One cannot quantize EM in the straightforward way by obtaining the Hamiltonian from the Lagrangian and then imposing canonical commutators because of gauge invariance.

Gauge invariance: Maxwell eqn's are invariant under $A^\mu \to A^\mu + \partial^\mu \chi$
so some of variables A^μ are redundant.

Need to reduce # of degrees of freedom by "picking a gauge"

Recall $\vec{B} = \vec{\nabla} \times \vec{A}$
$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$

We may choose a gauge in which $\phi = 0$ and $\vec{\nabla}\cdot\vec{A} = 0$
(called "Coulomb gauge" or "radiation gauge")
By doing so, we lose manifest Lorentz invariance, but it is still there.

For simplicity, consider a free electromagnetic field
w/no sources: $\rho = 0$, $\vec{J} = 0$
In which case Gauss's law is
$\vec{\nabla}\cdot\vec{E} = 0$ or $\nabla^2\phi + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = 0$ (*)
(need this later)

Start w/ general gauge w/ $\phi \neq 0$ and $\vec{\nabla}\cdot\vec{A} \neq 0$.

Define $\phi' = \phi + \frac{\partial \chi}{\partial t}$

$\vec{A}' = \vec{A} - \vec{\nabla}\chi$

Choose $\chi(\vec{x},t) = -\int_{t'=0}^{t'=t} \phi(\vec{x},t')\,dt'$

Then $\frac{\partial \chi}{\partial t} = -\phi(\vec{x},t)$

$\Rightarrow \quad \phi' = 0$

$\vec{A}' = \vec{A} - \int_{t'=0}^{t'=t} \vec{\nabla}\phi(\vec{x},t')\,dt'$

Now suppose $\vec{\nabla}\cdot\vec{A}' \neq 0$

recall P300 $\left[\begin{array}{l} V = \int \frac{k\rho}{|x-x'|} = \frac{1}{4\pi}\int \frac{\rho}{|x-x'|} \\ \nabla^2 V = -\frac{\rho}{\epsilon_0} = -\rho \end{array} \right.$

Define $\phi'' = \phi' + \frac{\partial \chi'}{\partial t} = \frac{\partial \chi'}{\partial t}$

$\vec{A}'' = \vec{A}' - \vec{\nabla}\chi'$

Choose $\chi' = -\frac{1}{4\pi}\int d^3x' \, \frac{\vec{\nabla}\cdot\vec{A}'(\vec{x}',t)}{|\vec{x}-\vec{x}'|} \Rightarrow \nabla^2\chi' = +\vec{\nabla}\cdot\vec{A}'$

Then $\vec{\nabla}\cdot\vec{A}'' = \vec{\nabla}\cdot\vec{A}' - \nabla^2\chi' = 0$

But maybe we messed up ϕ

$\phi'' = \frac{\partial \chi'}{\partial t} = -\frac{1}{4\pi}\int d^3x' \, \frac{\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A}'(x',t)}{|x-x'|} = 0$

But Gauss's law $\Rightarrow \nabla^2\phi' + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}') = 0$ and $\phi' = 0$ so

Therefore after 2 gauge transformations, we have

$\phi'' = 0$ and $\vec{\nabla}\cdot\vec{A}'' = 0$.

We now drop the primes, and assume $A^0 = 0, \vec{\nabla}\cdot\vec{A} = 0$

$$A^0 = 0$$
$$\vec{\nabla}\cdot\vec{A} = 0$$

Maxwell's eqns for a free field ($j^\mu = 0$)

$$0 = \partial_\mu F^{\mu\nu} = \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu)$$
$$= \partial^2 A^\nu - \partial^\nu \underbrace{(\partial_\mu A^\mu)}$$
$$\partial_0 A^0 + \vec{\nabla}\cdot\vec{A} = 0$$

$\Rightarrow \partial^2 A^\nu = 0$ wave eqn (KG eqn for massless particle)

Let's solve $\partial^2 \vec{A} = 0$ (e.o.m)
$\vec{\nabla}\cdot\vec{A} = 0$ (gauge choice)

Ansatz: $\boxed{\vec{A} = \vec{\epsilon}\, e^{-ik\cdot x}}$

$$\partial^2 \vec{A} = -k^2 \vec{\epsilon}\, e^{-ikx}$$
$$0 = k^2 = (k^0)^2 - (\vec{k})^2 \Rightarrow k^0 = \pm|\vec{k}| = \pm\omega_k$$

for $\mu = 0$

We can restrict $k^0 = \omega_k$ if we include both

$$\vec{\epsilon}\, e^{-ik\cdot x} \quad \text{and} \quad \vec{\epsilon}\, e^{ik\cdot x}$$

$$\vec{\nabla}\cdot\vec{A} = i\vec{k}\cdot\vec{\epsilon}\, e^{-ik\cdot x}$$

$$\vec{k}\cdot\vec{\epsilon} = 0$$

$\vec{\epsilon}$ must be orthogonal to $\vec{k}$

2 polarizations

Given $\vec{k}$, choose two unit vectors $\vec{\epsilon}_\lambda(\vec{k})$, $\lambda = 1, 2$

perpendicular to $\vec{k}$ & to each other.

