[Repeat earlier calculations in covariant notation.]

Recall action for a real scalar field
(Scalar means Lorentz scalar, i.e. ϕ is invariant under Lz tx.)

$$S = \int d^4x \ \mathcal{L}$$

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - V(\phi)$$

Since $\partial_\mu \phi = \left(\frac{\partial\phi}{\partial t}, \vec{\nabla}\phi\right)$, we have

$$(\partial\phi)^2 = (\partial\phi)\cdot(\partial\phi) = \eta_{\mu\nu}(\partial^\mu\phi)(\partial^\nu\phi) = (\partial^\mu\phi)(\partial_\mu\phi)$$
$$= \eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi)$$
$$= \left(\frac{\partial\phi}{\partial t}\right)^2 - (\vec{\nabla}\phi)^2$$

Therefore
$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - V(\phi)$ which is Lorentz invariant

---

Also under a Lorentz transformation $x' = \Lambda x$
we have $d^4x' = |\det\Lambda|\, d^4x$. [Jacobian]

$O(3,1)$
$SO(3,1)$

Since $|\det\Lambda| = 1$, the measure d^4x is invariant

$\Lambda^T\eta\Lambda = \eta \Rightarrow \det\Lambda^T \det\eta \det\Lambda = \det\eta$
$(\det\Lambda)^2 = 1$

$$\Rightarrow S = \int d^4x \left[\frac{1}{2}(\partial\phi)^2 - V(\phi)\right] \text{ is Lorentz invariant}$$

- Principle of least action: S is stationary under a variation $\delta\phi$ that vanishes at the boundaries (i.e. at t_0 and t_1, and also as $|\vec{x}| \to \infty$)

$$\delta S = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) \right]$$

As shown before $\delta(\partial_\mu \phi) = \partial_\mu (\delta\phi)$.
Integrate by parts using 4d divergence theorem

$$\delta S = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi + \begin{pmatrix} \text{vanishing} \\ \text{surface} \\ \text{term} \end{pmatrix} \right]$$

Since $\delta S = 0$ for arbitrary variation $\delta\phi$

$$\Rightarrow \quad \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0$$

Euler Lagrange eqn for fields

For $\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - V(\phi)$

$$\frac{\partial \mathcal{L}}{\partial\phi} = -\frac{\partial V}{\partial\phi}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \frac{\partial}{\partial(\partial_\mu\phi)}\left[\frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi\right]$$

$$= \frac{1}{2}\eta^{\alpha\beta}\delta^\mu_\alpha\,\partial_\beta\phi + \frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\phi\,\delta^\mu_\beta$$

$$= \frac{1}{2}\eta^{\mu\beta}\partial_\beta\phi + \frac{1}{2}\eta^{\alpha\mu}\partial_\alpha\phi$$

$$= \frac{1}{2}\partial^\mu\phi + \frac{1}{2}\partial^\mu\phi$$

$$= \partial^\mu\phi$$

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) = -\frac{\partial V}{\partial\phi} - \partial_\mu\partial^\mu\phi = 0$$

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\phi + \frac{\partial V}{\partial\phi} = 0$$

$$\ddot{\phi} - \nabla^2\phi + \frac{\partial V}{\partial\phi} = 0$$

If $V = \frac{1}{2}\mu^2\phi^2$, $\ddot{\phi} - \nabla^2\phi + \mu^2\phi = 0$ (Klein Gordon eqn)

→ Recall symmetry of Lagrangian $L(q_i, \dot{q}_i) \Rightarrow Q = \sum \frac{\partial L}{\partial \dot{q}_i} \frac{\delta q_i}{\alpha}$ w/ $\frac{dQ}{dt} = 0$.

Now consider not an arbitrary variation but a symmetry of the Lagrangian density

i.e. $$\mathcal{L}(\phi + \delta\phi, \partial_\mu\phi + \delta(\partial_\mu\phi)) = \mathcal{L}(\phi, \partial_\mu\phi)$$

$$0 = \delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi)$$

Use $\delta(\partial_\mu\phi) = \partial_\mu(\delta\phi)$ and E-L eqn to write

$$= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu(\delta\phi)$$

$$= \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right]$$

We now define the current j^μ associated w/ the symmetry

$$j^\mu \equiv \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\frac{\delta\phi}{\alpha}$$

α = infinitesimal param. associated w/ variation

and we have proved that

$$\partial_\mu j^\mu = 0$$

we say: "the current is conserved"

Noether's theorem for fields: continuous symmetry ⇒ conserved current.

