Dirac Lagrangian

Can we obtain Dirac eqn $(i\not\partial - \mu)\psi = 0$ as Euler-Lagrange eqn

$$\frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \right) = 0 \ ?$$

Consider $\mathcal{L} = \overline{\Psi}(i\not\partial - \mu)\psi = \overline{\Psi}(i\gamma^\mu \partial_\mu \psi - \mu\psi)$

where ψ and $\overline{\Psi}$ are independent (complex) fields.

What is $\overline{\Psi}$? We'll see.

(Later we'll see how ψ & $\overline{\Psi}$ are related.)

Vary $\overline{\Psi}$ $\Rightarrow$ $\dfrac{\partial \mathcal{L}}{\partial \overline{\Psi}} - \partial_\mu \left(\dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \overline{\Psi})} \right) = 0$

$$\left.\begin{aligned} \frac{\partial \mathcal{L}}{\partial \overline{\Psi}} &= (i\not\partial - \mu)\psi \\ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \overline{\Psi})} &= 0 \end{aligned}\right\} \Rightarrow (i\not\partial - \mu)\psi = 0$$

Vary ψ

$$\left.\begin{aligned} \frac{\partial \mathcal{L}}{\partial \psi} &= -\overline{\Psi}\mu \\ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} &= \overline{\Psi} i\gamma^\mu \end{aligned}\right\} \quad \begin{aligned} -\overline{\Psi}\mu - \partial_\mu(\overline{\Psi} i\gamma^\mu) &= 0 \\ \overline{\Psi}(-\mu - i\gamma^\mu \overleftarrow{\partial}_\mu) &= 0 \\ \overline{\Psi}(-i\overleftarrow{\not\partial} - \mu) &= 0 \end{aligned}$$

Another way to derive this.

$$S = \int d^4x\, \mathcal{L} = \int d^4x\, \bar{\psi}(i\partial\!\!\!/ - \mu)\psi \underset{\substack{\uparrow \\ [\text{IBP}]}}{=} \int d^4x\, \bar{\psi}(-i\overleftarrow{\partial\!\!\!/} - \mu)\psi$$

$$\left.\begin{aligned} \frac{\partial \mathcal{L}}{\partial \psi} &= \bar{\psi}(i\overleftarrow{\partial\!\!\!/} - \mu) \\ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} &= 0 \end{aligned}\right\} \Rightarrow \bar{\psi}(i\overleftarrow{\partial\!\!\!/} - \mu) = 0$$

Consider $0 = (i\partial\!\!\!/ - \mu)\psi = i\gamma^\mu \partial_\mu \psi - \mu\psi$

Take hermitian conjugate $(\mu^* = \mu)$

$$0 = -i(\partial_\mu \psi)^\dagger (\gamma^\mu)^\dagger - \psi^\dagger \mu$$

From HW: $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$

$$0 = -i\psi^\dagger \overleftarrow{\partial_\mu}(\gamma^0\gamma^\mu\gamma^0) - \psi^\dagger\mu$$

Multiply on right by γ^0 & use $(\gamma^0)^2 = 1$

$$0 = \psi^\dagger\gamma^0(-i\overleftarrow{\partial\!\!\!/} - \mu)$$

This suggests that $\bar{\psi} = \psi^\dagger\gamma^0$

[In fact, one can show that $\int d^4x\, \psi^\dagger\gamma^0(i\partial\!\!\!/ - \mu)\psi$ is Lorentz invariant, & is a suitable action]

Invariance under space-time translation

$\Rightarrow$ conserved energy, momentum tensor

$$T_c^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \partial^\nu \psi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} \partial^\nu \bar{\psi} - \eta^{\mu\nu} \mathcal{L}$$

$$= \bar{\psi} i\gamma^\mu \partial^\nu \psi + 0 - \eta^{\mu\nu} \bar{\psi} \underbrace{(i\not{\partial} - \mu)\psi}_{0}$$

$$T_c^{\mu\nu} = \bar{\psi} i\gamma^\mu \partial^\nu \psi$$

$$T_c^{00} = \bar{\psi} i\gamma^0 \partial^0 \psi$$

$$= \psi^\dagger i\partial^0 \psi$$

$$\mathcal{H} = T^{00} = \bar{\psi}\, i\gamma^0 \partial^0 \psi = i\psi^\dagger \dot{\psi}$$

