We want to consider a <u>complex</u> scalar field.
We could do this by writing $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$

$\boxed{\text{Two real scalar fields}}$

$$\mathcal{L} = \sum_{i=1}^{2} \left[\frac{1}{2} (\partial\phi_i)^2 - \frac{1}{2}\mu_i^2 \phi_i^2 \right] - V_{int}(\phi_1, \phi_2)$$

[trivial to generalize to more fields]

$V_{int}(\phi_1, \phi_2)$ can couple the fields together $\Rightarrow$ mutual interactions

If $V_{int} = 0 \Rightarrow 2$ independent free fields

Canonical momentum density $\pi_i = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_i)} = \partial^0 \phi_i = \dot{\phi}_i$

energy momentum tensor $T^{\mu\nu} = \sum_i (\partial^\mu \phi_i \, \partial^\nu \phi_i) - \eta^{\mu\nu} \mathcal{L}$

$$\mathcal{H} = T^{00} = \sum_i \left[\frac{1}{2}\pi_i^2 + \frac{1}{2}(\nabla\phi_i)^2 + \frac{1}{2}\mu_i \phi_i^2 \right] + V_{int}(\phi_1, \phi_2)$$

$$T^{0j} = \sum_i \left[\pi_i \, \partial^j \phi_i \right]$$

Impose equal time canonical commutation relations

$$[\phi_i(x,t), \phi_j(x',t)] = 0$$
$$[\pi_i(x,t), \pi_j(x',t)] = 0$$
$$[\phi_i(x,t), \pi_j(x',t)] = i\hbar\, \delta_{ij}\, \delta^{(3)}(x - x')$$

Define $\omega_{ik} = \sqrt{k^2 + \mu_i^2}$

mode expansion (as before)

$$\phi_i(x,t) = \int \tilde{dk} \left[a_i(k,t)\, e^{i\vec{k}\cdot\vec{x}} + a_i^\dagger(k,t)\, e^{-i\vec{k}\cdot\vec{x}} \right]$$

$$\Rightarrow \left[\tilde{a}_i(k,t), \tilde{a}_j^\dagger(k',t) \right] = \hbar\, \delta_{ij}\, (2\pi)^3\, (2\omega_{ik})\, \delta^{(3)}(k+k')$$

others vanish.

$$H = \int d^3x\, \mathcal{H} = \int \tilde{dk} \sum_i \omega_{ik}\, a_i^\dagger(k,t)\, a_i(k,t) + \int d^3x\, V_{int}(\phi_1, \phi_2)$$

$$p^j = \int d^3x\, T^{0j} = \int \tilde{dk} \sum_i k^j a_i^\dagger(k,t)\, a_i(k,t)$$

Free fields ($V_{int} = 0$)

$$\Rightarrow \tilde{a}_i(k,t) = e^{-i\omega_{ik}t}\, \tilde{a}_i(k)$$

Vacuum state defined by $\tilde{a}_i(k)|0\rangle = 0$ for all k and $i = 1,2$

$\tilde{a}_i^\dagger(k)|0\rangle$ = state of one particle of type i of momentum $\vec{p} = \hbar\vec{k}$, energy $E = \hbar\omega_{ik} = \sqrt{(\hbar k)^2 + (\hbar\mu_i)^2}$ and mass $m_i = \hbar\mu_i$

Restrict: ① $\mu_1 = \mu_2$

② $V_{int}(\phi_1, \phi_2)$ depends only on combination $\phi_1^2 + \phi_2^2$

$$\mathcal{L} = \frac{1}{2}(\partial\phi_1)^2 + \frac{1}{2}(\partial\phi_2)^2 - \frac{1}{2}\mu^2(\phi_1^2 + \phi_2^2) + V_{int}(\phi_1, \phi_2)$$

?