[better keep → on $\vec{k}$ due to $\pm\vec{k}$ below]

$$\vec{k}\cdot\vec{\epsilon}_\lambda = 0$$

$$\vec{\epsilon}_\lambda \cdot \vec{\epsilon}_{\lambda'} = \delta_{\lambda\lambda'}$$

Also, let $\vec{\epsilon}_1$, $\vec{\epsilon}_2$, $\hat{k}$ form a right handed basis

$$\Rightarrow \quad \vec{\epsilon}_1 \times \vec{\epsilon}_2 = \hat{k}$$

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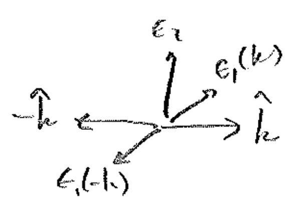
most general solution

$$\vec{A} = \int d\vec{k} \sum_{\lambda=1}^{2} \left[a_\lambda(k)\, e^{-ik\cdot x} + a_\lambda^*(k)\, e^{ik\cdot x} \right] \vec{\epsilon}_\lambda(\vec{k}) \Big|_{k^0=\omega_k}$$

(we require $\vec{A}$ to be real)

It will be useful to establish a relation between $\vec{\epsilon}_\lambda(\vec{k})$ & $\vec{\epsilon}_\lambda(-\vec{k})$

We require $\begin{cases} \vec{\epsilon}_1(-\vec{k}) = -\vec{\epsilon}_1(\vec{k}) \\ \vec{\epsilon}_2(-\vec{k}) = +\vec{\epsilon}_2(\vec{k}) \end{cases}$



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This guarantees $\{\vec{\epsilon}_1(-k), \vec{\epsilon}_2(-k), -\hat{k}\}$ is right handed

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Observe that $\vec{\epsilon}_\lambda(\vec{k})\cdot\vec{\epsilon}_{\lambda'}(-\vec{k}) = (-1)^\lambda \delta_{\lambda\lambda'}$ $\quad(\lambda = 1, 2)$

$$\vec{E} = -\underbrace{\vec{\nabla}\phi}_{0} - \frac{\partial \vec{A}}{\partial t}$$

$$= \int d\vec{k} \sum_{\lambda=1}^{2} \left[a_\lambda(k)\,(i\omega) e^{-ik\cdot x} + a_\lambda^*(k)(-i\omega) e^{ik\cdot x} \right] \vec{\epsilon}_\lambda(\vec{k})$$

$$= \int d\vec{k} \sum_{\lambda=1}^{2} i\omega \left[a_\lambda(k) e^{-ik\cdot x} - a_\lambda^*(k) e^{ik\cdot x} \right] \vec{\epsilon}_\lambda(\vec{k})$$

$\vec{\epsilon}_\lambda$ is the polarization vector of the plane polarized EM wave

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$$\vec{B} = \vec{\nabla}\times\vec{A}$$

$$= \int d\vec{k} \sum_{\lambda=1}^{2} \left[a_\lambda(k)\,(i\vec{k}) e^{-ik\cdot x} + a_\lambda^*(k)(-i\vec{k}) e^{ik\cdot x} \right] \vec{\epsilon}_\lambda(k)$$

$$= \int d\vec{k} \sum_{\lambda=1}^{2} i \left[a_\lambda(k) e^{-ik\cdot x} - a_\lambda^*(k) e^{ik\cdot x} \right] (\vec{k}\times\vec{\epsilon}_\lambda)$$

Observe that $\vec{B} \parallel \vec{k}\times\vec{E}$, as expected.