Recall

$$\psi = \int \widetilde{dk} \sum_s \left[a_s(k)\, u_{ks}\, e^{-ik\cdot x} + b_s^\dagger(k)\, v_{ks}\, e^{ik\cdot x} \right]\Big|_{k^0 = \omega_k}$$

$$i\psi^\dagger = \int \widetilde{dk} \sum_s \left[i a_s^\dagger(k)\, u_{ks}^\dagger\, e^{ik\cdot x} + i b_s(k)\, v_{ks}^\dagger\, e^{-ik\cdot x} \right]$$

$$\dot{\psi} = \int \widetilde{dk} \sum_s \left[-i\omega_k a_s(k)\, u_{ks}\, e^{-ikx} + i\omega_k b_s^\dagger(k)\, v_{ks}\, e^{ik\cdot x} \right]$$

$$H = \int d^3x\, \mathcal{H} = \int d^3x\ i\psi^\dagger \dot{\psi}$$

$$= \int \widetilde{dk}\, \widetilde{dk'} \sum_{s,s'} \Big[\omega_{k'} a_s^\dagger(k)\, a_{s'}(k')\, \underbrace{u_{ks}^\dagger u_{k's'}}_{(2\omega_k)\delta_{ss'}} \underbrace{\int d^3x\ e^{i(k-k')\cdot x}}_{(2\pi)^3 \delta^{(3)}(\vec{k}-\vec{k}')}$$

$$- \omega_{k'} b_s(k)\, b_{s'}^\dagger(k')\, \underbrace{v_{ks}^\dagger v_{k's'}}_{(2\omega_k)\delta_{ss'}} \underbrace{\int d^3x\ e^{-i(k-k')\cdot x}}_{(2\pi)^3 \delta^{(3)}(\vec{k}-\vec{k}')}$$

$$- \omega_{k'} a_s^\dagger(k)\, b_{s'}^\dagger(k')\, \underbrace{u_{ks}^\dagger v_{k's'}}_{u_{ks}^\dagger v_{-ks'}} \underbrace{\int d^3x\ e^{i(k+k')\cdot x}}_{(2\pi)^3 e^{i(\omega_k+\omega_{k'})t} \delta^{(3)}(\vec{k}+\vec{k}')}$$

$$+ \omega_{k'} b_s(k)\, a_{s'}(k')\, \underbrace{v_{ks}^\dagger u_{k's'}}_{v_{ks}^\dagger u_{-ks'}} \underbrace{\int d^3x\ e^{-i(k+k')\cdot x}}_{(2\pi)^3 e^{-i(\omega_k+\omega_{k'})t} \delta^{(3)}(\vec{k}+\vec{k}')} \Big]$$

[HW: compute p^i]

Consider

$$u^{\dagger}_{ks} v_{-k,s'} = \left(\sqrt{\omega - s|k|}\,\chi^{\dagger}_{ks},\ \sqrt{\omega + s|k|}\,\chi^{\dagger}_{ks}\right) \begin{pmatrix} \sqrt{\omega - s'|k|}\,\chi_{-k,s'} \\ -\sqrt{\omega + s'|k|}\,\chi_{-ks'} \end{pmatrix}$$

$$= \left(\sqrt{\omega - s|k|}\sqrt{\omega - s'|k|} - \sqrt{\omega + s|k|}\sqrt{\omega + s'|k|}\right) \chi_{ks}\,\chi_{-k,s'}$$

Now $\chi^{\dagger}_{ks}\chi_{ks'} = \delta_{ss'}$

but $\chi_{-k,s'} \sim \chi_{k,-s'}$ spin up wrt. $\vec{k}$ is spin down wrt $-\vec{k}$

so $\chi_{ks}\,\chi_{-ks'} \neq 0$ iff $s' = -s$

If $s' = -s$, prefactor vanishes, so $u^{\dagger}_{ks} v_{-k,s'} = 0$

Similarly $v^{\dagger}_{ks} u_{-k,s'} = 0$ [Peskin Schroeder 3.65]

Therefore

$$\mathbf{H} = \int dk \sum_s \left[\omega_k\, a^{\dagger}_s(k)\, a_s(k) - \omega_k\, b_s(k)\, b^{\dagger}_s(k)\right]$$

$$= \int dk \sum_s \Big[\omega_k\, a^{\dagger}_s(k)\, a_s(k) - \omega_k \underbrace{[b_s(k), b^{\dagger}_s(k)]}_{\text{infinite zero point energy}} - \omega_k\, b^{\dagger}_s(k)\, b_s(k)\Big]$$

This is not correct!

$a^{\dagger}_s$ creates positive energy states $\hbar\omega_k$

[Houston, we have a problem!]

Experimentally, electrons (+ other spin ½ particles) obey Pauli exclusion principle (a.k.a. Fermi-Dirac statistics): can't have more than one fermion in same state.

