Lorentz covariance

notational reformulation to make Lorentz symmetry manifest

Position 4-vector

$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$$

The up position of the index is significant

Lorentz transformations (boosts + rotations)

Under a Lorentz transformation Λ

$$x' = \underset{\substack{\uparrow \\ 4\times 4 \text{ matrix}}}{\Lambda}\ \underset{\uparrow \text{ column vector}}{x}$$

e.g. Boost in $+x$ direction

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$\beta = \frac{v}{c}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$x'^\mu = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu x^\nu \underset{\uparrow}{=} \Lambda^\mu{}_\nu x^\nu$$

Einstein summation convention
(repeated indices, one up + one down, are summed over 0 to 3)

(Let $c=1$ again.)

A 4-vector is any quantity A^μ that transforms as

$$A'^\mu = \Lambda^\mu{}_\nu A^\nu$$

Examples: displacement $\Delta x^\mu = (\Delta t, \Delta \vec{x})$

4-velocity $u^\mu = \frac{dx^\mu}{d\tau} = (\gamma, \gamma\vec{\beta})$

4-momentum $p^\mu = m u^\mu = (E, \vec{p})$

4-wavevector $k^\mu = (\omega, \vec{k})$

Observe: $p^\mu = \hbar k^\mu$

Lorentz invariants

Spacetime interval between two points

$$\Delta s^2 = (\Delta t)^2 - (\Delta \vec{x})^2$$

$$= (\Delta t, \Delta x, \Delta y, \Delta z) \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}}_{\eta \,=\, \text{Minkowski metric}} \begin{pmatrix} \Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}$$

$$= \Delta x^{\mu} \, \eta_{\mu\nu} \, \Delta x^{\nu}$$ [using Einstein]

$$= \eta_{\mu\nu} \, \Delta x^{\mu} \, \Delta x^{\nu}$$ [left right position unimportant]

We are using "mostly-minus" (or West coast) metric convention

Relativists (GR) use "mostly-plus" (East coast)

$$\eta = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$(\Delta s)^2 = -(\Delta t)^2 + (\Delta \vec{x})^2$$

Lorentz transformation is one the leaves Δs^2 invariant

$$(\Delta s')^2 = \eta_{\mu\nu}\,\Delta x'^{\mu}\,\Delta x'^{\nu}$$

$$= \eta_{\mu\nu}\left(\Lambda^{\mu}{}_{\alpha}\,\Delta x^{\alpha}\right)\left(\Lambda^{\nu}{}_{\beta}\,\Delta x^{\beta}\right)$$

$$= \left(\Lambda^{\mu}{}_{\alpha}\,\eta_{\mu\nu}\,\Lambda^{\nu}{}_{\beta}\right)\Delta x^{\alpha}\,\Delta x^{\beta}$$

$$(\Delta s)^2 = \eta_{\alpha\beta}\,\Delta x^{\alpha}\,\Delta x^{\beta}$$

$$\Rightarrow \boxed{\Lambda^{\mu}{}_{\alpha}\,\eta_{\mu\nu}\,\Lambda^{\nu}{}_{\beta} = \eta_{\alpha\beta}}$$

matrices don't care about up or down, but they do care about left + right.

In matrix notation $\Lambda^T \eta \Lambda = \eta$

where Λ^T = transpose of matrix $\Lambda^{\mu}{}_{\alpha}$

For any two 4-vectors A^μ & B^μ

$\underbrace{\eta_{\mu\nu} A^\mu B^\nu}_{}$ is Lorentz invariant. [show]

$A \cdot B$

Also $A^2 \equiv A \cdot A = \eta_{\mu\nu} A^\mu A^\nu$

$$(\Delta x)^2 = \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu = (\Delta s)^2$$

$$p^2 = \eta_{\mu\nu} p^\mu p^\nu = E^2 - \vec{p}^{\,2} = m^2$$

$$k^2 = \eta_{\mu\nu} k^\mu k^\nu = \omega^2 - \vec{k}^{\,2} = 0$$

$$k \cdot x = \eta_{\mu\nu} k^\mu x^\nu = \omega t - \vec{k} \cdot \vec{x}$$

Covariant 4-vectors

A 4-vector A^μ w/ index up is called a contravariant 4-vector

Define a covariant 4-vector A_μ w/ index down by

$$A_\mu = \eta_{\mu\nu} A^\nu \qquad \text{[called lowering indices]}$$

A_μ contains no new information because

$$A_0 = \eta_{00} A^0 + \sum \eta_{0i} A^i = A^0$$

$$A_i = \eta_{i0} A^0 + \sum_j \eta_{ij} A^j = -A^i$$

e.g. $x_\mu = (t, -\vec{x})$

$p_\mu = (E, -\vec{p})$

Observe: $A \cdot B = \eta_{\mu\nu} A^\mu B^\nu = A^\mu B_\mu$

Also $= \eta_{\nu\mu} A^\mu B^\nu = A_\nu B^\nu$

Define inverse metric $\eta^{\mu\nu}$ as $\eta^{\mu\lambda}\eta_{\lambda\nu} = \delta^{\mu}{}_{\nu}$

where $\delta^{\mu}{}_{\nu} = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$ $\Rightarrow$ $\eta^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

"accidentally" the same as $\eta_{\mu\nu}$ [not true in GR]

