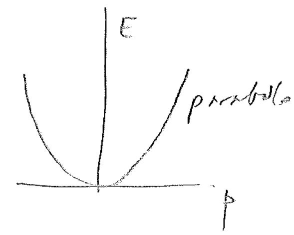
spin ½ particles

[eg. electrons]

Schrodinger eqn $i\hbar \frac{\partial \psi}{\partial t} = H\psi$

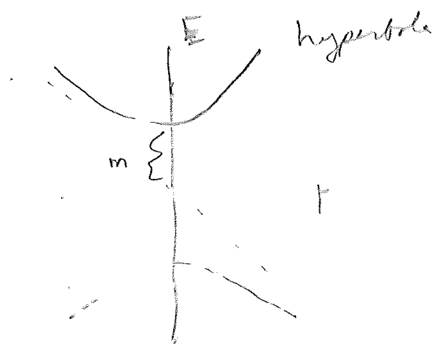


Free nonrelativistic particle: $E = \frac{p^2}{2m}$

In position space: $\vec{p} = \frac{\hbar}{i}\vec{\nabla}$

$$\Rightarrow H = \frac{1}{2m}\left(\frac{\hbar}{i}\vec{\nabla}\right)^2 = -\frac{\hbar^2}{2m}\nabla^2$$

$$\Rightarrow \text{t.d.S.e.} \quad i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi$$



Free relativistic particle: $E^2 = \vec{p}^2 + m^2$
$(c = 1)$

$$0 = E^2 - (\vec{p}^2 + m^2)$$
$$= (E + \sqrt{\vec{p}^2 + m^2})(E - \sqrt{\vec{p}^2 + m^2})$$

negative energy sol / positive energy solut

[similar for $p \ll m$; $E = m + \frac{p^2}{2m}$]

choose $E = \sqrt{\vec{p}^2 + m^2}$

This would give

$$i\hbar \frac{\partial \psi}{\partial t} = \sqrt{-\nabla^2 + m^2}\,\psi \qquad \text{[disgusting]}$$

Dirac: tried to f.a. w. square roots

$$0 = E^2 - (\vec{p}^2 + m^2)$$

$$= (E + \vec{\alpha}\cdot\vec{p} + \beta m)(E - \vec{\alpha}\cdot\vec{p} - \beta m)$$

$$\Rightarrow (\vec{\alpha}\cdot\vec{p} + \beta m)^2 = p^2 + m^2$$

only works if $\vec{\alpha}, \beta$ are anticommuting matrices ($\vec{\alpha}\beta = -\beta\vec{\alpha}$)
to kill the cross term

choose a second root

$$E = \vec{\alpha}\cdot\vec{p} + \beta m$$

$$\Rightarrow H = \frac{\hbar}{i}\vec{\alpha}\cdot\vec{\nabla} + \beta m$$

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{\hbar}{i}\vec{\alpha}\cdot\vec{\nabla}\psi + \beta m\psi \qquad \text{(Dirac eqn)}$$

We'll try an equivalent, more covariant, approach

$0 = E^2 - \vec{p}^2 - m^2 = p^2 - m^2$ where $p^2 = p_\mu p^\mu$

$p^2 - m^2 = (\sqrt{p^2} + m)(\sqrt{p^2} - m)$ still has square roots: $\sqrt{p^2} \overset{?}{=} p$

Instead try

$p^2 - m^2 = (\not{p} + m)(\not{p} - m)$

where $\not{p}$ is linear in p^μ

$\not{p} \equiv \gamma_0 p^0 + \gamma_1 p^1 + \gamma_2 p^2 + \gamma_3 p^3 = \gamma_\mu p^\mu$ for some objects γ_μ

$p^2 - m^2 = \not{p}\not{p} + \underbrace{m\not{p} - \not{p}m}_{0} - m^2$

$p^2 = \not{p}\not{p}$

$(p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2 = (\gamma_0 p^0 + \gamma_1 p^1 + \gamma_2 p^2 + \gamma_3 p^3)(\gamma_0 p^0 + \gamma_1 p^1 + \gamma_2 p^2 + \gamma_3 p^3)$

$= \gamma_0^2 (p^0)^2 + \gamma_1^2 (p^1)^2 + \gamma_2^2 (p^2)^2 + \gamma_3^2 (p^3)^2$

$+ (\gamma_0\gamma_1 + \gamma_1\gamma_0) p^0 p^1$ + etc.