$|k, s\rangle = a_s^\dagger(k)|0\rangle$ = fermion w/ mom. k + helicity s

$|k, s; k', s'\rangle = a_s^\dagger(k)\, a_{s'}^\dagger(k')|0\rangle$ = 2 fermion state

But $|k, s; k, s\rangle$ should be impossible.

In 1928, Jordan + Wigner proposed that creation operators for fermions <u>anticommute</u> w/ one another

$$a_s^\dagger(k)\, a_{s'}^\dagger(k') = -\, a_{s'}^\dagger(k')\, a_s^\dagger(k)$$

$$\Rightarrow |k, s; k', s'\rangle = -\,|k', s'; k, s\rangle$$

(antisymmetric under exchange)

$$\Rightarrow |k, s; k, s\rangle = -\,|k, s; k, s\rangle = 0.$$

Define anticommutator $\{P, Q\} = PQ + QP$.

Thus $\{a_s^\dagger(k), a_{s'}^\dagger(k)\} = \{b_s^\dagger(k), b_{s'}^\dagger(k')\} = 0$

Also $\{a_s(k), a_{s'}(k')\} = \{b_s(k), b_{s'}(k')\} = 0$

Instead of $[a, a^+]$ and $[b, b^+]$, we impose

$$\{a_s(k), a^+_{s'}(k')\} = \{b_s(k), b^+_{s'}(k')\} = \hbar (2\pi)^3 (2\omega_k) \delta^{(3)}(k-k') \delta_{ss'}$$
$$\{a_s(k), b_{s'}(k')\} = \{a_s(k), b^+_{s'}(k')\} = 0, \text{ etc.}$$

Thus

$$H = \int d\tilde{k} \sum_s [\omega_k a^+_s(k) a_s(k) - \omega_k b_s(k) b^+_s(k)]$$
$$= \int d\tilde{k} \sum_s [\omega_k a^+_s(k) a_s(k) - \omega_k \underbrace{\{b_s(k), b^+_s(k)\}}_{\hbar (2\omega_k)(\text{vol space})} + \omega_k b^+_s(k) b_s(k)]$$

$$= \int d\tilde{k} \sum_s \omega_k [a^+_s(k) a_s(k) + b^+_s(k) b_s(k)] - \underbrace{\int \frac{d^3k}{(2\pi)^3} \sum_s \hbar\omega_k}_{\text{negative zero point energy}}$$

positive energy for both degrees of freedom (particle + antiparticle)

negative zero point energy ($\frac{\hbar\omega_k}{2}$ per degree of freedom (ie particle + antiparticle))

[Perhaps zero cosmological constant is result of equal # of bosonic + fermionic degrees of freedom

$\Rightarrow$ supersymmetry?]

Finally, let's check $\left[H,\ b^\dagger_{s'}(k')\right] = \hbar\omega_{k'}\, b^\dagger_{s'}(k')$

$$H\, b^\dagger_{s'}(k') = \int \widetilde{dk} \sum_s \omega_k \left(a^\dagger_s(k)\, a_s(k) + b^\dagger_s(k)\, b_s(k) \right) b^\dagger_{s'}(k')$$

$$= b^\dagger_{s'}(k') \int \widetilde{dk} \sum_s \omega_k \left(a^\dagger_s(k)\, a_s(k) + b^\dagger_s(k)\, b_s(k) \right)$$

$$+ \int \cancel{\frac{d^3k}{(2\pi)^3}}\, \cancel{\frac{1}{2\omega_k}} \sum_s \omega_k\, b^\dagger_s(k) \underbrace{\left\{ b_s(k),\ b^\dagger_{s'}(k') \right\}}_{\hbar\, \cancel{(2\pi)^3}\, \cancel{(2\omega_k)}\, \delta(k-k')\, \delta_{ss'}}$$

$$= b^\dagger_{s'}(k')\, H + \hbar\omega_{k'}\, b^\dagger_{s'}(k') \qquad \checkmark$$

Thus
$$H\, |k,s\rangle = H\, b^\dagger_s(k)\, |0\rangle$$
$$= \left(b^\dagger_s(k)\, H + \hbar\omega_k\, b^\dagger_s(k) \right) |0\rangle$$
$$= 0 + \hbar\omega_k\, |k,s\rangle$$

so $|k,s\rangle$ has energy $\hbar\omega_k$

Recall

$$\psi = \int d\tilde{k} \sum_s \left[a_s(k)\, u_{ks} e^{-ik\cdot x} + b_s^\dagger(k) v_{ks}\, e^{ik\cdot x} \right]$$

$$\psi^\dagger = \int d\tilde{k} \sum_s \left[a_s^\dagger(k)\, u_{ks}^\dagger\, e^{ik\cdot x} + b_s(k) v_{ks}^\dagger\, e^{-ik\cdot x} \right]$$

$$\{a_s(k), a_{s'}^\dagger(k')\} = \{b_s(k), b_{s'}^\dagger(k')\} = \hbar (2\pi)^3 (2\omega_k)\, \delta^{(3)}(\vec{k}-\vec{k}')\, \delta_{ss'}$$

