Minimal coupling prescription

Review:

Electromagnetism $\mathcal{L}(A) = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \mathcal{L}_{int}(A)$

$\Rightarrow$ e.o.m. $\partial_\mu F^{\mu\nu} + \frac{\partial \mathcal{L}_{int}}{\partial A_\nu} = 0$

Define $j^\nu = -\frac{\partial \mathcal{L}_{int}}{\partial A_\nu}$ then

$$\partial_\mu F^{\mu\nu} = j^\nu \quad \text{(Maxwell's eqns)} \qquad j^\nu = (\rho, \vec{j})$$

Theory of symmetry $\mathcal{L}(\phi)$ SO(2) or U(1)

current $j^\mu = \sum_i \frac{\partial \mathcal{L}(\phi)}{\partial(\partial_\mu \phi)} \frac{\delta \phi_i}{\alpha}$

Are these the same? Both are conserved

$$\underbrace{\partial_\nu \partial_\mu F^{\mu\nu}}_{0} = \partial_\nu j^\nu$$

Interaction between light & matter

Consider an electromagnetic field and complex scalar field

$$\mathcal{L} = \mathcal{L}_{EM}(F) + \mathcal{L}_{CS}(\phi, \partial\phi)$$

$$\mathcal{L}_{EM}(F) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$\mathcal{L}_{CS}(\phi, \partial\phi) = |\partial\phi|^2 - \mu^2 |\phi|^2 - V_{int}(|\phi|)$$

As written, ϕ & A^μ are completely independent fields....

Why choose a complex scalar field?
U(1) symmetric $\Rightarrow$ conserved charge Q
which we'd like to interpret as electric charge.
But charged particle creates electric field
so there must be some interaction between fields A^μ & ϕ

$$\mathcal{L} = \mathcal{L}_{EM}(F) + \mathcal{L}_{CS}(\phi, \partial\phi) + \mathcal{L}_{int}(A, \phi)$$

Question 1:

What is the form of $\mathcal{L}_{int}$?

Question 2.

Since $J^\mu = -\frac{\delta \mathcal{L}_{int}}{\delta A_\mu}$, how do we know

that $\partial_\mu \left(\frac{\delta \mathcal{L}_{int}}{\delta A_\mu} \right) = 0$?

gauge invariance

Recall that a gauge transformation $\Rightarrow$ $\delta A_\mu = \partial_\mu \chi$

but
$$\delta F_{\mu\nu} = \delta(\partial_\mu A_\nu - \partial_\nu A_\mu)$$
$$= \partial_\mu(\delta A_\nu) - \partial_\nu(\delta A_\mu)$$
$$= \partial_\mu \partial_\nu \chi - \partial_\nu \partial_\mu \chi$$
$$= 0$$

So $\mathcal{L}_{EM}(F) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ is gauge invariant

$$\delta \mathcal{L}_{EM}(F) = 0$$

What about $\mathcal{L}_{int}(A, \phi)$?

$$\delta \mathcal{L}_{int} = \frac{\partial \mathcal{L}_{int}}{\partial A_\mu} \delta A_\mu = -J^\mu \partial_\mu \chi \neq 0$$

Problem 1: $\mathcal{L}_{int}(A, \phi)$ is not gauge invariant

Perhaps we are not thinking broadly enough about gauge transformations.

Maybe ϕ as well as A^μ changes.

Can this solve problem 1?

How ~~does~~ should ϕ transform under a gauge transformation to solve problem 1?

$$\mathcal{L}_{CS}(\phi, \partial\phi) = (\partial\phi^*)(\partial\phi) - \mu^2 \phi^*\phi - V_{int}(|\phi|)$$

Recall U(1) symmetry of $\mathcal{L}_{CS}$

$$\phi \to e^{-i\alpha}\phi \qquad \Rightarrow \phi^*\phi \to \phi^*\phi$$
$$\phi^* \to e^{i\alpha}\phi$$

$\mathcal{L}_{CS}$ is invariant under a global transformation.

