Electromagnetism

Maxwell's eqns

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \qquad \text{(Gauss)}$$

$$\vec{\nabla}\times\vec{B} - \mu_0\epsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{j} \qquad \text{(Ampere)}$$

Recall $\mu_0 \epsilon_0 = \frac{1}{c^2} = 1$

$$\vec{\nabla}\times\vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \qquad \text{(Faraday)}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \text{(no monopoles)}$$

These equations are self consistent iff current is conserved

Consider $\frac{\partial \rho}{\partial t} = \epsilon_0 \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{E})$

$$= \epsilon_0 \nabla \cdot \frac{\partial \vec{E}}{\partial t}$$

$$= \nabla\cdot\left(\frac{1}{\mu_0}\vec{\nabla}\times\vec{B} - j\right)$$

$$= -\vec{\nabla}\cdot\vec{j} \qquad \text{because div curl} = 0.$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0$$

Define 4-current: $j^\mu = (\rho, \vec{j}) \Rightarrow \partial_\mu j^\mu = 0$ $\qquad [c=1]$

Suggests current is associated w/ a continuous symmetry (U(1) phase symmetry)

Since div curl = 0

$$\vec{\nabla}\cdot\vec{B} = 0 \quad \Rightarrow \quad \vec{B} = \vec{\nabla}\times\vec{A}$$

$$\boxed{\vec{A} = \text{vector potential}}$$

$$\nabla\times E + \frac{\partial}{\partial t}(\nabla\times\vec{A}) = \vec{\nabla}\times\left(E + \frac{\partial\vec{A}}{\partial t}\right) = 0$$

Also curl grad = 0 so

$$\vec{E} + \frac{\partial\vec{A}}{\partial t} = -\vec{\nabla}\phi$$

arbitrary chosen

$$\boxed{\phi = \text{scalar potential}}$$

$$\Rightarrow \begin{cases} \vec{E} = -\vec{\nabla}\phi - \dfrac{\partial\vec{A}}{\partial t} \\ \vec{B} = \vec{\nabla}\times\vec{A} \end{cases}$$

Thus last two Maxwell eqns automatic

$\vec{A}$ and ϕ are not uniquely defined by $\vec{E}$ and $\vec{B}$.

Can add gradient to $\vec{A}$ w/o changing $\vec{B}$

$$\vec{A} \to \vec{A} - \vec{\nabla}\chi \quad \Rightarrow \quad \vec{B} = \vec{\nabla}\times\vec{A} \to \vec{\nabla}\times\vec{A} - \underbrace{\vec{\nabla}\times\vec{\nabla}\chi}_{0} = \vec{B}$$

This doesn't change $\vec{E}$ either, provided

$$\phi \to \phi + \frac{\partial\chi}{\partial t}$$

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t} \to -\vec{\nabla}\left(\phi + \frac{\partial\chi}{\partial t}\right) - \frac{\partial}{\partial t}\left(\vec{A} - \vec{\nabla}\chi\right)$$

$$= -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t} = \vec{E}$$

"gauge transformation" $\left\{ \begin{array}{l} \vec{A} \to \vec{A} - \vec{\nabla}\chi \\ \phi \to \phi + \dfrac{\partial\chi}{\partial t} \end{array} \right.$

we say "gauge invariant"

has no effect on physical observables $\vec{E}$ and $\vec{B}$

$$\phi \to \phi + \partial^0\chi$$
$$A^i \to A^i - \partial_i\chi = A^i + \partial^i\chi$$

[c=1 here]

Define 4-vector potential $A^\mu = (\phi, A^i)$

gauge transformation: $A^\mu \to A^\mu + \partial^\mu\chi$

A^μ and η^μ transform as 4-vectors under Lz txs.

How do $\vec{E}$ + $\vec{B}$ transform? Are they part of 4-vectors? No!

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$$

$$E^i = -\partial_i A^0 - \partial_0 A^i$$

$$= \partial^i A^0 - \partial^0 A^i$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$B^3 = \partial_1 A^2 - \partial_2 A^1$$

$$= -\partial^1 A^2 + \partial^2 A^1$$

Suggests we define an antisymmetric 2nd rank tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$F^{00} = 0$$

$$F^{i0} = \partial^i A^0 - \partial^0 A^i = E^i$$

$$F^{21} = \partial^2 A^1 - \partial^1 A^2 = B^3$$

$$F^{\overset{\text{row}}{\mu}\overset{\text{column}}{\nu}} = \begin{pmatrix} 0 & & & \\ E^1 & 0 & & \\ E^2 & B^3 & 0 & \\ E^3 & & & 0 \end{pmatrix}$$

[HW: fill in]

[HW: show $\partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} + \partial^\lambda F^{\mu\nu} = 0$ $\Rightarrow$ last 2 maxwell eqns]

Last 2 Maxwell eqns automatically satisfied

when we write $\vec{E}, \vec{B}$ in terms of $\phi, \vec{A}$

ie $F^{\mu\nu}$ in terms of A^μ

$$\boxed{F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu}$$

Then $\partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} + \partial^\lambda F^{\mu\nu}$

$= \partial^\mu(\partial^\nu A^\lambda - \partial^\lambda A^\mu) + \partial^\nu(\partial^\lambda A^\mu - \partial^\mu A^\lambda) + \partial^\lambda(\partial^\mu A^\nu - \partial^\nu A^\mu)$

$= 0$

Also, observe that $F^{\mu\nu}$ is automatically gauge invariant

$A^\mu \to A^\mu + \partial^\mu \chi$

$F^{\mu\nu} \to \partial^\mu(A^\nu + \partial^\nu \chi) - \partial^\nu(A^\mu + \partial^\mu \chi) = F^{\mu\nu} + \underbrace{(\partial^\mu\partial^\nu - \partial^\nu\partial^\mu)}_{0}\chi$

First 2 Maxwell eqns

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$$

$$\vec{\nabla}\times\vec{B} - \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t} = \mu_0\vec{J}$$

$\mu_0\epsilon_0 = \frac{1}{c^2}$. We already set $c = 1$

Also set $\epsilon_0 = 1$ (Heaviside-Lorentz units) $\Rightarrow \mu_0 = 1$

[Coleman: these are the units God uses, so we'll use them too]

$$\left.\begin{aligned} \vec{\nabla}\cdot\vec{E} &= \rho \\ \vec{\nabla}\times\vec{B} - \frac{\partial\vec{E}}{\partial t} &= \vec{J} \end{aligned}\right\}$$

HW: show that these are both expressed by

$$\boxed{\partial_\mu F^{\mu\nu} = j^\nu}$$