All other anticommutators vanish

From this we may deduce <u>equal time anticommutators</u>

$$\{\psi(\vec{x},t), \psi(\vec{x}',t)\} = 0$$

$$\{\psi^\dagger(\vec{x},t), \psi^\dagger(\vec{x}',t)\} = 0$$

And [although we don't supply the details]

$$\{\psi_\alpha(\vec{x},t), \psi_\beta^\dagger(\vec{x}',t)\} = \hbar\, \delta_{\alpha\beta}\, \delta^{(3)}(\vec{x}-\vec{x}')$$

← (see next pages)

Recall $\mathcal{L} = \bar{\Psi}(i\not{\partial} - \mu)\psi = \psi^\dagger \gamma^0 (i\gamma^0 \partial_0 + i\vec{\gamma}\cdot\vec{\nabla} - \mu)\psi$

$$\pi \equiv \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} = i\psi^\dagger (\gamma^0)^2 = i\psi^\dagger$$

$$\{\psi_\alpha(\vec{x},t), \pi_\beta(\vec{x}',t)\} = i\hbar\, \delta_{\alpha\beta}\, \delta^{(3)}(\vec{x}-\vec{x}')$$

analogous to $[\phi_i(\vec{x},t), \pi_j(\vec{x}',t)] = i\hbar\, \delta_{ij}\, \delta^{(3)}(\vec{x}-\vec{x}')$

[end of class presentation spinor QFT]

$$\mathcal{L} = \bar{\Psi}(i\not{\partial} - m)\psi$$

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} = \bar{\psi}\, i\gamma^0 = i\psi^\dagger$$

If we require $\{\psi(x), \pi(x')\} = i\,\delta(x-x')$

then $\{\psi(x), \psi^\dagger(x')\} = \delta(x-x')$

(Really $\{\psi_\alpha(x), \psi^\dagger_\beta(x')\} = \delta_{\alpha\beta}\,\delta(x-x')$.)

Let's see if this follows from the anticommutation relations

$$\{b_{ps}, b^\dagger_{p's'}\} = (2\pi)^3\, 2\omega\, \delta(\vec{p}-\vec{p}\,')\,\delta_{ss'}$$

$$\{d_{ps}, d^\dagger_{p's'}\} = (2\pi)^3\, 2\omega\, \delta(\vec{p}-\vec{p}\,')\,\delta_{ss'}$$

$$\psi_\alpha(x) = \int \widetilde{dp} \sum_s \left(b_{ps}(u_{ps})_\alpha e^{-ip\cdot x} + d^\dagger_{ps}(v_{ps})_\alpha e^{ip\cdot x} \right)$$

$$\psi^\dagger_\beta(x') = \int \widetilde{dp'} \sum_{s'} \left(b^\dagger_{p's'}(u^\dagger_{p's'})_\beta e^{ip'\cdot x'} + d_{p's'}(v^\dagger_{p's'})_\beta e^{-ip'\cdot x'} \right)$$

$$\{\psi_\alpha(x), \psi^\dagger_\beta(x')\} = \int \widetilde{dp}\, \widetilde{dp'} \sum_{s,s'} \Big[\{b_{ps}, b^\dagger_{p's'}\}(u_{ps})_\alpha (u^\dagger_{p's'})_\beta e^{+i(p'\cdot x' - p\cdot x)}$$

$$+ \{d^\dagger_{ps}, d_{p's'}\}(v_{ps})_\alpha (v^\dagger_{p's'})_\beta e^{i(p\cdot x - p'\cdot x')} \Big]$$

$\widetilde{dp} = \frac{d^3p}{(2\pi)^3 2\omega}$

$\left(\begin{matrix} \vec{p} = \vec{p}' \\ s = s' \end{matrix} \Rightarrow \omega = \omega' \right)$

$$= \int \widetilde{dp} \Big[\underbrace{\sum_s (u_{ps})_\alpha (u^\dagger_{ps})_\beta}_{\underbrace{(\not{p}+m)\gamma^0}_{E - \vec{p}\cdot\vec{\gamma}\gamma^0 + m\gamma^0}} e^{ip\cdot(x'-x)} + \underbrace{\sum_s (v_{ps})_\alpha (v^\dagger_{ps})_\beta}_{\underbrace{(\not{p}-m)\gamma^0}_{E - \vec{p}\cdot\vec{\gamma}\gamma^0 - m\gamma^0}} e^{ip\cdot(x-x')} \Big]$$

