Quantum field theory

classical mechanics

$x_i(t)$, $p_i(t)$

classical field theory

$\phi(x,t)$, $\pi(x,t)$

quantum mechanics

$X_i(t)$, $P_i(t)$

operators

quantum field theory
(Heisenberg picture: but we omit subscript H)

$\Phi(x,t)$, $\Pi(x,t)$

operators

Recall x is a parameter labelling the field (analogous to i)
so it is not an operator (just like t)

If ϕ = real field, then Φ = hermitian operator
(hermitian quantum field)

We'll actually use ϕ, π for quantum fields, not Φ, Π

How do classical fields evolve in time? Acc. to e.o.m. (i.e. Euler-Lagrange) QΦ-2

How do quantum fields evolve in time? Via the unitary time evolution operator

$$U(t) = e^{-i\frac{Ht}{\hbar}}$$

$$H = \int d^3x \ \mathcal{H}$$

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi)$$

Unitary time evolution operator $U = e^{-i\frac{Ht}{\hbar}}$

Heisenberg picture fields obey

$$\phi(x,t) = U(t)^\dagger \ \phi(x,0) \ U(t)$$

Schrodinger picture field $\phi(x)$

Since $\dfrac{dU}{dt} = -\dfrac{i}{\hbar} H U$

$$\Rightarrow \quad \frac{\partial\phi}{\partial t} = \frac{1}{i\hbar}[\phi, H]$$

$$\frac{\partial\pi}{\partial t} = \frac{1}{i\hbar}[\pi, H]$$

Heisenberg eqns for quantum fields

How do quantum fields differ from classical fields?
They do not necessarily commute.

Canonical commutation relations

In Schrodinger picture, recall for QM

$$[x_i, x_j] = 0$$
$$[p_i, p_j] = 0$$
$$[x_i, p_j] = i\hbar \delta_{ij}$$

take these to be fundamental relations defining x_i & p_i

For QFT, we postulate

$$[\phi(x,0), \phi(x',0)] = 0$$
$$[\pi(x,0), \pi(x',0)] = 0$$
$$[\phi(x,0), \pi(x',0)] = i\hbar\, \delta^{(3)}(x-x')$$

Recall we proved $[A,B] = C \Rightarrow [A_H(t), B_H(t)] = C_H(t)$

Heisenberg picture equal-time commutation relations

$$[\phi(x,t), \phi(x',t)] = 0$$
$$[\pi(x,t), \pi(x',t)] = 0$$
$$[\phi(x,t), \pi(x',t)] = i\hbar\, \delta^{(3)}(x-x')$$

We do not assume anything about unequal time commutation relations

eg $[\phi(x,t), \phi(x',t')]$

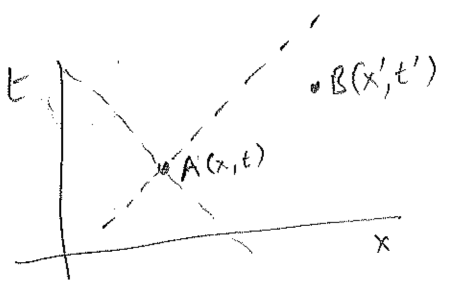
They are complicated & depend on the dynamics of the particular theory (ie on $U(t)$ + $\therefore H$)

We want our QFT to be consistent w/ special relativity

$\mathcal{L}$ and $\therefore$ e.o.m. are invariant under Lorentz transf. ✓

But canonical comm. relations involve equal time
which is a frame dependent statement

Consider operators $A(x,t)$ and $B(x',t')$ at spacelike separated events



A & B occur at different times in one frame
but at equal times in another

(def. of spacelike: $\exists$ a frame in which occur at same time)

Consider $[A(x,t), B(x',t')]$

If this vanishes at equal times in some frame
then it vanishes at unequal times in any Lorentz transformed frame

If $[A(x,t), B(x',t)] = 0$ for $x \neq x'$
then $[A(x,t), B(x',t')] = 0$ for spacelike separated points
i.e. $(t-t')^2 < |\vec{x}-\vec{x}'|^2$

Thus we require
$$\left.\begin{array}{l}[\phi(x,t), \phi(x',t')] = 0 \\ [\phi(x,t), \pi(x',t')] = 0\end{array}\right\} \text{ if } (t-t')^2 < |x-x'|^2$$

Another way to think about it

If $[A, B] \neq 0$ then measurements of each observable affects result of the other (e.g. S_z + S_x)

I.e measure A, then B, then A again, can get a different result for A the 2nd time.

But if A + B are spacelike-separated then measurement of B cannot affect measurement of A (otherwise information has travelled > speed of light)