$\Rightarrow \gamma_0^2 = 1, \quad \gamma_1^2 = \gamma_2^2 = \gamma_3^2 = -1$

$\gamma_0\gamma_1 + \gamma_1\gamma_0 = 0$ etc. $\Rightarrow \gamma_0\gamma_1 = -\gamma_1\gamma_0$

γ_μ are anticommuting objects (e.g. matrices) called Dirac matrices

Define anticommutator $\{A, B\} = AB + BA$

Then $\{\gamma_\mu, \gamma_\nu\} = 0$ if $\mu \neq \nu$

$\{\gamma_\mu, \gamma_\mu\} = 2\gamma_\mu^2$ if $\mu = \nu$ but $\gamma_\mu^2 = \eta_{\mu\mu}$

$$\Rightarrow \boxed{\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}\mathbb{1}}$$

Clifford algebra

If γ_μ obey these relations, then $\not{p}\not{p} = p^2$

$$0 = p^2 - m^2$$

$$= (\not{p} + m)\underbrace{(\not{p} - m)}_{\text{choose this root}}$$

$$0 = \not{p} - m$$

Now $\vec{p} \rightarrow \frac{\hbar}{i} \vec{\nabla}$

$$p^i \rightarrow \frac{\hbar}{i} \frac{\partial}{\partial x^i} = \frac{\hbar}{i} \partial_i = -\frac{\hbar}{i} \partial^i = i\hbar \partial^i$$

Also $i\hbar \partial_0 \psi = H\psi$

If ψ is an energy eigenstate $H\psi = E\psi$

$$i\hbar \partial_0 \psi = E\psi$$

$$p^0 = E = i\hbar \partial_0 = i\hbar \partial^0$$ ← valid when acting on an energy eigenstate

Together we have

$$\boxed{p^\mu = i\hbar \partial^\mu}$$

Therefore acting on an energy eigenstate, we have

$$(i\hbar \not{\partial} - m)\psi = (i\hbar \gamma_\mu \partial^\mu - m)\psi$$

$$= (\gamma_\mu p^\mu - m)\psi$$

$$= (\not{p} - m)\psi$$

$$= 0$$

Let $m = \mu\hbar$

$$\boxed{(i\not{\partial} - \mu)\psi = 0}$$ Dirac eqn (written covariantly)

We need a set of four Dirac gamma matrices $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ that satisfy the Clifford algebra $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$

The smallest matrices for which this is possible are 4×4.

The Dirac gamma matrices are not unique because if γ_μ obey Clifford algebra so does the set $\gamma_\mu' = D^{-1}\gamma_\mu D$ (similarity transformation) for any invertible 4×4 matrix D.

Proof:

$$\{\gamma_\mu', \gamma_\nu'\} = \gamma_\mu'\gamma_\nu' + \gamma_\nu'\gamma_\mu'$$

$$= D^{-1}\gamma_\mu D D^{-1}\gamma_\nu D + D^{-1}\gamma_\nu D D^{-1}\gamma_\mu D$$

$$= D^{-1}(\underbrace{\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu}_{2\eta_{\mu\nu}\mathbb{1}}) D$$

$$= 2\eta_{\mu\nu}\mathbb{1}$$

[γ^μ do not transform under LT transformations but given $\Lambda^\mu{}_\nu$ we may find $D(\Lambda) \ni D^{-1}(\Lambda)\gamma^\mu D(\Lambda) = \Lambda^\mu{}_\nu\gamma^\nu$]

One convenient representation is the Weyl representation (Peskin Schroeder p.41)

$$\gamma_0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \qquad \gamma_i = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

where $\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and $0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