Let $\vec{p} \to -\vec{p}$ in 2nd term

$$= \int \widetilde{dp} \Big[(E - \vec{p}\cdot\vec{\gamma}\gamma^0 + m\gamma^0) e^{iE(t'-t)}$$
$$+ (E + \vec{p}\cdot\vec{\gamma}\gamma^0 - m\gamma^0) e^{iE(t-t')} \Big] e^{-i\vec{p}\cdot(\vec{x}' - \vec{x})}$$

Now if $t = t'$ then $\vec{p}$ + m terms drop out

$$= \int \widetilde{dp}\, 2E\, e^{-i\vec{p}\cdot(\vec{x}' - \vec{x})} = \int \frac{d^3p}{(2\pi)^3 (2E)} (2E)\, e^{-i\vec{p}\cdot(\vec{x}' - \vec{x})}$$

$$= \delta(\vec{x}' - \vec{x})\, \delta_{\alpha\beta} \qquad \text{QED}$$

(2-15-17) Extra

additional verification (after course in 2016 completed)

- Completeness.

(cf Peskin + Schroeder, p. 48 for slightly different approach)

Recall $\chi_{p+} = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{i\phi} \end{pmatrix}$ $\qquad \chi_{p-} = \begin{pmatrix} -\sin\frac{\theta}{2} e^{-i\phi} \\ \cos\frac{\theta}{2} \end{pmatrix}$

$$\chi_{p+}\chi_{p+}^\dagger + \chi_{p-}\chi_{p-}^\dagger = \begin{pmatrix} c \\ se^{i\phi} \end{pmatrix}(c,\ se^{-i\phi}) + \begin{pmatrix} -se^{-i\phi} \\ c \end{pmatrix}(-se^{i\phi},\ c)$$

$$= \begin{pmatrix} c^2+s^2 & 0 \\ 0 & c^2+s^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}$$

$$\chi_{p+}\chi_{p+}^\dagger - \chi_{p-}\chi_{p-}^\dagger = \begin{pmatrix} c^2-s^2 & 2sce^{-i\phi} \\ 2sce^{i\phi} & s^2-c^2 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta\cos\phi - i\sin\theta\sin\phi \\ \sin\theta\cos\phi + i\sin\theta\sin\phi & -\cos\theta \end{pmatrix}$$

$$= \frac{p_z}{p}\sigma_z + \frac{p_x}{p}\sigma_x + \frac{p_y}{p}\sigma_y$$

$$= \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}$$

Recall $u_{ps} = \begin{pmatrix} \sqrt{E - s|\vec{p}|}\, \chi_{ps} \\ \sqrt{E + s|\vec{p}|}\, \chi_{ps} \end{pmatrix}$

$$\sum u_{ps} u_{ps}^\dagger = \sum_s \begin{pmatrix} (E - s|p|)\, \chi_{ps}\chi_{ps}^\dagger & \sqrt{(E-s|p|)(E+s|p|)}\, \chi_{ps}\chi_{ps}^\dagger \\ \sqrt{(E-s|p|)(E+s|p|)}\, \chi_{ps}\chi_{ps}^\dagger & (E + s|p|)\, \chi_{ps}\chi_{ps}^\dagger \end{pmatrix}$$

$$= \begin{pmatrix} E\mathbb{1} - |p| \frac{\vec{\sigma}\cdot\vec{p}}{|p|} & m\mathbb{1} \\ m\mathbb{1} & E\mathbb{1} + |p| \frac{\vec{\sigma}\cdot\vec{p}}{|p|} \end{pmatrix}$$

$$\sum u_{ps} u_{ps}^\dagger = \begin{pmatrix} E - \vec{\sigma}\cdot\vec{p} & m \\ m & E + \vec{\sigma}\cdot\vec{p} \end{pmatrix}$$

$$\sum u_{ps}\bar{u}_{ps} = \sum u_{ps}u_{ps}^\dagger \gamma^0 = \begin{pmatrix} E - \vec{\sigma}\cdot\vec{p} & m \\ m & E + \vec{\sigma}\cdot\vec{p} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} m & E - \vec{\sigma}\cdot\vec{p} \\ E + \vec{\sigma}\cdot\vec{p} & m \end{pmatrix}$$