∴ spacelike separated operators must commute

$$[A(x,t), B(x',t')] = 0 \quad \text{if} \quad (t-t')^2 < |\vec{x}-\vec{x}'|^2$$

Conserved charges generate symmetries

Consider a symmetry of the Lagrangian density

$$\phi \to \phi + \delta\phi \quad \Rightarrow \quad \delta\mathcal{L} = 0$$

Classically there exists a conserved current

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\delta\phi}{\alpha} \quad \Rightarrow \quad \partial_\mu j^\mu = 0$$

$$j^0 = \frac{\partial \mathcal{L}}{\partial \dot\phi} \frac{\delta\phi}{\alpha} = \dot\phi \frac{\delta\phi}{\alpha} = \pi \frac{\delta\phi}{\alpha}$$

$$Q = \int d^3x \, j^0 = \int d^3x' \, \pi \frac{\delta\phi}{\alpha}$$

We now show that the operator Q <u>generates</u> the symmetry transformation of the quantum fields

$$[\phi(x,t), Q] = i\hbar \frac{\delta\phi}{\alpha}$$

Proof: $\left[\phi(x,t), \int d^3x \, \pi(x',t) \frac{\delta\phi}{\alpha}(x',t)\right]$

$$= \int d^3x \, \underbrace{[\phi(x,t), \pi(x',t)]}_{i\hbar\,\delta^{(3)}(x-x')} \frac{\delta\phi}{\alpha}(x',t)$$

$$= i\hbar \frac{\delta\phi}{\alpha}(x,t)$$

[HW: Compute $[\phi(x,t), P^\mu]$ where P^μ = 4-momentum of field]

If use mostly plus metric

change

$$\mathcal{L} = -\tfrac{1}{2}(\partial\phi)^2 - V(\phi)$$

~~we~~ change

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)} = -\partial^\mu\phi$$

inserted to make $E > 0$

$$T^{\mu\nu} = -\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L}\right]$$

$$= -[-\partial^\mu\phi\,\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L}]$$

change

$$= \partial^\mu\phi\,\partial^\nu\phi + \eta^{\mu\nu}\mathcal{L}$$

$$T^{00} = \tfrac{1}{2}\dot\phi^2 + \tfrac{1}{2}(\nabla\phi)^2 + V(\phi)$$

$$T^{0i} = \partial^0\phi\,\partial^i\phi$$

Now $\Pi = \dfrac{\partial\mathcal{L}}{\partial(\partial_0\phi)} = -\partial^0\phi = +\partial_0\phi = \dot\phi$

$$T^{0i} = -\dot\phi\,\partial^i\phi = -\pi\,\partial^i\phi$$

$$[\phi, P^i] = -i\hbar\,\partial^i\phi$$ ~~[illegible]~~

$$[\phi, H] = i\hbar\,\partial_0\phi = -i\hbar\,\partial^0\phi$$

$$\boxed{[\phi, P^\mu] = -i\hbar\,\partial^\mu\phi}$$

[HW]

(a) $$p^i = \int d^3x \; \pi \, \partial^i \phi$$

$$[\phi(x,t), P^i] = \int d^3x' \; \underbrace{[\phi(x,t), \pi(x',t)]}_{i\hbar\,\delta(x-x')} \partial^i \phi(x',t)$$

$$\boxed{[\phi(x,t), P^i] = i\hbar\, \partial^i \phi(x,t)}$$

(b) since $i\hbar\, \partial_0 \phi = [\phi, H]$ by Heisenberg eqn

$$i\hbar\, \partial^0 \phi = [\phi, P^0]$$

$$\therefore \boxed{[\phi(x,t), P^\mu] = i\hbar\, \partial^\mu \phi}$$

(c) $$H = \int d^3x \left[\tfrac{1}{2}\pi^2 + \tfrac{1}{2}(\nabla\phi)^2 + V(\phi) \right]$$

$$[\phi(x,t), H] = \int d^3x \; \tfrac{1}{2}[\phi(x,t), \pi^2(x',t)]$$

$$= \int d^3x \, \tfrac{1}{2}\Big([\phi(x,t), \pi(x',t)]\pi(x',t) + \pi(x',t)[\phi(x,t), \pi(x',t)]\Big)$$

$$= \int d^3x \; i\hbar\, \delta(x-x')\, \pi(x',t)$$

$$= i\hbar\, \pi(x,t)$$

Combine w/ Heisenberg equation to conclude

$$\boxed{\pi(x,t) = \frac{\partial\phi}{\partial t}}$$

Challenge: use $[\pi(x,t), H] = i\hbar \frac{\partial\pi}{\partial t}$ to find the other Hamilton eqn

$$\frac{\partial\pi}{\partial t} = \nabla^2\phi - \frac{\partial V}{\partial\phi}$$

[HW]

Recall the mode expansion

$$\phi(x,t) = \int \tilde{dk} \left[\tilde{a}(k,t)\, e^{i\vec{k}\cdot\vec{x}} + \tilde{a}^{\dagger}(k,t)\, e^{-i\vec{k}\cdot\vec{x}} \right]$$

$$\Pi(x,t) = \int \tilde{dk}\, (-i\omega_k) \left[\tilde{a}(k,t)\, e^{i\vec{k}\cdot\vec{x}} - \tilde{a}^{\dagger}(k,t)\, e^{-i\vec{k}\cdot\vec{x}} \right]$$

(Before we used reality of ϕ to establish a^*; here we use hermiticity of ϕ to $\to a^{\dagger}$)

and their inverses

$$\tilde{a}(k,t) = \int d^3x\; e^{-i\vec{k}\cdot\vec{x}} \left[\omega_k \phi(x,t) + i\Pi(x,t) \right]$$

$$\tilde{a}^{\dagger}(k,t) = \int d^3x\; e^{-i\vec{k}\cdot\vec{x}} \left[\omega_k \phi(x,t) - i\Pi(x,t) \right]$$

Here ϕ + Π are Hermitian fields
$\tilde{a}^{\dagger}$ is the Hermitian conjugate of $\tilde{a}$

(N.B. we have not used Hamilton's eqns for ϕ, π
which have not yet been established for quantum fields)