$\mathcal{L}$ is invariant under a "rotation of fields"

$$\begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix} \longrightarrow \underbrace{\begin{pmatrix}\cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha\end{pmatrix}}_{R = \text{rotation matrix}}\begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix} \Rightarrow \phi_1^2 + \phi_2^2 \text{ invariant}$$

Not a physical rotation but an "internal symmetry"

$SO(2)$ = special orthogonal in 2 dimensions

orthogonal $\Rightarrow \boxed{R^T R = \mathbb{1}}$

$\det(R^T R) = (\det R)^2 = 1 \Rightarrow \det R = \pm 1$

special $\Rightarrow$ restrict to $\boxed{\det R = +1}$

If more scalar fields $(\phi_1, \ldots, \phi_N)$ can impose $SO(N)$ symmetry

Infinitesimal rotation (α small)

$$\begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix} \longrightarrow \begin{pmatrix}1 & \alpha \\ -\alpha & 1\end{pmatrix}\begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix} = \begin{pmatrix}\phi_1 + \alpha\phi_2 \\ \phi_2 - \alpha\phi_1\end{pmatrix}$$

$$\begin{cases}\delta\phi_1 = \alpha\phi_2 \\ \delta\phi_2 = -\alpha\phi_1\end{cases}$$

SO(2) symmetry ⟹ conserved current

$$j^\mu = \sum \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \frac{\delta \phi_i}{\alpha}$$

$$= (\partial^\mu \phi_1) \frac{\delta \phi_1}{\alpha} + (\partial^\mu \phi_2) \frac{\delta \phi_2}{\alpha}$$

$$= (\partial^\mu \phi_1)\, \phi_2 - (\partial^\mu \phi_2)\, \phi_1$$

$$Q = \int d^3x \, j^0 = \int d^3x \left(\pi_1 \phi_2 - \pi_2 \phi_1 \right)$$

[HW: show $Q = i \int \widetilde{dk} \left[a_1^\dagger(k,t)\, a_2(k,t) - a_2^\dagger(k,t)\, a_1(k,t) \right]$]

NB $\omega_{1k} = \omega_{2k} = \omega_k$ since $\mu_1 = \mu_2$

$[\phi_i, \pi_j] \sim \delta_{ij}$

$$Q = \int d^3x \, (\pi_1 \phi_2 - \pi_2 \phi_1) = \int d^3x \, (\pi_1 \phi_2 - \phi_1 \pi_2)$$

$$\phi_i = \int d\tilde{k} \left[\tilde{a}_i(k,t) e^{i\vec{k}\cdot\vec{x}} + \tilde{a}_i^\dagger(k,t) e^{-i\vec{k}\cdot\vec{x}} \right]$$

$$\pi_i = \int d\tilde{k} \left[-i\omega_k \tilde{a}_i(k,t) e^{i\vec{k}\cdot\vec{x}} + i\omega_k \tilde{a}_i^\dagger(k,t) e^{-i\vec{k}\cdot\vec{x}} \right]$$

$$Q = \int d\tilde{k}\, d\tilde{k}' \Big[\tilde{a}_1(k,t)\tilde{a}_2(k',t) \underbrace{\int d^3x \, e^{i(\vec{k}+\vec{k}')\vec{x}}}_{(2\pi)^3 \delta(\vec{k}+\vec{k}')} \big(\underbrace{-i\omega_k + i\omega_{k'}}_{\to 0} \big)$$

$$+ \tilde{a}_1^\dagger(k,t) \tilde{a}_2^\dagger(k',t) \underbrace{\int d^3x \, e^{-i(\vec{k}+\vec{k}')\vec{x}}}_{(2\pi)^3 \delta(\vec{k}+\vec{k}')} \big(\underbrace{i\omega_k - i\omega_{k'}}_{\to 0} \big)$$

$$+ a_1(k,t) \tilde{a}_2^\dagger(k',t) \underbrace{\int d^3x \, e^{i(\vec{k}-\vec{k}')\vec{x}}}_{(2\pi)^3 \delta(\vec{k}-\vec{k}')} \big(\underbrace{-i\omega_k - i\omega_{k'}}_{-2i\omega_k} \big)$$

$$+ \tilde{a}_1^\dagger(k,t) a_2(k',t) \underbrace{\int d^3x \, e^{-i(\vec{k}-\vec{k}')x}}_{(2\pi)^3 \delta(\vec{k}-\vec{k}')} \big(\underbrace{i\omega_k + i\omega_{k'}}_{2i\omega_k} \big)$$

$$= \int d\tilde{k} \, \frac{1}{2\omega_k} \, 2i\omega_k \left[-\tilde{a}_1(k,t) \tilde{a}_2^\dagger(k,t) + \tilde{a}_1^\dagger(k,t) a_2(k,t) \right]$$

$$= i \int d\tilde{k} \left[\tilde{a}_1^\dagger(k,t) a_2(k,t) - \tilde{a}_2^\dagger(k,t) \tilde{a}_1(k,t) \right]$$

$$= i \int d\tilde{k} \left[\tilde{a}_1^\dagger(k) a_2(k) - \tilde{a}_2^\dagger(k) \tilde{a}_1(k) \right]$$