Quantization

$$\vec{A} = \int \widetilde{dk} \sum_{\lambda=1}^{2} \left[a_\lambda(k)\, e^{-ik\cdot x} + a_\lambda^\dagger(k)\, e^{ik\cdot x} \right] \vec{\epsilon}_\lambda(\vec{k})$$

operators

Impose equal time commutators (analogous to real scalar)

$$[\, a_\lambda(k),\, a^\dagger_{\lambda'}(k')\,] = \hbar\, (2\pi)^3\, (2\omega_k)\, \delta^{(3)}(\vec{k}-\vec{k}')\, \delta_{\lambda\lambda'}$$

where $\omega_k = |\vec{k}|$

$a_\lambda^\dagger(k)$ creates a plane-polarized photon w/ momentum $\vec{k}$ and polarization $\vec{\epsilon}_\lambda(\vec{k})$

$$H = \int d^3x\, T_B^{00} = \int \widetilde{dk} \sum_\lambda \omega_k\, a_\lambda^\dagger(k)\, a_\lambda(k) \qquad [\text{Hw}]$$

$$p^i = \int d^3x\, T_B^{0i} = \int \widetilde{dk} \sum_\lambda k^i\, a_\lambda^\dagger(k)\, a_\lambda(k)$$

Define $|\vec{k}, \lambda\rangle = a_\lambda^\dagger(k)|0\rangle$

Then $|\vec{k}, \lambda\rangle$ represents photon w/ momentum $\hbar\vec{k}$ + energy $\hbar\omega$

(where $E = c|\vec{p}|$ as expected of a massless particle)

+ polarization $\vec{\epsilon}_\lambda(\vec{k})$

HW

$$\vec{E} = \sum_\lambda \int \widetilde{dk} \quad i\omega\, \vec{\epsilon}_{k\lambda} \left[a_{k\lambda} e^{-ik\cdot x} - a^\dagger_{k\lambda} e^{ik\cdot x} \right]$$

$$\vec{B} = \sum_\lambda \int \widetilde{dk} \quad i\vec{k} \times \vec{\epsilon}_{k\lambda} \left[a_{k\lambda} e^{-ik\cdot x} - a^\dagger_{k\lambda} e^{ik\cdot x} \right]$$

$$(\vec{k}\times\vec{\epsilon}_{k\lambda})\cdot(\vec{k}'\times\vec{\epsilon}_{k'\lambda'}) = (\vec{k}\cdot\vec{k}')(\vec{\epsilon}_{k\lambda}\cdot\vec{\epsilon}_{k'\lambda'}) - (\vec{k}\cdot\vec{\epsilon}_{k'\lambda'})(\vec{k}'\cdot\vec{\epsilon}_{k\lambda})$$

Then

$$H = \frac{1}{2}\int d^3x\,(\vec{E}^2 + \vec{B}^2)$$

$$= -\frac{1}{2}\sum_{\lambda\lambda'}\int \widetilde{dk}\,\widetilde{dk}' \left[\omega\omega' \vec{\epsilon}_{k\lambda}\cdot\vec{\epsilon}_{k'\lambda'} + \vec{k}\cdot\vec{k}'\, \vec{\epsilon}_{k\lambda}\cdot\vec{\epsilon}_{k'\lambda'} - \vec{k}\cdot\vec{\epsilon}_{k'\lambda'}\, \vec{k}'\cdot\vec{\epsilon}_{k\lambda} \right]$$

$$\times \Big[a_{k\lambda} a_{k'\lambda'} \underbrace{\int d^3x\, e^{-i(k+k')\cdot x}}_{(2\pi)^3 e^{-2i\omega t}\,\delta(\vec{k}+\vec{k}')} + a^\dagger_{k\lambda} a^\dagger_{k'\lambda'} \underbrace{\int d^3x\, e^{i(k+k')\cdot x}}_{(2\pi)^3 e^{2i\omega t}\,\delta(\vec{k}+\vec{k}')}$$

$$- a_{k\lambda} a^\dagger_{k'\lambda'} \underbrace{\int d^3x\, e^{i(k'-k)\cdot x}}_{(2\pi)^3\,\delta(\vec{k}-\vec{k}')} - a^\dagger_{k\lambda} a_{k'\lambda'} \underbrace{\int d^3x\, e^{i(k-k')\cdot x}}_{(2\pi)^3\,\delta(\vec{k}-\vec{k}')} \Big]$$

Since $\vec{k}' = \pm\vec{k}$, we have $\vec{k}\cdot\vec{\epsilon}_{k'\lambda'} = \pm\vec{k}'\cdot\vec{\epsilon}_{k'\lambda'} = 0$

$\vec{k}'\cdot\vec{\epsilon}_{k\lambda} = \pm\vec{k}\cdot\vec{\epsilon}_{k\lambda} = 0$ So last term $= 0$

For $\vec{k}' = \vec{k}$, $(\vec{k}\cdot\vec{k}')(\vec{\epsilon}_{k\lambda}\cdot\vec{\epsilon}_{k'\lambda'}) = \vec{k}^2\, \vec{\epsilon}^{\,2}_{k\lambda} = \vec{k}^2$