Calculate equal-time commutators for $\tilde{a}$ + $\tilde{a}^\dagger$

$$[\tilde{a}(k,t), \tilde{a}(k',t)] = \int d^3x\, d^3x'\, e^{-i(\vec{k}\cdot\vec{x}+\vec{k}'\cdot\vec{x}')} [\omega_k \phi(x,t) + i\pi(x,t), \omega_{k'}\phi(x',t) + i\pi(x',t)]$$

$$[\phi,\phi] = [\pi,\pi] = 0 \quad \text{so}$$

$$= \int d^3x\, d^3x'\, e^{-i(\vec{k}\cdot\vec{x}+\vec{k}'\cdot\vec{x}')} \Big(i\omega_k \underbrace{[\phi(x,t), \pi(x',t)]}_{i\hbar\,\delta^{(3)}(x-x')} + i\omega_{k'} \underbrace{[\pi(x,t), \phi(x',t)]}_{-i\hbar\,\delta^{(3)}(x-\vec{x}')} \Big)$$

$$= \underbrace{\int d^3x\, e^{-i(\vec{k}+\vec{k}')\vec{x}}}_{(2\pi)^3\,\delta(\vec{k}+\vec{k}')} \underbrace{(-\hbar\omega_k + \hbar\omega_{k'})}_{0}$$

$$\Rightarrow k' = -\vec{k} \Rightarrow \omega_{k'} = \omega_k$$

$$[\tilde{a}(k,t), \tilde{a}(k',t)] = 0$$

Likewise

$$[\tilde{a}^\dagger(k,t), \tilde{a}^\dagger(k',t)] = 0$$

by a similar calculation or by

$$[A,B] = 0 \Rightarrow [A,B]^\dagger = 0 \Rightarrow (AB-BA)^\dagger = 0$$

$$\Rightarrow B^\dagger A^\dagger - A^\dagger B^\dagger = 0 \Rightarrow [A^\dagger, B^\dagger] = 0$$

$$[\tilde{a}(k,t),\ \tilde{a}^{+}(k',t)]$$

$$\doteq \int d^3x\, d^3x'\, e^{-i\vec{k}\cdot\vec{x} + i\vec{k}'\cdot\vec{x}'}\Big(-i\omega_k \underbrace{[\phi(x,t),\pi(x',t)]}_{i\hbar\,\delta(x-x')} + i\omega_{k'} \underbrace{[\pi(x',t),\phi(x,t)]}_{-i\hbar\,\delta(x-x')}\Big)$$

$$= \underbrace{\int d^3x\ e^{i(\vec{k}'-\vec{k})\cdot\vec{x}}}_{(2\pi)^3\,\delta^{(3)}(\vec{k}-\vec{k}')} \underbrace{(\hbar\omega_k + \hbar\omega_{k'})}_{2\hbar\omega_k}$$

$$\Downarrow$$

$$\vec{k}' = \vec{k} \Rightarrow \omega_{k'} = \omega_k$$

$$\boxed{[\tilde{a}(k,t),\ \tilde{a}^{+}(k',t)] = \hbar\,(2\pi)^3\,(2\omega_k)\,\delta^{(3)}(\vec{k}-\vec{k}')}$$

Can use these commutators in the mode expansion to recover $[\phi,\pi]$ etc.

HW

$$\phi(x,t) = \int \tilde{dk} \left[\tilde{a}(k,t) e^{i\vec{k}\cdot\vec{x}} + \tilde{a}^{\dagger}(k,t) e^{-i\vec{k}\cdot\vec{x}} \right]$$

$$\pi(x,t) = \int \tilde{dk} \, (-i\omega_k) \left[\tilde{a}(k,t) e^{i\vec{k}\cdot\vec{x}} - \tilde{a}^{\dagger}(k,t) e^{-i\vec{k}\cdot\vec{x}} \right]$$

Consider

$$[\phi(x,t), \pi(x',t)] = \int \tilde{dk} \, \tilde{dk}' (-i\omega_k') \Big\{ [\tilde{a}^{\dagger}(k,t), \tilde{a}(k',t)] e^{-ik\cdot x + ik'\cdot x'} - [\tilde{a}(k,t), \tilde{a}^{\dagger}(k',t)] e^{ik\cdot x - ik'\cdot x'} \Big\}$$

$$= \int \frac{d^3k}{(2\pi)^3 (2\omega_k)} (-i\omega_k) \Big\{ -\hbar \, e^{ik(x'-x)} - \hbar \, e^{ik(x-x')} \Big\}$$

$$= i\hbar \, \delta^{(3)}(x-x')$$

Consider

$$[\phi(x,t), \phi(x',t)] = \int \tilde{dk} \, \tilde{dk}' \Big\{ [\tilde{a}^{\dagger}(k,t), \tilde{a}(k',t)] e^{-ikx + ik'x'} + [\tilde{a}(k,t), \tilde{a}^{\dagger}(k',t)] e^{ikx - ik'x'} \Big\}$$

$$= \int \frac{d^3k}{(2\pi)^3 (2\omega_k)} \left[-\hbar \, e^{-ik(x-x')} + \hbar \, e^{ik(x-x')} \right]$$

let $k \to -k$ in 2nd term $\Rightarrow$ vanishes!

Similar for

$[\pi(x,t), \pi(x',t)] = \ldots$

Let's compute H in terms of $\tilde{a}$ and $\tilde{a}^{\dagger}$.

