so $m = \sqrt{2}\,\hbar\mu$ ← (Higgs boson)

$V = \frac{\lambda}{4}(\phi^2 - v^2)^2 + \text{const}$

$\frac{\partial V}{\partial \phi} = \lambda(\phi^2 - v^2)\phi$

$\frac{\partial^2 V}{\partial \phi^2} = \lambda(\phi^2 - v^2) + 2\lambda\phi^2$

$= 2\lambda v^2$

More generally: $\partial^2 h + \left(\frac{\partial^2 V}{\partial \phi^2}\right)\Big|_{\phi = v} h + \ldots$

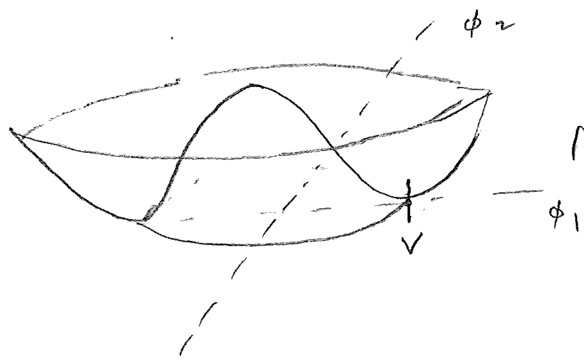
so $m^2_{\text{Higgs}} = \hbar^2 \left(\frac{\partial^2 V}{\partial \phi^2}\right)\Big|_{\phi = v}$

<u>Complex scalar field</u> $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$

$$\mathcal{L} = |\partial\phi|^2 - V(|\phi|^2)$$

Consider

$$V(|\phi|^2) = -\mu^2|\phi|^2 + \lambda|\phi|^4$$
$$= -\frac{\mu^2}{2}(\phi_1^2+\phi_2^2) + \frac{\lambda}{4}(\phi_1^2+\phi_2^2)^2$$



Mexican hat potential
SO(2) symmetry
= U(1) symmetry

If $\phi_2 = 0$, same potential as before, w/ minima $\phi_1 = \pm v$, $v = \frac{\mu}{\sqrt{\lambda}}$

$V(|\phi|^2)$ is minimized on the circle $\phi_1^2 + \phi_2^2 = v^2$ $\Rightarrow$

$$\Rightarrow \quad \phi_1 = v\cos\theta$$
$$\phi_2 = v\sin\theta$$

or equivalently $\phi = \frac{1}{\sqrt{2}}(v\cos\theta + i v\sin\theta) = \frac{1}{\sqrt{2}} v e^{i\theta}$

Can rewrite (up to additive constant)

$$V(|\phi|^2) = \frac{\lambda}{4}(\phi_1^2+\phi_2^2 - v^2)^2 = \lambda\left(|\phi|^2 - \frac{v^2}{2}\right)^2$$

As before shift the field to minimum.

Arbitrarily choose minimum at $\theta = 0$

$$\begin{cases} \phi_1 = v + h \\ \phi_2 = \eta \end{cases}$$

SO(2) symmetry is hidden ("broken")

or $$\phi = \frac{1}{\sqrt{2}}(v + h + i\eta)$$

$$|\phi|^2 = \frac{1}{2}(v+h)^2 + \frac{1}{2}\eta^2$$

$$|\phi|^2 - \frac{v^2}{2} = vh + \frac{1}{2}h^2 + \frac{1}{2}\eta^2$$

$$V(|\phi|^2) = \lambda\left(|\phi|^2 - \frac{v^2}{2}\right)^2$$

$$= \lambda\left(vh + \frac{1}{2}h^2 + \frac{1}{2}\eta^2\right)^2$$

$$= \underbrace{\lambda v^2}_{\mu^2} h^2 + \text{terms cubic \& quartic in } h \text{ \& } \eta$$

$$\partial_\mu \phi = \frac{1}{\sqrt{2}}(\partial_\mu h + i\partial_\mu \eta)$$

$$|\partial\phi|^2 = \frac{1}{2}(\partial h)^2 + \frac{1}{2}(\partial\eta)^2$$

$$\mathcal{L} = |\partial\phi|^2 + V(|\phi|^2)$$

$$= \underbrace{\frac{1}{2}(\partial h)^2 + \frac{1}{2}(2\mu^2)h^2}_{\text{real scalar w/ mass } m = \sqrt{2}\,\hbar\mu \text{ (Higgs boson)}} + \underbrace{\frac{1}{2}(\partial\eta)^2}_{\text{real massless scalar (Goldstone boson)}} + \text{cubic + quartic self interactions}$$

↓

talk about trough

Abelian Higgs model

Complex scalar field w/ spontaneously broken $U(1)$ symmetry minimally coupled to an abelian vector field

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + |D\phi|^2 - V(|\phi|^2)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$V(|\phi|^2) = \lambda(|\phi|^2 - \tfrac{1}{2}v^2)^2$$

Let $\phi = \frac{1}{\sqrt{2}}(v + h + i\eta)$

$$D_\mu\phi = \partial_\mu\phi + iqA_\mu\phi$$

$$= \frac{1}{\sqrt{2}}(\partial_\mu h + i\partial_\mu\eta + iqA_\mu[v+h+i\eta])$$

$$= \frac{1}{\sqrt{2}}(\partial_\mu h - q\eta A_\mu) + \frac{i}{\sqrt{2}}(\partial_\mu\eta + qvA_\mu + qhA_\mu)$$

$$|D_\mu\phi|^2 = \frac{1}{2}(\partial_\mu h - q\eta A_\mu)^2 + \frac{1}{2}(\partial_\mu\eta + qvA_\mu + qhA_\mu)^2$$

As before

$$V(|\phi|^2) = \lambda(vh + \tfrac{1}{2}h^2 + \tfrac{1}{2}\eta^2)^2$$

$\mathcal{L}$ has quadratic, cubic & quartic terms.
Keep only quadratic

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu h)^2 + \frac{1}{2}(\partial_\mu\eta + qvA_\mu)^2 + \lambda v^2 h^2$$

has weird cross term
$A^\mu\partial_\mu\eta$

Define a new field

$$B_\mu = A_\mu + \frac{1}{qv}\partial_\mu \eta$$

Then

$$\frac{1}{2}(\partial_\mu \eta + qv A_\mu)^2 = \frac{1}{2}(qv)^2 B_\mu B^\mu$$

Observe

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \partial_\mu \left(B_\nu - \frac{1}{qv}\partial_\nu \eta\right) - \partial_\nu \left(B_\mu - \frac{1}{qv}\partial_\mu \eta\right)$$

$$= \partial_\mu B_\nu - \partial_\nu B_\mu \quad \Rightarrow \text{gauge transformat}$$

$$\mathcal{L} = \underbrace{-\frac{1}{4}(\partial_\mu B_\nu - \partial_\nu B_\mu)^2 + \frac{1}{2}(qv)^2 B_\mu B^\mu}_{\substack{\text{vector field } B_\mu \\ \text{w/ mass } \mu_B = qv}} + \underbrace{\frac{1}{2}(\partial_\mu h)^2 + \frac{1}{2}(2\lambda v^2)h^2}_{\substack{\text{Higgs field w/ mass} \\ \mu = \sqrt{2\lambda}\, v}}$$

Even though $\mathcal{L}$ is gauge invariant, vector field gets a mass.

massless vector field A_μ eats a goldstone boson η and gets heavy

[Under a gauge tx:
$A^\mu \to A^\mu - \partial^\mu \chi$
$\eta \to \eta - qv\chi$
$B^\mu \to B^\mu$]