Complex scalar field

Two real scalar fields w/ SO(2) symmetry can be profitably recast as a single complex scalar field

Let $\boxed{\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)}$ $\qquad \frac{1}{\sqrt{2}}$ for later convenience

mass term $\frac{1}{2}\mu^2(\phi_1^2 + \phi_2^2) = \mu(\phi_1 - i\phi_2)(\phi_1 + i\phi_2) = \mu\phi^*\phi = \mu|\phi|^2$

$$\mathcal{L} = (\partial_\mu\phi^*)(\partial^\mu\phi) - \mu^2\phi^*\phi - V_{int}(|\phi|^2)$$

When varying $\mathcal{L}$, treat ϕ and ϕ^* as independent variables (even though they are related by complex conjugation)

$$0 = \frac{\partial\mathcal{L}}{\partial\phi^*} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\phi^*}\right)$$

$$= -\mu^2\phi - \underbrace{V_{int}'(|\phi|^2)}_{\substack{\text{prime = differentiate} \\ \text{w.r.t. argument } |\phi|^2}} \underbrace{\frac{\partial|\phi|^2}{\partial\phi^*}}_{\phi} - \partial_\mu(\partial^\mu\phi)$$

$$\Rightarrow \partial^2\phi + \mu^2\phi + V_{int}'(|\phi|^2)\,\phi = 0$$

$$0 = \frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) \Rightarrow \partial^2\phi^* + \mu^2\phi^* + V_{int}'(|\phi|^2)\phi^* = 0$$

(c.c. of previous eqn)

HW: [show these eqns are equivalent to those derived from $\mathcal{L}(\phi_1, \phi_2)$]

show E-L eqns from $\mathcal{L}(\phi, \phi^\dagger)$ same as from $\mathcal{L}(\phi_1, \phi_2)$

$$\partial^2 \phi + \mu^2 \phi + V_{int}'(|\phi|^2)\, \phi = 0$$

$$\Rightarrow \quad \partial^2 \phi_i + \mu^2 \phi_i + V_{int}'(|\phi|^2)\, \phi_i = 0 \qquad i = 1, 2$$

$$\mathcal{L}(\phi_1, \phi_2) = \Sigma \left[\tfrac{1}{2}(\partial \phi_i)^2 - \tfrac{1}{2}\mu^2 \phi_i^2 \right] - V_{int}\left(\tfrac{1}{2}(\phi_1^2 + \phi_2^2) \right)$$

$$0 = \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right] - \frac{\partial \mathcal{L}}{\partial \phi_i} = \partial^2 \phi_i + \mu^2 \phi_i - V_{int}'\left(\tfrac{1}{2}(\phi_1^2 + \phi_2^2) \right) \underbrace{\frac{\partial\left(\tfrac{1}{2}(\phi_1^2 + \phi_2^2) \right)}{\partial \phi_i}}_{\phi_i}$$

Canonical momenta π & π^* (independent)

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0\phi)} = \dot{\phi}^*$$

$$\pi^* = \frac{\partial \mathcal{L}}{\partial(\partial_0\phi^*)} = \dot{\phi}$$

Legendre transform to obtain Hamiltonian

$$\mathcal{H} = \pi\dot{\phi} - \pi^*\dot{\phi}^* - \mathcal{L}$$

$$= \pi^*\pi + \nabla\phi^*\cdot\nabla\phi + \mu\,\phi^*\phi + V_{int}(|\phi|^2)$$

$$= |\pi|^2 + |\nabla\phi|^2 + \mu|\phi|^2 + V_{int}(|\phi|^2)$$

Energy-momentum tensor

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\partial^\nu\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi^*)}\partial^\nu\phi^* - \eta^{\mu\nu}\mathcal{L}$$

$$= (\partial^\mu\phi^*)(\partial^\nu\phi) + (\partial^\mu\phi)(\partial^\nu\phi^*) - \eta^{\mu\nu}\mathcal{L} \quad \Rightarrow \quad T^{\mu\nu} = T^{\nu\mu}$$

$$T^{00} = \mathcal{H}$$

$$T^{0j} = \dot{\phi}^*\partial^j\phi + \dot{\phi}\,\partial^j\phi^* = \pi\,\partial^j\phi + \pi^*\partial^j\phi^*$$