σ_i are Pauli matrices

$$\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Observe that $\sigma_i^2 = \mathbb{1}$
- Observe that σ_i are hermitian matrices: $\sigma_i^\dagger = \sigma_i$

where $A^\dagger \equiv (A^T)^*$

[HW: show $\not{\partial}\not{\partial} = \partial^2$

show $(\gamma_\mu)^\dagger = \gamma_0 \gamma_\mu \gamma_0$

show $\gamma^\mu \gamma_\nu \gamma_\mu = a \gamma_\nu$ & find the constant a. (-2)]

[HW: Show γ_μ obey Clifford algebra]

The Pauli matrices satisfy

$$\sigma_i \sigma_j = \delta_{ij} \mathbb{1} + i \epsilon_{ijk} \sigma_k \quad \text{(Einstein summation)}$$

where ϵ_{ijk}: totally antisymmetric Levi-Civita symbol

$$\begin{cases} \epsilon_{123} = 1 \\ \epsilon_{213} = \epsilon_{321} = \epsilon_{132} = -1 \\ \epsilon_{231} = \epsilon_{312} = 1 \end{cases}$$

$\epsilon_{ijk} = 0$ if any two indices are equal

Check: $i = j = 1$: $\sigma_1^2 = \mathbb{1}$ $\quad = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓

$i=1, j=2$ $\quad \sigma_1 \sigma_2 = i(\underbrace{\epsilon_{121}}_{0} \sigma_1 + \underbrace{\epsilon_{122}}_{0} \sigma_2 + \underbrace{\epsilon_{123}}_{1} \sigma_3)$

$$= i\sigma_3$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \begin{pmatrix} i & \\ & -i \end{pmatrix} = i\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

compute commutators, anticommutators, familiar ...

$$[\sigma_i, \sigma_j] = \sigma_i \sigma_j - \sigma_j \sigma_i = i\epsilon_{ijk}\sigma_k - i\epsilon_{jik}\sigma_k = 2i\epsilon_{ijk}\sigma_k$$

$$\left[\tfrac{\hbar}{2}\sigma_i, \tfrac{\hbar}{2}\sigma_j\right] = \tfrac{\hbar}{2} 2i\epsilon_{ijk}\tfrac{\hbar}{2}\sigma_k = i\hbar\epsilon_{ijk}\tfrac{\hbar}{2}\sigma_k$$

$$[S_i, S_j] = i\hbar\epsilon_{ijk} S_k \quad \text{(well-known angular momentum c.r.)}$$

$$\{\sigma_i, \sigma_j\} = \sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}\mathbb{1}$$

Let's solve Dirac eqn $(i\partial\!\!\!/ - \mu)\psi = 0$

where recall $\partial\!\!\!/ = \gamma_\mu \partial^\mu$ and ψ: 4-component spinor $\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}$

Observe that if ψ obeys Dirac, it also obeys Klein Gordon

$(i\partial\!\!\!/ - \mu)\psi = 0$

$(-i\partial\!\!\!/ - \mu)(i\partial\!\!\!/ - \mu)\psi$. $\quad (\underbrace{\partial\!\!\!/\partial\!\!\!/ + \mu^2)\psi = (\partial^2}_{HW} + \mu^2)\psi = 0$ ← Klein Gordon

— Recall general solution of Klein Gordon for a complex scalar

$$\phi(x) = \int \widetilde{dk}\left[a(k) e^{-ik\cdot x} + b^*(k) e^{ik\cdot x}\right]\Big|_{k^0 = \omega_k}$$

— Consider the ansatz

$$\psi(x) = \int \widetilde{dk}\left[a(k) e^{-ikx} u_k + b^*(k) e^{ikx} v_k \right]\Big|_{k^0 = \omega_k}$$

u_k, v_k: constant 4-component spinors

$$(i\partial\!\!\!/ - \mu)\psi = \int \widetilde{dk}\left[a(k) e^{-ikx}(k\!\!\!/ - \mu)u_k + b^*(k) e^{ikx}(-k\!\!\!/ - \mu)v_k\right]\Big|_{k^0 = \omega_k}$$

This vanishes provided that

$$k\!\!\!/ u_k = \mu u_k$$

$$k\!\!\!/ v_k = -\mu v_k$$

where $k^0 = \omega_k$ is understood.