Called global because

α = const, indep of position $\Rightarrow$ same everywhere

Consider instead a local transformation

$\alpha = q\chi(x)$ $\Rightarrow$ where χ is the gauge transf. param

[q will turn out to be the charge of ϕ]

$$\phi \to e^{-iq\chi}\phi$$
$$\phi^* \to e^{iq\chi}\phi^*$$

$|\phi|^2$ is still invariant, but not $|\partial\phi|^2$

$$\partial_\mu\phi \to \partial_\mu(e^{-iq\chi}\phi)$$
$$= e^{-iq\chi}\partial_\mu\phi + e^{-iq\chi}(-iq\partial_\mu\chi)\phi$$
$$= e^{-iq\chi}(\partial_\mu\phi - iq(\partial_\mu\chi)\phi)$$

$$|\partial_\mu\phi|^2 \to |\partial_\mu\phi - iq(\partial_\mu\chi)\phi|^2$$

Problem 2: $\mathcal{L}_{CS}$ not invariant under a local transformation

(2 questions, 2 problems)

[Well of course we don't need $\mathcal{L}_{CS}(\phi)$ + $\mathcal{L}_{int}(\phi, A)$ separately gauge invariant, just the sum of them]

Consider a gauge transformation acting on both fields:

$$\begin{cases} A^\mu \longrightarrow A^\mu + \partial^\mu \chi \\ \phi \longrightarrow e^{-iq\chi}\phi \end{cases}$$

Consider the combination $\partial_\mu\phi + iqA_\mu\phi$

$$\partial_\mu\phi + iqA_\mu\phi \longrightarrow \partial_\mu\left(e^{-iq\chi}\phi\right) + iq\left(A_\mu + \partial_\mu\chi\right)\left(e^{-iq\chi}\phi\right)$$

$$= e^{-iq\chi}\left[\partial_\mu\phi - iq(\partial_\mu\chi)\phi + iqA_\mu\phi + iq(\partial_\mu\chi)\phi\right]$$

$$= e^{-iq\chi}\left(\partial_\mu\phi + iqA_\mu\phi\right)$$

Define this combination as gauge covariant derivative.

$$\boxed{D_\mu\phi = \partial_\mu\phi + iqA_\mu\phi}$$

We have shown $D_\mu\phi \longrightarrow e^{-iq\chi}D_\mu\phi$

Observe: $(D_\mu\phi)^*(D^\mu\phi)$ is invariant.

Minimal coupling prescription

Replace all derivatives in $\mathcal{L}_{CS}(\phi, \partial\phi)$ w/ gauge covariant derivatives

$$\mathcal{L} = \mathcal{L}_{EM}(F) + \mathcal{L}_{CS}(\phi, D\phi)$$

$$\mathcal{L}_{CS} = |D_\mu \phi|^2 - \mu^2 |\phi|^2 - V_{int}(|\phi|)$$

$\Rightarrow \mathcal{L}$ is invariant under gauge transformations of A and ϕ

Problems 1+2 solved

What is $\mathcal{L}_{int}(A, \phi)$? Consider

$$(D^\mu\phi)^*(D_\mu\phi) = (\partial^\mu\phi^* - iq A^\mu \phi^*)(\partial_\mu\phi + iq A_\mu \phi)$$

$$= (\partial^\mu\phi^*)(\partial_\mu\phi) + \underbrace{iq A_\mu(\partial^\mu\phi^*)\phi - iq A^\mu \phi^* \partial_\mu\phi + q^2 A^\mu A_\mu \phi^*\phi}_{\mathcal{L}_{int}(A, \phi)}$$

$$\mathcal{L} = \mathcal{L}_{EM}(F) + \mathcal{L}_{CS}(\phi, \partial\phi) + \mathcal{L}_{int}(A, \phi)$$

Question 1 answered! Answer is not obvious!

$$j^\mu = -\frac{\delta \mathcal{L}_{int}}{\delta A_\mu} = iq\, \phi^*(\partial^\mu\phi) - iq\, \phi(\partial^\mu\phi^*) - 2q^2 A^\mu \phi^*\phi$$

Question 2: Is this current conserved? Yes, due to global symmetry

Rewrite $\mathcal{L}$

$$\mathcal{L} = \mathcal{L}_{EM}(F) + (\partial^\mu \phi^* - iqA^\mu \phi^*)(\partial_\mu \phi + iqA_\mu \phi) - \mu^2 |\phi|^2 - V_{int}(|\phi|)$$

$\mathcal{L}$ is gauge invariant (local symmetry)
but still possess a global $U(1)$ symmetry

$$\begin{cases} \phi \to e^{-i\alpha} \phi \\ A^\mu \to A^\mu \end{cases} \quad \text{ie} \quad \begin{matrix} \delta\phi = -i\alpha\phi \\ \delta A^\mu = 0 \end{matrix}$$