[show the agree. w/ previous expression for T^{0j}]

$$P^\nu \doteq \int d^3x\, T^{0\nu} = \text{conserved 4-momentum of the field}$$

SO(2) symmetry

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \to \begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$$

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \longrightarrow \frac{1}{\sqrt{2}}\left[(\phi_1\cos\alpha + \phi_2\sin\alpha) + i(-\phi_1\sin\alpha + \phi_2\cos\alpha)\right]$$

$$= \frac{1}{\sqrt{2}}\left[\phi_1\underbrace{(\cos\alpha - i\sin\alpha)}_{e^{-i\alpha}} + \phi_2\underbrace{(\sin\alpha + i\cos\alpha)}_{ie^{-i\alpha}}\right]$$

$$= \frac{1}{\sqrt{2}}e^{-i\alpha}(\phi_1 + i\phi_2)$$

$\phi \longrightarrow e^{-i\alpha}\phi$ U(1) transformation
↳ unitary 1×1 matrix (ie complex phase)

$\phi^* \to e^{i\alpha}\phi^*$

$\Rightarrow \phi^*\phi$ is invariant

[more generally, an SO(2N) symmetry of real fields
$\Rightarrow$ U(N) symmetry of complex fields]

Infinitesimal phase transformation
$\phi \to (1 - i\alpha + \dots)\phi$ $\Rightarrow$ $\delta\phi = -i\alpha\phi$, $\delta\phi^* = i\alpha\phi^*$

Conserved current

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\frac{\delta\phi}{\alpha} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\frac{\delta\phi^*}{\alpha} = (\partial^\mu\phi^*)(-i\phi) + (\partial^\mu\phi)(i\phi^*)$$

$$= -i\left[(\partial^\mu\phi^*)\phi - (\partial^\mu\phi)\phi^*\right]$$

$$\boxed{j^0 = -i(\pi\phi - \pi^*\phi^*)}$$

Impose equal-time commutation relations

$$[\phi(x,t), \pi(x',t)] = i\hbar\, \delta^{(3)}(x-x')$$

$$[\phi^\dagger(x,t), \pi^\dagger(x',t)] = i\hbar\, \delta^{(3)}(x-x')$$

$$[\phi(x,t), \pi^\dagger(x',t)] = 0$$

$$[\phi^\dagger(x,t), \pi(x,t)] = 0$$

all others vanish

[HW: verify these follow from c.c.r of real scalar fields]

ϕ, π are not hermitian

but we say they are <u>observable</u>

in the sense that their $\underset{\text{herm}}{\underline{\text{real}}}$ + $\underset{\text{antiherm}}{\underline{\text{imaginary}}}$ parts

are observable

<u>Mode decomposition</u>

[see my FPS notes for details]

Carefully retracing the steps we used for a real scalar field, we obtain the mode decomposition for a complex scalar field

$$\phi(x,t) = \int \widetilde{dk}\, [\, \tilde{a}(k,t)\, e^{i\vec{k}\cdot\vec{x}} + \tilde{b}^\dagger(k,t)\, e^{-i\vec{k}\cdot\vec{x}} \,]$$

except that we use $b^\dagger$ instead of $a^\dagger$ for the 2nd term because ϕ is no longer hermitian. $[\, \tilde{c}(k,t) = \tilde{b}^\dagger(-k,t) \,]$

$$\phi^\dagger(x,t) = \int \widetilde{dk}\, [\, \tilde{b}(k,t)\, e^{i\vec{k}\cdot\vec{x}} + \tilde{a}^\dagger(k,t)\, e^{-i\vec{k}\cdot\vec{x}} \,]$$

(If $b = a$, then $\phi^\dagger = \phi$)