Recall $\mathcal{H} = \frac{1}{2}\left[\mu^2\phi^2 + (\nabla\phi)^2 + \pi^2\right] + V_{int}(\phi)$

$$\phi = \int \widetilde{dk}\left[\tilde{a}(k,t)\, e^{i\vec{k}\cdot\vec{x}} + \tilde{a}^{\dagger}(k,t)\, e^{-i\vec{k}\cdot\vec{x}}\right]$$

$$\vec{\nabla}\phi = \int \widetilde{dk}\left[i\vec{k}\,\tilde{a}(k,t)\, e^{i\vec{k}\cdot\vec{x}} - i\vec{k}\,\tilde{a}^{\dagger}(k,t)\, e^{-i\vec{k}\cdot\vec{x}}\right]$$

$$\pi = \int \widetilde{dk}\left[-i\omega_k\,\tilde{a}(k,t)\, e^{i\vec{k}\cdot\vec{x}} + i\omega_k\,\tilde{a}^{\dagger}(k,t)\, e^{-i\vec{k}\cdot\vec{x}}\right]$$

Then $H = \int d^3x\, \mathcal{H}$

$$= \frac{1}{2}\int \widetilde{dk}\,\widetilde{dk'}\Big[\tilde{a}(k,t)\,\tilde{a}(k',t)\underbrace{\int d^3x\, e^{i(\vec{k}+\vec{k}')\cdot\vec{x}}}_{(2\pi)^3\delta(\vec{k}+\vec{k}')}\left(\mu^2 - \vec{k}\cdot\vec{k}' - \omega_k\omega_{k'}\right)$$

$$+ \tilde{a}^{\dagger}(k,t)\,\tilde{a}^{\dagger}(k',t)\underbrace{\int d^3x\, e^{-i(\vec{k}+\vec{k}')\cdot\vec{x}}}_{(2\pi)^3\delta(\vec{k}+\vec{k}')}\left(\mu^2 - \vec{k}\cdot\vec{k}' - \omega_k\omega_{k'}\right)$$

$$+ \tilde{a}(k,t)\,\tilde{a}^{\dagger}(k',t)\underbrace{\int d^3x\, e^{i(\vec{k}-\vec{k}')\cdot\vec{x}}}_{(2\pi)^3\delta(\vec{k}-\vec{k}')}\left(\mu^2 + \vec{k}\cdot\vec{k}' + \omega_k\omega_{k'}\right)$$

$$+ \tilde{a}^{\dagger}(k,t)\,\tilde{a}(k',t)\underbrace{\int d^3x\, e^{-i(\vec{k}-\vec{k}')\cdot\vec{x}}}_{(2\pi)^3\delta(\vec{k}-\vec{k}')}\left(\mu^2 + \vec{k}\cdot\vec{k}' + \omega_k\omega_{k'}\right)\Big]$$

$+ \int d^3x\, V_{int}(\phi) \longleftarrow$ (a mess which we won't calculate)

Observe

$$\int d\tilde{k}' (2\pi)^3 \delta(\vec{k}'+\vec{k}) f(k') = \int \frac{d^3k'}{(2\pi)^3 2\omega_{k'}} (2\pi)^3 \delta(\vec{k}'+\vec{k}) f(k')$$

$$= \frac{f(-k)}{2\omega_k} \qquad (\text{since } \omega_{-k} = \omega_k)$$

$$\int d\tilde{k}' \, (2\pi)^3 \, \delta(\vec{k}-\vec{k}') f(k') = \frac{f(k)}{2\omega_k} \quad \text{so}$$

$$H = \frac{1}{2}\int d\tilde{k} \frac{1}{2\omega_k}\Big[\tilde{a}(k,t)\tilde{a}(-k,t)(\underbrace{\mu^2+\vec{k}^2-\omega_k^2}_{0})$$

$$+ \tilde{a}^\dagger(k,t)\tilde{a}^\dagger(-k,t)(\underbrace{\mu^2+\vec{k}^2-\omega_k^2}_{0})$$

$$+ \tilde{a}(k,t)\tilde{a}^\dagger(k,t)(\underbrace{\mu^2+\vec{k}^2+\omega_k^2}_{2\omega_k^2})$$

$$+ \tilde{a}^\dagger(k,t)\tilde{a}(k,t)(\underbrace{\mu^2+\vec{k}^2+\omega_k^2}_{2\omega_k^2})\Big] + \int d^3x \, V_{int}(\phi)$$

$$= \int d\tilde{k} \, \frac{\omega_k}{2}\Big(\underbrace{\tilde{a}(k,t)\tilde{a}^\dagger(k,t) + \tilde{a}^\dagger(k,t)\tilde{a}(k,t)}_{[\tilde{a}(k,t),\, \tilde{a}^\dagger(k,t)] + \tilde{a}^\dagger(k,t)\tilde{a}(k,t)}\Big) + \int d^3x \, V_{int}(\phi)$$

$$H = \int d\tilde{k} \left(\omega_k \tilde{a}^\dagger(k,t)\,\tilde{a}(k,t) + \frac{\omega_k}{2} \underbrace{[\tilde{a}(k,t), \tilde{a}^\dagger(k,t)]}_{} \right) + \int d^3x\, V_{int}$$

our first infinity because $\delta(0) = \infty$

This is one we should have expected.