Why do we say that current is conserved?

Let $\underset{\substack{\uparrow \\ \text{4-current} \\ \text{density}}}{j^\mu} = (\underset{\substack{\uparrow \\ \text{charge} \\ \text{density}}}{\rho}, \underset{\substack{\uparrow \\ \text{3-current} \\ \text{density}}}{\vec{j}})$

$$\partial_\mu j^\mu = \partial_0 j^0 + \vec{\nabla}\cdot\vec{j} = \underbrace{\frac{d\rho}{dt} + \vec{\nabla}\cdot\vec{j} = 0}_{\text{called continuity eqn in P3000}}$$

Continuity eqn implies that the amount of charge in a fixed volume changes only if current flows in or out of the boundary of the volume.

$\leftarrow \partial V =$ boundary of V

charge $Q = \int_V d^3x \, \rho$

$$\frac{dQ}{dt} = \int d^3x \, \frac{\partial \rho}{\partial t} = -\int d^3x \, \vec{\nabla}\cdot\vec{j} \underset{\substack{\uparrow \\ \text{divergence} \\ \text{theorem}}}{=} -\oint_{\partial V} \vec{j}\cdot d\vec{A}$$

= (current flowing into the volume)

If no current through boundary (e.g. ∂V at ∞) then $\frac{dQ}{dt} = 0 \Rightarrow Q$ is conserved.

~~Rewrite continuity eqn in Lorentz covariant form~~

$$\cancel{0 = \frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = \partial_\mu j^\mu \quad \text{where } j^\mu = (\rho, \vec{j})}$$

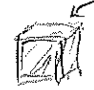
This is valid provided j^μ transforms as a 4-vector. W/o a full proof, let's see that this is plausible

$$j'^\mu = \Lambda^\mu{}_\nu j^\nu$$

Consider a boost in $+x$ direction $\Lambda = \begin{pmatrix} \gamma & -\gamma v & & \\ -\gamma v & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

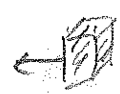
$$\rho' = \gamma\rho - \gamma v j_x$$
$$j'_x = -\gamma v \rho + \gamma j_x$$

Suppose in unprimed frame $j_x = 0$



charge in box $= \rho\, dx\, dy\, dz$

then in primed frame $\rho' = \gamma\rho$, $j'_x = -\gamma v \rho$ ✓



box is Lorentz contracted by factor γ

$\rho \uparrow$ by γ

$\int j^0 d^3x$

$$Q = \int \rho\, d^3x$$

↑

Lorentz invariant

$\int j^0 d^3x$

dx

$\int \frac{d^3k}{(2\pi)^3 (2\omega)}$

$d^3k \quad dk_x dk_y dk_z$

ω

$dk'_x \frac{\partial k'_x}{\partial x}$

Spacetime translation invariance

The previous approach must be generalized for spacetime translation symmetry because $\mathcal{L}$ is not invariant but rather changes due to its dependence on the field.

$$x^\nu \to x^\nu + a^\nu \qquad a^\nu = (\underset{\substack{\uparrow \\ \text{shift in time}}}{a^0}, \underset{\text{spatial shift}}{\vec{a}})$$

$$\phi(x) \to \phi(x+a) = \phi(x) + \underbrace{a^\nu \partial_\nu \phi}_{\delta\phi} + \dots \leftarrow O(a^2) \text{ terms}$$

$$\partial_\mu \phi(x) \to \partial_\mu \phi(x+a) = \partial_\mu \phi(x) + \underbrace{a^\nu \partial_\nu (\partial_\mu \phi)}_{\delta(\partial_\mu \phi)} \underset{\substack{\uparrow \\ \text{observe}}}{=} \partial_\mu(\delta\phi)$$

$$\mathcal{L}(x) \to \mathcal{L}(x+a) = \mathcal{L}(x) + \underbrace{a^\nu \partial_\nu \mathcal{L}}_{\delta\mathcal{L}} + \dots$$