For $\vec{k}' = -\vec{k}$, $(\vec{k}\cdot\vec{k}')(\vec{\epsilon}_{k\lambda}\cdot\vec{\epsilon}_{k'\lambda'}) = -\vec{k}^2\, \vec{\epsilon}_{k\lambda}\cdot\epsilon_{-k,\lambda'} = \underbrace{-\vec{k}^2 (-1)^\lambda \delta_{\lambda\lambda'}}_{\text{But this isn't too relevant}}$

except in the case of momentum when we need to know that

$$\vec{\epsilon}_{-k,\lambda}\cdot\vec{\epsilon}_{k,\lambda'} = \vec{\epsilon}_{k\lambda}\cdot\vec{\epsilon}_{-k,\lambda'}$$

$$H = -\frac{1}{2}\sum_{\lambda}\sum_{\lambda'}\int d\vec{k}\ \frac{1}{2\omega}\Big[e^{-2i\omega t} a_{k\lambda} a_{-k\lambda'} (\underbrace{\omega^2 - \vec{k}^2}_{0}) \vec{\epsilon}_{k\lambda'} \vec{\epsilon}_{-k,\lambda'}$$

$$+ e^{2i\omega t} a^{\dagger}_{k\lambda} a^{\dagger}_{-k,\lambda} (\underbrace{\omega^2 - \vec{k}^2}_{0}) \vec{\epsilon}_{k\lambda'} \epsilon_{-k,\lambda'}$$

$$- a_{k\lambda} a^{\dagger}_{k\lambda'} (\underbrace{\omega^2 + \vec{k}^2}_{2\omega^2}) \delta_{\lambda\lambda'}$$

$$- a^{\dagger}_{k\lambda} a_{k\lambda'} (\underbrace{\omega^2 + \vec{k}^2}_{2\omega^2}) \delta_{\lambda\lambda'} \Big]$$

$$= \sum_{\lambda}\int d\vec{k}\ \frac{\omega}{2}(a_{k\lambda} a^{\dagger}_{k\lambda} + a^{\dagger}_{k\lambda} a_{k\lambda})$$

$$\boxed{H = \sum_{\lambda}\int d\vec{k}\ \omega\, a^{\dagger}_{k\lambda} a_{k\lambda}}$$

HW

$$\vec{P} = \int d^3x\, (\vec{E} \times \vec{B})$$

$$= -\sum_{\lambda\lambda'} \int \widetilde{dk}\, \widetilde{dk'}\, \omega \underbrace{\vec{\epsilon}_{k\lambda} \times (\vec{k}' \times \vec{\epsilon}_{k'\lambda'})} \quad [\text{ as before }]$$

$$\underbrace{\vec{\epsilon}_{k\lambda} \cdot \vec{\epsilon}_{k'\lambda'}}\, \vec{k}' - \underbrace{\vec{\epsilon}_{k\lambda} \cdot \vec{k}'}_{0 \text{ if } \vec{k}' = \pm\vec{k}}\, \vec{\epsilon}_{k'\lambda'}$$

$$\begin{cases} \delta_{\lambda\lambda'} & \text{if } \vec{k}' = \vec{k} \\ (-)^\lambda \delta_{\lambda\lambda'} & \text{if } \vec{k}' = -\vec{k} \end{cases}$$

$$= -\sum_{\lambda} \int \widetilde{dk}\, \frac{1}{2\omega} \omega \vec{k} \left[-a_{k\lambda} a^\dagger_{k\lambda} - a^\dagger_{k\lambda} a_{k\lambda} \right]$$

$$= \sum_{\lambda} \int \widetilde{dk}\, \frac{\vec{k}}{2} \left(a_{k\lambda} a^\dagger_{k\lambda} + a^\dagger_{k\lambda} a_{k\lambda} \right)$$

$$\boxed{\vec{P} = \sum_{\lambda} \int \widetilde{dk}\, \vec{k}\, a^\dagger_{k\lambda} a_{k\lambda}}$$

$$-\sum_{\lambda} \int \widetilde{dk}\, \frac{1}{2\omega}\, \omega (-\vec{k}) (-)^\lambda e^{-2i\omega t} \underbrace{a_{k\lambda} a_{-k\lambda}}_{= a_{-k\lambda} a_{k\lambda}}$$

we need to use $\boxed{\vec{\epsilon}_{k\lambda} \cdot \vec{\epsilon}_{-k,\lambda'} = \vec{\epsilon}_{-k,\lambda} \cdot \vec{\epsilon}_{k,\lambda'}}$ to show this "odd"

so integrand is odd under $\vec{k} \to -\vec{k}$ & $\therefore$ integral vanishes