Suppose we add a constant to the Hamiltonian density
(overall scale of energy is not observable)

$$\mathcal{H} \to \mathcal{H} + c$$

Then $H = \int d^3x\, \mathcal{H} \quad \to \quad H + c\,(\text{vol})$

$$\to H + \underbrace{\int d^3x \cdot c}_{c\cdot(\text{vol})}$$

But we have been integrating over all space, so vol $= \infty$.

$$[\tilde{a}(k,t), \tilde{a}^\dagger(k',t)] = \hbar\,(2\pi)^3\,(2\omega_k)\,\delta^{(3)}(k-k')$$

$$= \hbar\,(2\omega_k) \int d^3x\; e^{i(\vec{k}-\vec{k}')\cdot\vec{x}}$$

$$\xrightarrow[k'\to k]{} \hbar\,(2\omega_k) \underbrace{\int d^3x \cdot 1}_{\text{vol}}$$

$$H = \int d\tilde{k} \left(\omega_k \tilde{a}^\dagger(k,t)\,\tilde{a}(k,t) + \hbar\omega_k^2 \cdot (\text{vol}) \right) + \int d^3x\, V_{int}$$

A second infinity is lurking here.

$$(\text{vol}) \int d\tilde{k}\ \hbar \omega_k^2$$

$$= (\text{vol}) \int \frac{d^3k}{(2\pi)^3\, 2\omega_k}\ \hbar \omega_k^2$$

$$= (\text{vol}) \int \frac{d^3k}{(2\pi)^3} \left(\frac{1}{2}\hbar \omega_k \right)$$

↑ zero-point energy for each oscillator (w/ frequency ω_k)

There are an infinite # (continuous) oscillators so an infinite zero point energy.

Specifically, $\omega_k = \sqrt{k^2+\mu^2}$

$$\int d^3k\ \omega_k = \int_0^\infty k^2\, dk\, d\Omega_k\ \sqrt{k^2+\mu^2}$$

$$= \lim_{\Lambda\to\infty} 4\pi \int_0^\Lambda dk\ k^2 \sqrt{k^2+\mu^2}$$

(Perhaps there is a cutoff. E.g. in solids, Debye wavelength)

$$\approx \lim_{\Lambda\to\infty} 4\pi \int_0^\Lambda dk\ k^3$$

k large ⇒ ultraviolet

$$= \lim_{\Lambda\to\infty} \pi \Lambda^4 \quad \text{(ultraviolet divergence)}$$

H contains a term $\sim (\text{vol})\,\Lambda^4$

which diverges as $\text{vol} \to \infty$ and $\Lambda \to \infty$.

But its just a constant (unobservable shift)

so we ignore (drop) it.

$$\therefore \quad H = \int d\tilde{k}\,\left[\omega_k\,\hat{a}^\dagger(k,t)\,\hat{a}(k,t)\right] + \int d^3x\, V_{int}(\phi)$$

Due to spacetime translation invariance,
not only energy but also momentum is conserved

step

$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - V(\phi)$

Recall $T^{\mu\nu} = \partial^\mu\phi\,\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L}$

$P^\nu = \int d^3x\; T^{0\nu}$

$E = P^0 = \int d^3x\; T^{00} = \int d^3x\; \mathcal{H}$

$$P^i = \int d^3x\, T^{0i} = \int d^3x\; \partial^0\phi\,\partial^i\phi = \int d^3x \left(-\underbrace{\partial_0\phi}_{\pi}\;\underbrace{\partial_i\phi}_{\frac{\partial\phi}{\partial x^i}}\right)$$

$$\boxed{\vec{P} = -\int d^3x\;\pi(x,t)\,\vec{\nabla}\phi}$$

HW: show using mode expansions

$$\boxed{\vec{P} = \int \widetilde{dk}\;\vec{k}\;\hat{a}^\dagger(k,t)\,\hat{a}(k,t)}$$

Hint: do not to drop any infinite constants
in this case; they vanish automatically

(HW)

$$\vec{P} \equiv -\int d^3x \; \pi(x,t)\,\vec{\nabla}\phi(x,t)$$

$$\pi = \int \widetilde{dk}\left[-i\omega_k\,\tilde{a}(k,t)\,e^{i\vec{k}\cdot\vec{x}} + i\omega_k\,\tilde{a}^\dagger(k,t)\,e^{-i\vec{k}\cdot\vec{x}}\right]$$

$$\vec{\nabla}\phi = \int \widetilde{dk}\left[i\vec{k}\,\tilde{a}(k,t)\,e^{i\vec{k}\cdot\vec{x}} - i\vec{k}\,\tilde{a}^\dagger(k,t)\,e^{-i\vec{k}\cdot\vec{x}}\right]$$

$$\vec{P} = \int \widetilde{dk}\,\widetilde{dk}'\Big[\tilde{a}(k,t)\tilde{a}(k',t)\underbrace{\int d^3x\; e^{i(\vec{k}+\vec{k}')\cdot\vec{x}}}_{(2\pi)^3\delta(\vec{k}+\vec{k}')}(-\omega_k\vec{k}')$$

$$+\,\tilde{a}^\dagger(k,t)\tilde{a}^\dagger(k',t)\underbrace{\int d^3x\; e^{-i(\vec{k}+\vec{k}')\cdot\vec{x}}}_{(2\pi)^3\delta(\vec{k}+\vec{k}')}(-\omega_k\vec{k}')$$