Also

$$\pi(x,t) = \int \widetilde{dk}\, (-i\omega_k)\, [\, \tilde{b}(k,t)\, e^{+i\vec{k}\cdot\vec{x}} - \tilde{a}^\dagger(k,t)\, e^{-i\vec{k}\cdot\vec{x}} \,]$$

$$\pi^\dagger(x,t) = \int \widetilde{dk}\, (-i\omega_k)\, [\, \tilde{a}(k,t)\, e^{i\vec{k}\cdot\vec{x}} - \tilde{b}^\dagger(k,t)\, e^{-i\vec{k}\cdot\vec{x}} \,]$$

show we may invert these to obtain

$$\tilde{a}(k,t) = \int d^3x\, e^{-i\vec{k}\cdot\vec{x}}\, [\, \omega_k\, \phi(x,t) + i\pi^\dagger(x,t) \,]$$

$$\tilde{b}(k,t) = \int d^3x\, e^{-i\vec{k}\cdot\vec{x}}\, [\, \omega_k\, \phi^\dagger(x,t) + i\pi(x,t) \,]$$

HW [Find expressions for $\tilde{a}$ & $\tilde{b}$ in terms of $\tilde{a}_1$ & $\tilde{a}_2$:

$$a = \frac{1}{\sqrt{2}}(a_1 + i a_2), \qquad a^\dagger = \frac{1}{\sqrt{2}}(a_1^\dagger - i a_2^\dagger)$$

$$b = \frac{1}{\sqrt{2}}(a_1 - i a_2), \qquad b^\dagger = \frac{1}{\sqrt{2}}(a_1^\dagger + i a_2^\dagger) \,]$$

1.2.12 Interacting complex scalar field

For a complex scalar field

$$\mathcal{H} = \pi^*\pi + \nabla\phi^* \cdot \nabla\phi + \bar{m}^2\phi^*\phi + V_{\text{int}}(\phi, \phi^*) \tag{1.86}$$

gives Hamilton's equations

$$\dot{\phi} = \frac{\partial \mathcal{H}}{\partial \pi} = \pi^*, \qquad \dot{\pi^*} = -\frac{\partial \mathcal{H}}{\partial \phi^*} = \nabla^2\phi - \bar{m}^2\phi - \frac{\partial V_{\text{int}}(\phi, \phi^*)}{\partial \phi^*} \tag{1.87}$$

We now Fourier decompose the complex fields[3]

$$\phi(\mathbf{x}, t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \phi_{\mathbf{k}}(t), \qquad \pi^*(\mathbf{x}, t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} (\pi^*)_{\mathbf{k}}(t) \tag{1.88}$$

which can be inverted to give

$$\phi_{\mathbf{k}}(t) = \int d^3\mathbf{x}\; e^{-i\mathbf{k}\cdot\mathbf{x}} \phi(\mathbf{x}, t), \qquad (\pi^*)_{\mathbf{k}}(t) = \int d^3\mathbf{x}\; e^{-i\mathbf{k}\cdot\mathbf{x}} \pi^*(\mathbf{x}, t) \tag{1.89}$$

The equations of motion for the Fourier components are

$$\frac{d\phi_{\mathbf{k}}}{dt} = (\pi^*)_{\mathbf{k}}, \qquad \frac{d(\pi^*)_{\mathbf{k}}}{dt} = -\omega_{\mathbf{k}}^2\phi_{\mathbf{k}} - \int d^3\mathbf{x}\, e^{-i\mathbf{k}\cdot\mathbf{x}} \frac{\partial V_{\text{int}}(\phi, \phi^*)}{\partial \phi^*}, \qquad \omega_{\mathbf{k}} \equiv \sqrt{\mathbf{k}^2 + \bar{m}^2} \tag{1.90}$$

As before, we are inspired to define

$$\tilde{a}_{\mathbf{k}}(t) = \frac{\beta_{\mathbf{k}}}{2}\left[\phi_{\mathbf{k}}(t) + \frac{i}{\omega_{\mathbf{k}}}(\pi^*)_{\mathbf{k}}(t)\right], \qquad \tilde{c}_{\mathbf{k}}(t) = \frac{\beta_{\mathbf{k}}}{2}\left[\phi_{\mathbf{k}}(t) - \frac{i}{\omega_{\mathbf{k}}}(\pi^*)_{\mathbf{k}}(t)\right] \tag{1.91}$$

We leave $\beta_{\mathbf{k}}$ arbitrary for now, but require that it depend only on $|\mathbf{k}|$.