Bjorken & Drell

$$\psi = \sum_{s=\pm} \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\frac{m}{E}} \left[b u e^{-ip\cdot x} + d^\dagger v e^{ip\cdot x} \right]$$

Srednicki

using the mostly-plus metric

$$\psi = \sum_{s=\pm} \int \widetilde{dp} \left[b u e^{ip\cdot x} + d^\dagger v e^{-ip\cdot x} \right] \qquad (37.30)$$

Coleman

$$\psi = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2E_p}} \sum \left[b u e^{-ip\cdot x} + c^\dagger v e^{ip\cdot x} \right] \qquad (21.6)$$

Peskin & Schroeder

$$\psi = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \sum \left[a u e^{-ip\cdot x} + b^\dagger v e^{ip\cdot x} \right) \qquad (3.99)$$

Schwartz

$$\psi = \sum_s \int \frac{d^3p}{(2\pi)^3\sqrt{2\omega_p}} \left[a u e^{-ip\cdot x} + b^\dagger v e^{ip\cdot x} \right)$$

Recall $\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\gamma_i = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix}$ in Weyl representation

$$\not{k} = \gamma_\mu k^\mu = \gamma_0 \omega + \gamma_i k^i = \begin{pmatrix} 0 & \omega - \sigma_i k^i \\ \omega + \sigma_i k^i & 0 \end{pmatrix} \qquad \sigma_i k^i = \vec{\sigma}\cdot\vec{k}$$

Let $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ where u_1, u_2 are 2-component spinors

Then $\not{k} u = \mu u \Rightarrow \begin{pmatrix} 0 & \omega - \vec{\sigma}\cdot\vec{k} \\ \omega + \vec{\sigma}\cdot\vec{k} & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \mu u_1 \\ \mu u_2 \end{pmatrix}$

Then
$$(\omega - \vec{\sigma}\cdot\vec{k})\, u_2 = \mu\, u_1$$
$$(\omega + \vec{\sigma}\cdot\vec{k})\, u_1 = \mu\, u_2$$

now do DE-10 $\longrightarrow$

<u>Choose</u> $u_1 = N \chi_{ks}$ where $\vec{\sigma}\cdot\hat{k}\, \chi_{ks} = s\, \chi_{ks}$

so $\vec{\sigma}\cdot\vec{k}\, \chi_{ks} = s|\vec{k}|\, \chi_{ks}$

2nd eqn: $u_2 = \frac{N}{\mu}(\omega + \vec{\sigma}\cdot\vec{k})\chi_{ks} = \frac{N}{\mu}(\omega + s|k|)\chi_{ks}$

Check that 1st eqn holds

$$(\omega - \vec{\sigma}\cdot\vec{k})\frac{N}{\mu}(\omega + s|k|)\chi_{ks} = \frac{N}{\mu}\underbrace{(\omega - s|k|)(\omega + s|k|)}_{\underbrace{\omega^2 - k^2}_{\mu^2} \text{ since } s^2 = 1}\chi_{ks}$$

$$= \mu N \chi_{ks} = \mu u_1 \quad \checkmark$$

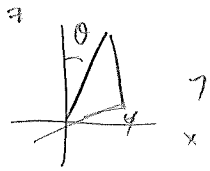
(Skip to DE-11)

The Pauli matrices have eigenvalues $s = \pm 1$ and its eigenvectors are 2-component spinors $\chi_s = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$

e.g. $\sigma_z \chi_s = s \chi_s$

$$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \pm\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \Rightarrow \chi_+ = \begin{pmatrix}1\\ 0\end{pmatrix},\ \chi_- = \begin{pmatrix}0\\ 1\end{pmatrix}$$

Let $\hat{k}$ denote the unit vector parallel to $\vec{k}$



$$\hat{k} = \begin{pmatrix}\sin\theta\cos\phi\\ \sin\theta\sin\phi\\ \cos\theta\end{pmatrix}$$