$$+\,\tilde{a}(k,t)\tilde{a}^\dagger(k',t)\underbrace{\int d^3x\; e^{i(\vec{k}-\vec{k}')\cdot\vec{x}}}_{(2\pi)^3\delta(\vec{k}-\vec{k}')}(\omega_k\vec{k}')$$

$$+\,\tilde{a}^\dagger(k,t)\tilde{a}^\dagger(k',t)\underbrace{\int d^3x\; e^{-i(\vec{k}-\vec{k}')\cdot\vec{x}}}_{(2\pi)^3\delta(k-k')}(\omega_k\vec{k}')\Big]$$

$$= \int \widetilde{dk}\,\frac{1}{2\omega_k}\,\omega_k\vec{k}\Big[\underbrace{\tilde{a}(k,t)\tilde{a}(-k,t)+\tilde{a}^\dagger(k,t)\tilde{a}^\dagger(-k,t)}_{\text{Vanish on integration}} + \underbrace{\tilde{a}(k,t)\tilde{a}^\dagger(k,t)}_{\tilde{a}^\dagger(k,t)\tilde{a}(k,t)\,+\,\underbrace{2\hbar\omega_k\,(\text{vol})}_{\text{Vanish on integration}}} + \tilde{a}^\dagger(k,t)\tilde{a}(k,t)\Big]$$

$$\vec{P} = \int \widetilde{dk}\;\vec{k}\,\tilde{a}^\dagger(k,t)\,\tilde{a}(k,t)$$

Next, consider a free field theory $(V_{int}(\phi) = 0)$

$$H_{free} = \int d\tilde{k} \quad \omega_k \, \hat{a}^\dagger(k,t) \, \hat{a}(k,t)$$

Consider time evolution of $\hat{a}(k,t)$

$$i\hbar \frac{d\hat{a}(k,t)}{dt} = [\hat{a}(k,t), H]$$

$$= \int d\tilde{k}' \, \omega_{k'} \, [\hat{a}(k,t), \hat{a}^\dagger(k',t)] \, \hat{a}(k',t)$$

$$= \int \frac{d^3k'}{(2\pi)^3(2\omega_{k'})} \, \omega_{k'} \, \hbar \, (2\pi)^3 (2\omega_k) \, \delta^{(3)}(k-k') \, \hat{a}(k',t)$$

$$= \hbar \omega_k \, \hat{a}(k,t)$$

$$\Rightarrow \hat{a}(k,t) = e^{-i\omega_k t} \, \hat{a}(k,0) = e^{-i\omega_k t} \underbrace{\hat{a}(k)}_{\text{schrodinger picture operator}}$$

Also $\hat{a}^\dagger(k,t) = e^{i\omega_k t} \, \hat{a}^\dagger(k)$

$$\Rightarrow \begin{cases} H_{free} = \int d\tilde{k} \; \omega_k \, \hat{a}^\dagger(k) \, \hat{a}(k) & \text{(independent of } t\text{)} \\ \vec{P} = \int d\tilde{k} \; \vec{k} \, \hat{a}^\dagger(k) \, \hat{a}(k) \end{cases}$$

$$P^\mu \equiv \int d\tilde{k} \; k^\mu \, \hat{a}^\dagger(k) \, \hat{a}(k) \quad \text{for free field}$$

Define the ground state of the QFT $|0\rangle$ via

$$\tilde{a}_i(k)\,|0\rangle = 0 \quad \text{for all } \vec{k}$$

Consider $$H|0\rangle = \int d\tilde{k}\; \omega_k\, \tilde{a}^{\dagger}(k)\, \underbrace{a(k)\,|0\rangle}_{0} = 0\,|0\rangle$$

$$\vec{P}\,|0\rangle = \int d\tilde{k}\; \vec{k}\; a^{\dagger}(k)\, a(k)\,|0\rangle = 0\,|0\rangle$$

This state has zero energy (after subtracting zero point) and zero momentum.

so it is the vacuum state.

Define $|k\rangle = \tilde{a}^{+}(k)\,|0\rangle$

To show: $|k\rangle$ is an eigenstate of H and $\vec{P}$ (in a free field theory)

First, compute $[H, \tilde{a}^{+}(k)]$

$$= \int \widetilde{dk'}\, \omega_{k'}\, [\tilde{a}^{+}(k')\,\tilde{a}(k'), \tilde{a}^{+}(k)]$$

$$= \int \frac{d^3k'}{(2\pi)^3 (2\omega_{k'})}\, \omega_{k'}\, \tilde{a}^{+}(k')\, \underbrace{[\tilde{a}(k'), \tilde{a}^{+}(k)]}_{\hbar (2\pi)^3 (2\omega_k)\, \delta^{(3)}(k-k')} \Rightarrow \omega_k = \omega_{k'}$$

$$= \hbar\omega_k\, \tilde{a}^{+}(k)$$

Then

$$H\,|k\rangle = H\,\tilde{a}^{+}(k)\,|0\rangle$$

$$= [H, \tilde{a}^{+}(k)]\,|0\rangle + \tilde{a}^{+}(k)\, \underbrace{H|0\rangle}_{0}$$

$$= \hbar\omega_k\, \tilde{a}^{+}(k)\,|0\rangle$$

$$= \hbar\omega_k\, |k\rangle \qquad \text{i.e.} \quad E = \hbar\omega_k$$

Also, you will show $\vec{P}\,|k\rangle = \hbar\vec{k}\,|k\rangle$ i.e. $\vec{p} = \hbar\vec{k}$

$\therefore$ $|k\rangle$ represents a particle of mass $m = \sqrt{E^2 - p^2} = \hbar\sqrt{\omega_k^2 - \vec{k}^2} = \hbar\mu$

Quantize a field, get a particle!