Combine eqs. (1.90) and (1.91) to obtain the equations of motion

$$\begin{aligned} \frac{d\tilde{a}_{\mathbf{k}}}{dt} &= -i\omega_{\mathbf{k}}\tilde{a}_{\mathbf{k}} - \frac{i\beta_{\mathbf{k}}}{2\omega_{\mathbf{k}}} \int d^3\mathbf{x}\; e^{-i\mathbf{k}\cdot\mathbf{x}} \frac{\partial V_{\text{int}}(\phi, \phi^*)}{\partial \phi^*}, \\ \frac{d\tilde{c}_{\mathbf{k}}}{dt} &= \quad i\omega_{\mathbf{k}}\tilde{c}_{\mathbf{k}} + \frac{i\beta_{\mathbf{k}}}{2\omega_{\mathbf{k}}} \int d^3\mathbf{x}\; e^{-i\mathbf{k}\cdot\mathbf{x}} \frac{\partial V_{\text{int}}(\phi, \phi^*)}{\partial \phi^*}. \end{aligned} \tag{1.92}$$

We combine eqs. (1.91) and (1.89) to write

$$\begin{aligned} \tilde{a}_{\mathbf{k}}(t) &= \frac{\beta_{\mathbf{k}}}{2} \int d^3\mathbf{x}\; e^{-i\mathbf{k}\cdot\mathbf{x}} \left[\phi(\mathbf{x}, t) + \frac{i}{\omega_{\mathbf{k}}}\pi^*(\mathbf{x}, t)\right], \\ \tilde{c}_{\mathbf{k}}(t) &= \frac{\beta_{\mathbf{k}}}{2} \int d^3\mathbf{x}\; e^{-i\mathbf{k}\cdot\mathbf{x}} \left[\phi(\mathbf{x}, t) - \frac{i}{\omega_{\mathbf{k}}}\pi^*(\mathbf{x}, t)\right], \end{aligned} \tag{1.93}$$

[3] We use $(\pi^*)_{\mathbf{k}}$ so as not to confuse with $(\pi_{\mathbf{k}})^*$, which is different.

We now define

$$\tilde{c}_{\mathbf{k}}(t) = \tilde{b}^*_{-\mathbf{k}}(t) \tag{1.94}$$

so that we may rewrite eq. (1.93) as

$$\begin{aligned}
\tilde{a}_{\mathbf{k}}(t) &= \frac{\beta_{\mathbf{k}}}{2}\int d^3\mathbf{x}\, e^{-i\mathbf{k}\cdot\mathbf{x}}\left[\phi(\mathbf{x},t) + \frac{i}{\omega_{\mathbf{k}}}\pi^*(\mathbf{x},t)\right], \\
\tilde{b}_{\mathbf{k}}(t) &= \frac{\beta_{\mathbf{k}}}{2}\int d^3\mathbf{x}\, e^{-i\mathbf{k}\cdot\mathbf{x}}\left[\phi^*(\mathbf{x},t) + \frac{i}{\omega_{\mathbf{k}}}\pi(\mathbf{x},t)\right],
\end{aligned} \tag{1.95}$$

which by eq. (1.92) satisfy the equations of motion

$$\begin{aligned}
\frac{d\tilde{a}_{\mathbf{k}}}{dt} &= -i\omega_{\mathbf{k}}\tilde{a}_{\mathbf{k}} - \frac{i\beta_{\mathbf{k}}}{2\omega_{\mathbf{k}}}\int d^3\mathbf{x}\, e^{-i\mathbf{k}\cdot\mathbf{x}}\frac{\partial V_{\text{int}}(\phi,\phi^*)}{\partial\phi^*}, \\
\frac{d\tilde{b}_{\mathbf{k}}}{dt} &= -i\omega_{\mathbf{k}}\tilde{b}_{\mathbf{k}} - \frac{i\beta_{\mathbf{k}}}{2\omega_{\mathbf{k}}}\int d^3\mathbf{x}\, e^{-i\mathbf{k}\cdot\mathbf{x}}\left[\frac{\partial V_{\text{int}}(\phi,\phi^*)}{\partial\phi^*}\right]^*.
\end{aligned} \tag{1.96}$$