But $\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$ so this is $m + E\gamma^0 - \vec{p}\cdot\vec{\gamma} = \not{p} + m$

$$V_{ps} = \begin{pmatrix} \sqrt{E - s|p|}\, \chi_{ps} \\ -\sqrt{E + s|p|}\, \chi_{ps} \end{pmatrix}$$

$$\sum v_{ps} v_{ps}^\dagger = \begin{pmatrix} E - \vec{\sigma}\cdot\vec{p} & -m \\ -m & E + \vec{\sigma}\cdot\vec{p} \end{pmatrix} \Rightarrow \sum v_{ps}\bar{v}_{ps} = \begin{pmatrix} -m & E - \vec{\sigma}\cdot\vec{p} \\ E + \vec{\sigma}\cdot\vec{p} & -m \end{pmatrix} = \not{p} - m$$

$$\boxed{\begin{aligned} \sum_s u_{ps}\bar{u}_{ps} &= \not{p} + m \\ \sum_s v_{ps}\bar{v}_{ps} &= \not{p} - m \end{aligned}}$$

(11-30-22)

$$H = \int \widetilde{dk} \, [\, -\omega \, b \, b^{+} \,]$$

Now if $[b, b^{+}] = \hbar (2\pi)^2 (2\omega) \, \delta(k-k')$

this would imply that

$$[H, b^{+}] = -\omega \, [b, b^{+}] \, b^{+} = -\hbar\omega \, b^{+}$$

thus the creation op would lower the energy.

However we know that Heisenberg eqns give

$$[\psi, H] = i \partial_0 \psi$$

and so $\psi \sim \int \widetilde{dk} \, [\, b^{+} e^{ikx} \,]$

implies $[b^{+}, H] \sim -\hbar\omega \, b^{+}$

ie $[H, b^{+}] = \hbar\omega \, b^{+}$

The result is that $[\psi, \pi] = i \delta^{(3)}(x-y)$

implies $[b, b^{+}] = -\hbar \, (2\pi)^3 (2\omega) \, \delta(k-k')$

But now if $\langle 0 | 0 \rangle = 1$

then $\langle k | k \rangle = \langle 0 | b \, b^{+} | 0 \rangle < 0$

ie states have negative norm!

(10-31-22)

Why do we need to use anticommutators for the Dirac theory?

It boils down to the form of $\mathcal{L}$ as a first-order expression

ie $\mathcal{L} \sim \bar{\psi}(i\not{\partial} - m)\psi$

Peskin + Schroeder + Coleman (also Tong, who seems to use Coleman approach (+ b, c† notation)) are the only authors I know of who seriously attempt to quantize Dirac using commutators to suss out what is wrong.

Peskin + Schroeder parametrize $\psi \sim a e^{-ip\cdot x} + b e^{ip\cdot x}$

These give $[a, a^\dagger] \sim [b, b^\dagger] > 0$ (eq 3.88) and then b is a raising operator while a is a lowering operator

($[\psi, H] = i\partial_0 \psi \Rightarrow [a, H] = -E a, \; [b, H] = +E b$)

+ ∴ either $a^\dagger$ or $b^\dagger$ acting on vacuum lowers the energy, which is unbounded below

Coleman parametrize $\psi \sim a e^{-ipx} + b^\dagger e^{ipx}$

but while a + b are then both lowering op's, we get $[a, a^\dagger] > 0$ but $[b, b^\dagger] < 0$ (as P+S point out)

Then $\langle 0 | b b^\dagger | 0 \rangle = \langle k | k \rangle < 0$ (assuming $\langle 0|0\rangle = 1$)

So no longer have positive-definiteness of norm

(2017)

Why do we need anticommutators? (Peskin/Schroeder p.52)

$$\mathcal{L} = \bar{\Psi}(i\not{\partial} - m)\Psi$$

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \Psi)} = i\bar{\Psi}\gamma^0 = i\Psi^\dagger$$

If $[\Psi(x), \pi(x')]_= = i\delta(\vec{x}-\vec{x}')$ then

$$[\Psi_\alpha(x), \Psi_\beta^\dagger(x')]_= = \delta(\vec{x}-\vec{x}')\,\delta_{\alpha\beta}$$

General solution of Dirac eqn

$$\Psi(x) = \int d\tilde{p} \sum_s \left(a_{ps} u_{ps} e^{-ip\cdot x} + b_{ps} v_{ps} e^{ip\cdot x} \right)$$

$$\Psi^\dagger(x) = \int d\tilde{p} \sum_s \left(a_{ps}^\dagger u_{ps}^\dagger e^{ip\cdot x} + b_{ps}^\dagger v_{ps}^\dagger e^{-ip\cdot x} \right)$$

Let us assume

$$[a_{ps}, a_{p's'}^\dagger] = (2\pi)^3 (2\omega)\, \delta(p-p')\, \delta_{ss'}$$

$$[b_{ps}, b_{p's'}^\dagger] = (2\pi)^3 (2\omega)\, \delta(p-p')\, \delta_{ss'}$$

$$[a_{ps}, b_{p's'}^\dagger] = 0$$

$$[b_{ps}, a_{p's'}^\dagger] = 0$$

$a_p(t) = a_p e^{-iEt}$ $\qquad b_p(t) = b_p e^{iEt}$

But $[H, \phi] = -i\dot{\phi}$ so

$[H, a] = -Ea$ $\qquad [H, b] = Eb$

a = lowering $\qquad b$ = raising.