Consider $\vec{\sigma}\cdot\hat{k} = \sigma_x \sin\theta\cos\phi + \sigma_y \sin\theta\sin\phi + \sigma_z\cos\theta = \begin{pmatrix}\cos\theta & \sin\theta\, e^{-i\phi}\\ \sin\theta\, e^{i\phi} & -\cos\theta\end{pmatrix}$

$\vec{\sigma}\cdot\hat{k}$ has eigenvalues $s = \pm 1$ and eigenvectors χ_{ks}

$$\vec{\sigma}\cdot\hat{k}\,\chi_{ks} = s\,\chi_{ks}$$

$\chi_{k+} = \begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2} e^{i\phi}\end{pmatrix}$ is spin up (in direction of $\vec{k}$) (positive helicity)

$\chi_{k-} = \begin{pmatrix}-\sin\frac{\theta}{2} e^{-i\phi}\\ \cos\frac{\theta}{2}\end{pmatrix}$ is spin down (opposite direction of $\vec{k}$) (negative helicity)

The eigenvectors are orthonormal

$$\chi_{ks}^\dagger \chi_{ks'} = \delta_{ss'}$$

The space of 2-dimension spinors is spanned by $\chi_{k\pm}$

the most general $\chi = \alpha\chi_{k+} + \beta\chi_{k-}$

Choose normalization fact $N = \sqrt{\omega - s|k|}$.

Using $\mu = \sqrt{(\omega - s|k|)(\omega + s|k|)}$ we find

$$u_2 = \frac{N}{\mu}(\omega + s|k|)\chi_{ks} \quad \sqrt{\omega + s|k|}\,\chi_{ks}$$

hence

$$\boxed{u_{ks} = \begin{pmatrix} \sqrt{\omega - s|k|}\;\chi_{ks} \\ \sqrt{\omega + s|k|}\;\chi_{ks} \end{pmatrix}, \quad s = \pm 1}$$

2 solutions, one of each helicity

Recall that $\chi^\dagger_{ks}\chi_{ks'} = \delta_{ss'}$

It follows that $u^\dagger_{ks}u_{ks'} = (2\omega)\,\delta_{ss'}$

Proof: $u^\dagger_{ks} = \left(\sqrt{\omega - s|k|}\,\chi^\dagger_{ks},\ \sqrt{\omega + s|k|}\,\chi^\dagger_{ks}\right)$

$$u^\dagger_{ks}u_{ks'} = (\omega - s|k|)\underbrace{\chi^\dagger_{ks}\chi_{ks'}}_{\delta_{ss'}} + (\omega + s|k|)\underbrace{\chi^\dagger_{ks}\chi_{ks'}}_{\delta_{ss'}}$$

$$= 2\omega\,\delta_{ss'}$$

[P+S 3.60]

Let $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

Then $\not{k} v = -\mu v \implies$
$$(\omega - \vec{\sigma}\cdot\vec{k}) v_2 = -\mu v_1$$
$$(\omega + \vec{\sigma}\cdot\vec{k}) v_1 = -\mu v_2$$

Again, choose $v_1 = N \chi_{ks}$ w/ $N = \sqrt{\omega - s|k|}$

then $v_2 = -\frac{N}{\mu}(\omega + s|k|)\chi_{ks} = -\sqrt{\omega + s|k|}\,\chi_{ks}$

Hence
$$\boxed{v_{ks} = \begin{pmatrix} \sqrt{\omega - s|k|}\,\chi_{ks} \\ -\sqrt{\omega + s|k|}\,\chi_{ks} \end{pmatrix}, \quad s = \pm 1}$$

2 more solutions, one of each helicity

$v^\dagger_{ks} v_{ks'} = (2\omega)\,\delta_{ss'}$ as above [P&S, 3.63]

General solution of Dirac eqn $(i\not{\partial} - \mu)\psi = 0$ is

$$\psi = \int d\tilde{k} \sum_{s=\pm 1} \left[a_s(k)\, u_{ks}\, e^{-ik\cdot x} + b^*_s(k)\, v_{ks}\, e^{ik\cdot x} \right] \Big|_{k^0 = \omega_k}$$

when we quantize, coefficients a & b^* become annihilation & creation operators a and $b^\dagger$