$\tilde{a}^{+}(k)$ = particle creation operator

$\tilde{a}(k)$ = particle annihilation operator

Perhaps this will just be confusing

Consider

$$\tilde{a}(k)|k'\rangle = \tilde{a}(k)\,\tilde{a}^\dagger(k')|0\rangle$$

$$= \left([\tilde{a}(k), a^\dagger(k')] + \tilde{a}^\dagger(k')\,\tilde{a}(k) \right)|0\rangle$$

$$= \hbar(2\pi)^3\, 2\omega_k\, \delta^{(3)}(\vec{k}-\vec{k}')\,|0\rangle$$

If $k \neq k'$ then $\tilde{a}(k)|k'\rangle = 0$

If $k = k'$ then $\tilde{a}(k)|k\rangle \sim |0\rangle$

goes from state of one-particle to the vacuum

Define $|k, k'\rangle = \tilde{a}^\dagger(k)\, \tilde{a}^\dagger(k')\, |0\rangle$

$$H|k,k'\rangle = H\, \tilde{a}^\dagger(k)\, \tilde{a}^\dagger(k')\, |0\rangle$$

$$= \Big(\underbrace{[H, \tilde{a}^\dagger(k)]}_{\hbar\omega_k \tilde{a}^\dagger(k)}\, \tilde{a}^\dagger(k') + \tilde{a}^\dagger(k)\, \underbrace{[H, \tilde{a}^\dagger(k')]}_{\hbar\omega_{k'} \tilde{a}^\dagger(k')} + \tilde{a}^\dagger(k)\, \tilde{a}^\dagger(k')\, \underbrace{H|0\rangle}_{0}$$

$$= (\hbar\omega_k + \hbar\omega_{k'})\, |k, k'\rangle$$

$$\vec{P}\,|k, k'\rangle = (\hbar\vec{k} + \hbar\vec{k}')\, |k, k'\rangle$$

ie a state w/ $E = \hbar\omega_k + \hbar\omega_{k'}$ and $\vec{p} = \hbar\vec{k} + \hbar\vec{k}'$

ie a system of two particles.

Can create an arbitrary # of particles

Since $[\tilde{a}^\dagger(k), \tilde{a}^\dagger(k')] = 0$ we have $|k, k'\rangle = |k', k\rangle$

ie quantum state is symmetric . . .

(identical bosons)

Recall symmetry under spacetime translations $x^\mu \to x^\mu + a^\mu$ implies existence of conserved energy momentum density

$$\partial_\mu T^{\mu\nu} = 0$$

Four-momentum of field $P^\nu = \int d^3x \; T^{0\nu}$ obeys $\frac{dP^\nu}{dt}$

Then the operator P^ν generates space time translations of field

We already know this for the energy $P^0 = H$

$$\phi(x,t) = e^{\frac{iHt}{\hbar}} \phi(x,0)\, e^{-\frac{iHt}{\hbar}}$$

Therefore

$$\phi(x, t+a^0) = e^{\frac{ia^0 H}{\hbar}} \phi(x,t)\, e^{-\frac{ia^0 H}{\hbar}}$$

Analogously $\vec{P}$ generates spatial translation

$$\phi(\vec{x}+\vec{a}, t) = e^{-\frac{i\vec{a}\cdot\vec{P}}{\hbar}} \phi(\vec{x},t)\, e^{\frac{i\vec{a}\cdot\vec{P}}{\hbar}}$$

Using $a\cdot P = a^0 P^0 - \vec{a}\cdot\vec{P}$ we can write

$$\phi(x+a) = e^{\frac{ia\cdot P}{\hbar}} \phi(x)\, e^{-\frac{ia P}{\hbar}}$$

Let's verify: l.h.s $\phi(x+a) = \phi(x) + a^\mu \partial_\mu \phi + O(a^2)$

r.h.s $= (1 + \frac{i}{\hbar} a^\mu P_\mu)\phi(1 - \frac{i}{\hbar}a^\mu P_\mu) = \phi(x) - \frac{ia^\mu}{\hbar}[\phi, P_\mu] + O(a^2)$

In HW you showed $[\phi, P_\mu] = i\hbar \partial_\mu \phi$ so

$$= \phi(x) + a^\mu \partial_\mu \phi \quad \checkmark$$

Recall $\phi(x,t) = \int \widetilde{dk} \left[\hat{a}(k,t)\, e^{i\vec{k}\cdot\vec{x}} + \hat{a}^\dagger(k,t)\, e^{-i\vec{k}\cdot\vec{x}} \right]$

For a free field $\hat{a}(k,t) = \hat{a}(k)\, e^{-i\omega_k t}$

$$\hat{a}^\dagger(k,t) = \hat{a}^\dagger(k)\, e^{i\omega_k t}$$

so

$$\phi(x,t) = \int \widetilde{dk} \left[\hat{a}(k)\, e^{-ik\cdot x} + \hat{a}^\dagger(k) e^{ik\cdot x} \right] \Big|_{k^0 = \omega_k}$$

where $k\cdot x = k^0 t - \vec{k}\cdot\vec{x}$

Consider unequal time commutator.