... we calculate [illegible]

Then

$$[\psi(x)_\alpha, \psi^\dagger_\beta(x')] = \int d\tilde{p}\, d\tilde{p}' \sum_{s,s'} \Big[[a_{ps}, a^\dagger_{p's'}]\, u_{ps} u^\dagger_{p's'}\, e^{i(p'\cdot x' - p\cdot x)}$$

$$+ [b_{ps}, b^\dagger_{p's'}]\, v_{ps} v^\dagger_{p's'}\, e^{i(p\cdot x - p'\cdot x')} \Big]$$

Now $\begin{cases} \vec{p}\,' = \vec{p} & \Rightarrow \omega' = \omega \\ s' = s \end{cases}$

$$[\psi(x)_\alpha, \psi^\dagger_\beta(x')] = \int d\tilde{p} \Big[\underbrace{\sum_s (u_{ps})_\alpha (u^\dagger_{ps})_\beta}_{\underbrace{(\not{p}+m)\gamma^0}_{(\mathbb{1}E - \vec{p}\cdot\vec{\gamma}\,\gamma^0 + m\gamma^0)_{\alpha\beta}}} e^{ip\cdot(x'-x)} + \underbrace{\sum_s (v_{ps})_\alpha (v^\dagger_{ps})_\beta}_{\underbrace{(\not{p}-m)\gamma^0}_{(\mathbb{1}E - \vec{p}\cdot\vec{\gamma}\,\gamma^0 - m\gamma^0)_{\alpha\beta}}} e^{ip\cdot(x-x')} \Big]$$

$$(\mathbb{1}E - \vec{p}\cdot\vec{\gamma}\,\gamma^0 - m\gamma^0)_{\alpha\beta} \xrightarrow{\text{let } \vec{p} \to -\vec{p}} \mathbb{1}E + \vec{p}\cdot\vec{\gamma}\,\gamma^0 - m\gamma^0$$

$$= \int d\tilde{p} \Big[(E - \vec{p}\cdot\vec{\gamma}\,\gamma^0 + m\gamma^0) e^{iE(t'-t)} + (E + \vec{p}\cdot\vec{\gamma}\,\gamma^0 - m\gamma^0) e^{iE(t-t')} \Big] e^{i\vec{p}\cdot(\vec{x}-\vec{x}')}$$

Equal time $\Rightarrow$ γ^0 terms drop out

$$\int d\tilde{p}\, \delta_{\alpha\beta} 2E\, e^{i\vec{p}\cdot(\vec{x}-\vec{x}')} = \int \frac{d^3\vec{p}}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{x}')} = \delta_{\alpha\beta}\, \delta(\vec{x}-\vec{x}')$$

so this gives the expected commutator

Now

$$H = \int d^3x \quad i\psi^\dagger \partial^0 \psi$$

$$\psi^\dagger(x) = \int d\tilde{p}' \sum_s \left(a^\dagger_{p's'} u^\dagger_{p's'} e^{ip'\cdot x} + b^\dagger_{p's'} v^\dagger_{p's'} e^{-ip'\cdot x} \right)$$

$$i\partial^0 \psi(x) = \int d\tilde{p} \sum_s \left(E\, a_{ps} u_{ps} e^{-ip\cdot x} - E\, b_{ps} v_{ps} e^{ip\cdot x} \right)$$

$$\begin{aligned} H = \sum_{s,s'} \int d\tilde{p}'\, d\tilde{p} \Big[& E\, a^\dagger_{p's'} a_{ps} u^\dagger_{p's'} u_{ps} \int d^3x \; e^{i(p'-p)\cdot x} \xrightarrow{} (2\pi)^3 \delta(\vec{p}-\vec{p}') \\ & + E\, b^\dagger_{p's'} a_{ps} v^\dagger_{p's'} u_{ps} \int d^3x \; e^{-i(p'+p)\cdot x} \xrightarrow{} (2\pi)^3 \delta(\vec{p}+\vec{p}') e^{-2iEt} \\ & - E\, a^\dagger_{p's'} b_{ps} u^\dagger_{p's'} v_{ps} \int d^3x \; e^{i(p'+p)\cdot x} \xrightarrow{} (2\pi)^3 \delta(\vec{p}+\vec{p}') e^{2iEt} \\ & - E\, b^\dagger_{p's'} b_{ps} v^\dagger_{p's'} v_{ps} \int d^3x \; e^{i(p-p')\cdot x} \Big] \xrightarrow{} (2\pi)^3 \delta(\vec{p}-\vec{p}') \end{aligned}$$