However, the change in $\mathcal{L}$ is due only to its dependence on ϕ (and $\partial_\mu \phi$)

ie $\mathcal{L}(\phi(x), \partial_\mu \phi(x))$ not $\mathcal{L}(\phi(x), \partial_\mu \phi(x), x)$

Thus

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi} \delta \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi)$$

$$= \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) a_\nu \partial^\nu \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu (a_\nu \partial^\nu \phi)$$

$$= a_\nu \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi \right]$$

We show earlier that

$$\delta \mathcal{L} = a_\nu \partial^\nu \mathcal{L} = a_\nu \eta^{\nu\mu} \partial_\mu \mathcal{L} = a_\nu \partial_\mu (\eta^{\mu\nu} \mathcal{L})$$

Equating these

$$a_\nu \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi \right] = a_\nu \partial_\mu \left[\eta^{\mu\nu} \mathcal{L} \right]$$

or

$$0 = a_\nu \partial_\mu \Big[\underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - \eta^{\mu\nu} \mathcal{L}}_{\text{define this to be } T^{\mu\nu}} \Big]$$

valid for arbitrary (but constant) a^μ

We have shown that spacetime translation invariance

$$\Rightarrow \quad 0 = \partial_\mu T^{\mu\nu}$$

where $T^{\mu\nu} = \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - \eta^{\mu\nu} \mathcal{L}$

i.e. a set of 4 conserved currents,
one for each component of $a^\nu = (a^0; \vec{a})$

$$a^0 \Rightarrow \partial_\mu T^{\mu 0} = 0$$
$$a^i \Rightarrow \partial_\mu T^{\mu i} = 0 \qquad (i = 1, 2, 3)$$

The charges associated w/ these 4 currents
we will call P^ν

$$P^\nu = \int d^3x \; T^{0\nu}$$

and we know these charges are conserved

$$\frac{dP^\nu}{dt} = \int_V d^3x \; \partial_0 T^{0\nu} = -\int_V d^3x \; \partial_i T^{i\nu} = -\oint_{\partial V} T^{i\nu} dA^i$$

current density
through surface

[Take a closer look at $T^{\mu\nu}$]

Real scalar field: $\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - V(\phi)$

As before $\dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi$

$$T^{\mu\nu} = \partial^\mu \phi \, \partial^\nu \phi - \eta^{\mu\nu}\left[\tfrac{1}{2}(\partial\phi)^2 - V(\phi)\right]$$

observe T is symmetric under $\mu \leftrightarrow \nu$ $\quad (T^{\mu\nu} = T^{\nu\mu})$
but this will not always be the case.
But for GR, we want $T^{\mu\nu}$ to be symmetric so will need to modify def

$$T^{00} = \dot\phi^2 - \eta^{00}\left[\tfrac{1}{2}\dot\phi^2 - \tfrac{1}{2}(\nabla\phi)^2 - V(\phi)\right] =$$

$$= \tfrac{1}{2}\dot\phi^2 + \tfrac{1}{2}(\nabla\phi)^2 + V(\phi) = \text{Hamiltonian density} \quad (\dot\phi = \pi)$$

i.e. energy density

$$P^0 = \int d^3x \, T^{00} = \int d^3x \, \mathcal{H} = H = \text{energy of the field}$$

$$T^{i0} = (\partial^i \phi)\dot\phi = -(\partial_i \phi)\dot\phi = \text{energy current density}$$

P^ν is a 4-vector, namely the 4-momentum of the field

$$T^{0i} = \dot\phi \, \partial^i \phi = -\dot\phi(\partial_i \phi) = \text{3-momentum density}$$

$$P^i = \int d^3x \, T^{0i} = \int d^3x \, \dot\phi \, \partial^i \phi$$

$$T^{ij} = \text{momentum current density}$$

$T^{\mu\nu}$ = canonical energy-momentum tensor

Conserved by virtue of spacetime translation invariance