Noether's theorem $\Rightarrow$ conserved current $\partial_\mu j^\mu = 0$

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\delta\phi}{\alpha} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \frac{\delta\phi^*}{\alpha}$$

$$= (\partial^\mu \phi^* - iqA^\mu \phi^*)(-i\ \phi) + (\partial_\mu \phi + iqA_\mu \phi)(i\ \phi^*)$$

$$= i\,\phi^*(\partial_\mu \phi) - i\ \phi(\partial^\mu \phi^*) + 2\,q A^\mu \phi^* \phi$$

which is (up to constant) same as previous one

Nicer way to write: $j^\mu = (D^\mu \phi^*)(-i\ \phi) + (D_\mu \phi)(-i\ \phi^*)$

$$\underbrace{\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}}_{\partial_\mu(-F^{\mu\nu})} - \frac{\partial \mathcal{L}}{\partial A_\nu} \qquad - \jmath^\nu$$

Global symmetry: $\phi \to e^{-i\alpha}\phi$

$\phi^* \to e^{i\alpha}\phi^*$

$A^\mu \to A^\mu$

$$\jmath^\mu_{glob} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\frac{\delta\phi}{\alpha} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\frac{\delta\phi^*}{\alpha} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\delta A_\nu$$

$$= (D^\mu\phi^*)(-i\phi) + (D^\mu\phi)(i\phi^*) - F^{\mu\nu}\cdot 0$$

$$= -i(D^\mu\phi^*)\phi + i(D^\mu\phi)\phi^*$$

$$= -i(\partial^\mu\phi^*)\phi + i(\partial^\mu\phi)\phi^* - 2q A^\mu\phi^*\phi$$

Also $\jmath^\mu = -\frac{\delta\mathcal{L}_{int}}{\delta A_\mu} = -\frac{\delta}{\delta A_\mu}\left(+iqA^\mu\partial_\mu\phi^*\phi - iqA^\mu\phi^*\partial_\mu\phi + q^2A^\mu A_\mu\phi^*\phi\right) = q\,\jmath^\mu_{glob}$

Local symmetry $\phi \to e^{-iq\chi}\phi$ $\qquad \delta\phi = -iq\chi\phi$

$\phi^* \to e^{iq\chi}\phi^*$ $\qquad \delta\phi^* = iq\chi\phi^*$

$A_\mu \to A_\mu + \partial_\mu\chi$ $\qquad \delta A_\mu = \partial_\mu\chi$

$$\jmath^\mu_{loc} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\delta\phi^* + \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\delta A_\nu$$

$$= D^\mu\phi^*(-iq\chi\phi) + D^\mu\phi(iq\chi\phi^*) - F^{\mu\nu}\partial_\nu\chi$$

$$= \jmath^\mu_{glob}\, q\chi - F^{\mu\nu}\partial_\nu\chi$$

$$= \jmath^\mu\chi - F^{\mu\nu}\partial_\nu\chi$$

Then $\partial_\mu \jmath^\mu_{local} = \underbrace{(\partial_\mu \jmath^\mu)}_{0}\chi + \underbrace{\jmath^\nu\partial_\nu\chi - (\partial_\mu F^{\mu\nu})\partial_\nu\chi}_{0} - \underbrace{F^{\mu\nu}\partial_\mu\partial_\nu\chi}_{0}$

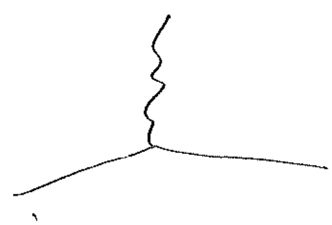
$$\jmath^\mu_{loc} = -\underbrace{\frac{\delta\mathcal{L}}{\delta A_\mu}}_{\partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\nu A_\mu)}}\chi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\partial_\nu\chi = \partial_\nu\left(-\frac{\partial\mathcal{L}}{\partial(\partial_\nu A_\mu)}\chi\right) \leftarrow \text{cf Coleman lectures p579n}$$

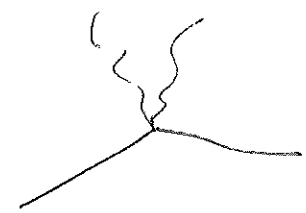
$$= \partial_\nu(F^{\mu\nu}\chi)$$

$$\to \partial_\mu \jmath^\mu_{loc} \qquad \partial_\nu\partial_\mu(F^{\mu\nu}\chi) \qquad 0$$

cubic + higher terms $\Rightarrow$ vertices in Feynman diagrams

$$\mathcal{L}_{int} \ni \quad A^\mu \phi^* \partial_\mu \phi \, , \qquad A^\mu A_\mu \phi^* \phi$$





"seagull"