$=$ (see p. L4-4,5)

$$H = \sum_s \int d\tilde{p} \left[E\, a^\dagger_{ps} a_{ps} - E\, b^\dagger_{ps} b_{ps} \right]$$

But then

$$[H, b] = [-E b^\dagger b, b] = -E [b^\dagger, b] b = E b$$

rather than $-Eb$ as expected for lowering op

$$[H, b^\dagger] = [-E b^\dagger b, b^\dagger] = -E b^\dagger [b, b^\dagger] = -E b^\dagger$$

rather than $E b^\dagger$ as expected for raising op

whereas

$$\left.\begin{aligned} [H, a] &= [E a^\dagger a, a] = E [a^\dagger, a] a = -E a \\ [H, a^\dagger] &= [E a^\dagger a, a^\dagger] = E a^\dagger [a, a^\dagger] = E a^\dagger \end{aligned}\right\} \text{ as expected}$$

At this point, consider switching to anticommutators w/ $[a, a^+]_\mp = 1$, $[b, b^+]_\mp = 1$

let's keep both possibilities

$$[AB, C]_- = ABC - CAB = A[B,C]_\mp \pm (ACB \mp CAB)$$
$$= A[B,C]_\mp \pm [A,C]_\mp B$$

Then

$$[H, a]_- = E[a^+a, a]_- = \pm E \underbrace{[a^+, a]_\mp}_{\mp [a, a^+]} a = -Ea$$

$$[H, a^+]_- = E[a^+a, a^+]_- = Ea^+ [a, a^+]_\mp = E a^+$$

correct w/ either!

$$[H, b]_- = -E[b^+b, b] = \mp E \underbrace{[b^+, b]_\mp}_{\mp [b, b^+]_\mp} b = Eb$$

$$[H, b^+]_- = -E[b^+b, b^+] = -E b^+ [b, b^+]_\mp = -E b^+$$

incorrect either way!

i.e. it looks like b is a raising operator & b^+ a lowering op

Let's try relabelling $b \leftrightarrow b^\dagger$ throughout

Then $\psi(x) = \int d\tilde{p} \sum_s [\, a_{ps} u_{ps} e^{-ip\cdot x} + b^\dagger_{ps} v_{ps} e^{ip\cdot x} \,]$

$[\psi_\alpha, \psi^\dagger_\beta]_\mp = \delta(x-x')\,\delta_{\alpha\beta} \Rightarrow [b^\dagger_{ps}, b_{p's'}]_\mp = (2\pi)^3 (2\omega)\, \delta(p-p')\,\delta_{ss'}$

$\cong \quad [b_{ps}, b^\dagger_{ps}]_\mp = \mp (2\pi)^3 (2\omega)\, \delta(p-p')\,\delta_{ss'}$

Also $H = \sum_s \int d\tilde{p}\, [\, E\, a^\dagger_{ps} a_{ps} - E\, b_{ps} b^\dagger_{ps} \,]$

Finally $[H, b^\dagger]_- = E b^\dagger$

$[H, b]_- = -Eb$

so this looks good either way!

as we can easily verify:

$[H, b^\dagger]_- = -E\,[b b^\dagger, b^\dagger]_- = \mp E [b, b^\dagger]_\mp\, b^\dagger = E b^\dagger$

$[H, b]_- = -E\,[b b^\dagger, b] = -E\, b\, [b^\dagger, b]_\mp = -Eb$

more directly $[\psi, H] = i\partial_0 \psi$ for both commutators + anticommutators

$\Rightarrow [H, a] = -Ea, \quad [H, b] = -Eb.$

$a_p(t) = a_p e^{-iEt} \qquad b^\dagger_p(t) = b_p e^{iEt}$

$[H, \phi] = -i\dot{\phi}$

$[H, a] = Ea \qquad [H, b^\dagger] \quad Eb^\dagger$

why must we choose anticommutators then?

(2022) Because $[b, b^\dagger] = -(2\pi)^3 (2\omega)\, \delta(\ )\, \delta_{ss'}$

implies ~~that~~ that $b^\dagger|0\rangle$ has negative norm! (relative to $|0\rangle$ + $a^\dagger|0\rangle$) for commutators

$\langle 0 | b b^\dagger | 0 \rangle = \langle 0 | [b, b^\dagger]_\mp | 0 \rangle = \mp^{\#} \langle 0 | 0 \rangle$