$$[\phi(x,t), \phi(x',t')] = \int \widetilde{dk}\, \widetilde{dk'} \Big([\hat{a}(k), \hat{a}^\dagger(k')]\, e^{-ik\cdot x + ik'\cdot x'} + [\hat{a}^\dagger(k), \hat{a}(k')]\, e^{ik\cdot x - ik'\cdot x'} \Big) \Big|_{k^0=\omega_k,\, k'^0 = \omega_{k'}}$$

$$= \int \widetilde{dk}\, \frac{d^3k'}{\cancel{(2\pi)^3}\cancel{(2\omega_{k'})}} \left(\hbar \cancel{(2\pi)^3}\cancel{(2\omega_k)}\, \delta^{(3)}(k-k') \right) \left(e^{-ik(x-x')} - e^{ik(x-x')} \right)$$

$$= \hbar \int \widetilde{dk} \left(e^{-ik\cdot(x-x')} - e^{ik\cdot(x-x')} \right) \Big|_{k^0 = \omega_k}$$

Observe that $[\phi(x,t), \phi(x',t')]$ is Lorentz invariant

because $\widetilde{dk}$ and $k\cdot(x-x')$ are

Evaluate it in any frame.

Let (x,t) + (x',t') be space like separated

$$(t-t')^2 < |\vec{x}-\vec{x}'|^2$$

Then $\exists$ frame in which $t=t'$.

In that frame

$$[\phi(x,t),\phi(x',t)] = \hbar \int \frac{d^3k}{(2\pi)^3(2\omega_k)} \left[e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} - e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')} \right]$$

$\underline{\text{not}} = \delta^{(3)}(\vec{x}-\vec{x}')$ due to ω_k!

But let $\vec{k}' = -\vec{k}$ in 2nd term

$$= \hbar \int \frac{d^3k}{(2\pi)^3(2\omega_k)} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} - \hbar \int \frac{d^3k'}{(2\pi)^3(2\omega_k')} e^{i\vec{k}'\cdot(\vec{x}-\vec{x}')}$$

$$= 0$$

So unequal time commutator vanishes for spacelike ~~separated~~ separated fields

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Eq time like: $\hbar \int \frac{d^3k}{(2\pi)^3(2\omega_k)} \Big[\underbrace{e^{-i\omega(t-t')} - e^{i\omega(t-t')}}_{2i \sin\omega_k(t-t')} \Big]$

- skip

Define $\phi(x) = \phi^{+}(x) + \phi^{-}(x)$ may not need these

$$\phi^{+}(x) = \int \widetilde{dk}\; a_k e^{-ik\cdot x}$$

$$\phi^{-}(x) = \int \widetilde{dk}\; a_k^{\dagger} e^{ik\cdot x}$$

$[\phi^{+}(x), \phi^{+}(x')] = 0$ because $[a_k, a_{k'}] = 0$

$[\phi^{-}(x), \phi^{-}(x')] = 0$ because $[a_k^{\dagger}, a_{k'}^{\dagger}] = 0$

Define $\Delta_{+}(x-x') = [\phi^{+}(x), \phi^{-}(x')]$

$$= \int \widetilde{dk}\,\widetilde{dk}'\, [a_k, a_{k'}^{\dagger}]\, e^{-ik\cdot x + ik'\cdot x'}$$

$$= \int \widetilde{dk}\, \frac{d^3k'}{(2\pi)^3(2\omega)}\, (2\pi)^3(2\omega)\,\delta(\vec{k}-\vec{k}')\, e^{-ik\cdot(x-x')}$$

$\delta(\omega-\omega')$

$$= \int \widetilde{dk}\; e^{-ik\cdot(x-x')}$$

$$= \int \frac{d^3k}{(2\pi)^3(2\omega)}\, e^{-ik\cdot(x-x')}$$

indeed, a fcn of $x-x'$

Because of this, answer is not $\delta(\vec{x}-\vec{x}')$

$$[\phi^{-}(x), \phi^{+}(x')] = -[\phi^{+}(x'), \phi^{-}(x)] = -\Delta_{+}(x'-x)$$

Consider (Problem 2016)

$$[\pi(x,t), \phi(0,0)] = \left[\frac{d\phi}{dt}(x,t), \phi(0,0)\right] = \frac{d}{dt}\Delta(x,t)$$

$$= \frac{d}{dt}\int \frac{d^3k}{(2\pi)^3(2\omega)}\left[e^{-i\omega t + i\vec{k}\cdot\vec{x}} - e^{i\omega t - i\vec{k}\cdot\vec{x}}\right]$$

$$= \int \frac{d^3k}{(2\pi)^3}\left[-\frac{i}{2}e^{-i\omega t + i\vec{k}\cdot\vec{x}} - \frac{i}{2}e^{-i\omega t - i\vec{k}\cdot\vec{x}}\right]$$

Again, consider equal time commutators

$$[\pi(x,0), \phi(0,0)] = -\frac{i}{2}\int \frac{d^3k}{(2\pi)^3}\left[e^{i\vec{k}\cdot x} + e^{-i\vec{k}\cdot\vec{x}}\right]$$

$$= -\frac{i}{2}\left[\delta(\vec{x}) + \delta(\cancel{-}\vec{x})\right]$$

$$= -i\delta(\vec{x})$$

vanishes except at $\vec{x}=0$

agrees w/ equal time commutator

problem

Finally $[\pi(x,t), \pi(0,0)]$

Can show that $[\pi(x,0), \pi(0,0)] = 0$