We may invert eq. (1.91) to obtain

$$\begin{aligned}
\phi_{\mathbf{k}}(t) &= \frac{1}{\beta_{\mathbf{k}}}\left[\tilde{a}_{\mathbf{k}}(t) + \tilde{c}_{\mathbf{k}}(t)\right] &&= \frac{1}{\beta_{\mathbf{k}}}\left[\tilde{a}_{\mathbf{k}}(t) + \tilde{b}^*_{-\mathbf{k}}(t)\right], \\
(\pi^*)_{\mathbf{k}}(t) &= -\frac{i\omega_{\mathbf{k}}}{\beta_{\mathbf{k}}}\left[\tilde{a}_{\mathbf{k}}(t) - \tilde{c}_{\mathbf{k}}(t)\right] &&= -\frac{i\omega_{\mathbf{k}}}{\beta_{\mathbf{k}}}\left[\tilde{a}_{\mathbf{k}}(t) - \tilde{b}^*_{-\mathbf{k}}(t)\right]
\end{aligned} \tag{1.97}$$

which we plug into eq. (1.88) to get

$$\begin{aligned}
\phi(\mathbf{x},t) &= \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{\beta_{\mathbf{k}}}\left[\tilde{a}_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{x}} + \tilde{b}^*_{\mathbf{k}}(t)e^{-i\mathbf{k}\cdot\mathbf{x}}\right], \\
\pi^*(\mathbf{x},t) &= \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{(-i\omega_{\mathbf{k}})}{\beta_{\mathbf{k}}}\left[\tilde{a}_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{x}} - \tilde{b}^*_{\mathbf{k}}(t)e^{-i\mathbf{k}\cdot\mathbf{x}}\right]
\end{aligned} \tag{1.98}$$

where we have let $\mathbf{k} \to -\mathbf{k}$ in the second term of the integral, and used $\beta_{-\mathbf{k}} = \beta_{\mathbf{k}}$.

Equal-time commutation relations

$$[\tilde{a}(k,t), \tilde{a}^\dagger(k',t)] = \hbar\ (2\pi)^3\ 2\omega_k\ \delta^{(3)}(k-k')$$
$$[\tilde{b}(k,t), \tilde{b}^\dagger(k',t)] = \hbar\ (2\pi)^3\ 2\omega_k\ \delta^{(3)}(k-k')$$

all other equal-time commutators vanish [verify using either $a, b \to a_1, a_2$ or using $[\phi, \pi]$ rels]

Plug in mode expansions to show

$$H = \int d^3x\, \mathcal{H} = \int \widetilde{dk}\ \ \omega_k \left[\tilde{a}^\dagger(k)\tilde{a}(k) + \tilde{b}^\dagger(k)\, b(k)\right] + \int d^3x\, V_{int}$$

$$\vec{P} = \int d^3x \left[-\pi \vec{\nabla}\phi - \pi^\dagger \vec{\nabla}\phi^\dagger\right] = \int \widetilde{dk}\ \ \vec{k}\left[\tilde{a}^\dagger(k)\tilde{a}(k) + \tilde{b}^\dagger(k)\, b(k)\right]$$

$$Q = \int d^3x \left[-i\left(\pi\phi - \pi^\dagger\phi^\dagger\right)\right] = \int \widetilde{dk}\ \left[\tilde{a}^\dagger(k)\, a(k) - \tilde{b}^\dagger(k)\, b(k)\right]$$

Can verify using either $a, b \leftrightarrow a_1, a_2$ relation

or using mode expansion and $[a, a^\dagger]$ relns

[HW: derive Q using previous expression in terms of a_1, b, etc]

[midterm: derive $\vec{P} = \int \widetilde{dk}\ \vec{k}\,[a^\dagger a - b^\dagger b]$ from $\vec{P} = \int d^3x\,[-\pi\nabla\phi - \pi^\dagger\nabla\phi^\dagger]$ using mode expansion]

Vacuum $|0\rangle$ defined by $\tilde{a}(k)|0\rangle = 0$

$\tilde{b}(k)|0\rangle = 0$

$$H|0\rangle = 0\,,\quad \vec{P}|0\rangle = 0,\quad Q|0\rangle = 0$$

$|\vec{k}\rangle_+ = \tilde{a}^\dagger(k)|0\rangle$

$|\vec{k}\rangle_- = \tilde{b}^\dagger(k)|0\rangle$

States created by $\tilde{a}^\dagger(k)$ and $\tilde{b}^\dagger(k)$ have equal momenta & energy but opposite charge

we can think of $b^\dagger$ as creating the antiparticle to $a^\dagger$

[HW: show $|\vec{k}\rangle_\pm$ are eigenstates of Q & find eigenvalues]