Can use $\eta^{\mu\nu}$ to raise indices

$$\eta^{\mu\nu} A_{\nu} = \eta^{\mu\nu}(\eta_{\nu\lambda} A^{\lambda}) = \delta^{\mu}{}_{\lambda} A^{\lambda} = A^{\mu}$$

$$A^0 = \eta^{00} A_0 + \sum \eta^{0i} A_i = A_0$$

$$A^i = \eta^{i0} A_0 + \sum \eta^{ij} A_j = -A_i$$

["the metric also raises" w/ apologies to Ernest Hemingway]

A contravariant vector $A = \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix}$ transforms as $A' = \Lambda A$

A covariant vector $\underset{\sim}{A} = \begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{pmatrix}$ transforms as $\underset{\sim}{A}' = (\eta \Lambda \eta) \underset{\sim}{A}$

[Prove in HW]

[Also in HW: show $\partial_{\mu}\phi$ transforms as $\partial\phi' = \Omega\, \partial\phi$

Chain rule $\Rightarrow$ $\Omega = (\Lambda^{-1})^T$ but $\Omega = \eta \Lambda \eta$]

[Proof $\Lambda^T \eta \Lambda = \eta$ $\Rightarrow$ $\Lambda^T \eta \Lambda \eta = 1$ $\Rightarrow$ $\eta \Lambda \eta = (\Lambda^T)^{-1}$]

Could define $A'_{\mu} = \Lambda_{\mu}{}^{\nu} A_{\nu}$ where $\Lambda_{\mu}{}^{\nu} = \eta_{\mu\alpha} \Lambda^{\alpha}{}_{\beta}\, \eta^{\beta\nu}$ $\Rightarrow$ $\tilde{\Lambda} = \eta \Lambda \eta$

$A'_{\mu} B'^{\mu} = \Lambda_{\mu}{}^{\alpha} \Lambda^{\mu}{}_{\beta} A_{\alpha} B^{\beta}$ $\Rightarrow$ $\Lambda_{\mu}{}^{\alpha} \Lambda^{\mu}{}_{\beta} = \delta^{\alpha}_{\beta}$ $\Rightarrow$ $\tilde{\Lambda}^T \Lambda = 1$ $\Rightarrow$ $\tilde{\Lambda}^T = \Lambda^{-1}$]

Gradient

$$\vec{\nabla}\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$

Define 4-gradient $\frac{\partial\phi}{\partial x^\mu} = \left(\frac{\partial\phi}{\partial t}, \vec{\nabla}\phi\right)$

Is it contravariant or covariant?

Could use chain rule, but here's a slick argument [in HW, use chain rule]

Scaler field is Lorentz invariant

$$\phi(x) = \phi'(x')$$

$$d\phi(x) = d\phi'(x')$$

$$\underset{\underset{\text{covariant}}{\uparrow}}{\frac{\partial\phi}{\partial x^\mu}} \; \underset{\underset{\text{contravariant}}{\uparrow}}{dx^\mu} = \frac{\partial\phi'}{\partial x'^\mu}\, dx'^\mu$$

Define $\underset{\text{index down}\nearrow}{\partial_\mu}\phi \equiv \frac{\partial\phi}{\partial x^\mu}$ [ie index goes from up to down when move it from denominator to num.]

Specifically $\partial_0\phi = \frac{\partial\phi}{\partial x^0} = \dot{\phi}$, $\partial_i\phi = \frac{\partial\phi}{\partial x^i} = (\vec{\nabla}\phi)^i$

Also $\partial^\mu\phi = \eta^{\mu\nu}\partial_\nu\phi = \left(\frac{\partial\phi}{\partial t}, -\vec{\nabla}\phi\right)$

Finally $\partial^2\phi = \partial\cdot\partial\phi = \partial_\mu\partial^\mu\phi = \ddot{\phi} - \nabla^2\phi$ (d'Alembertian operator)

HW

$x'^\mu = \Lambda^\mu{}_\nu x^\nu$ $\qquad x' = \Lambda x$

$\frac{\partial x'^\mu}{\partial x^\nu} = \Lambda^\mu{}_\nu$

$x^\mu = (\Lambda^{-1})^\mu{}_\nu x'^\nu$ $\qquad x = \Lambda^{-1} x'$

$\frac{\partial x^\mu}{\partial x'^\nu} = (\Lambda^{-1})^\mu{}_\nu$

$\frac{\partial x'^\mu}{\partial x'^\nu} = \frac{\partial x'^\mu}{\partial x^\lambda}\frac{\partial x^\lambda}{\partial x'^\nu} = \Lambda^\mu{}_\lambda (\Lambda^{-1})^\lambda{}_\nu = \delta^\mu{}_\nu$ $\qquad \Lambda\Lambda^{-1} = 1$

$\frac{\partial \phi}{\partial x'^\nu} = \frac{\partial \phi}{\partial x^\mu}\frac{\partial x^\mu}{\partial x'^\nu} = (\Lambda^{-1})^\mu{}_\nu \frac{\partial \phi}{\partial x^\mu}$

But $\frac{\partial \psi}{\partial x'^\nu} = \Omega_\nu{}^\mu \frac{\partial \psi}{\partial x^\mu}$ $\qquad (\partial'\phi) = \Omega(\partial\psi)$

So $\Omega_\nu{}^\mu = (\Lambda^{-1})^\mu{}_\nu$ [illegible] $\qquad \Omega = (\Lambda^{-1})^T$

$\Lambda^T \eta \Lambda = \eta$

$\Lambda^T \eta \Lambda \eta = 1$

$(\Lambda^T)^{-1} = \eta \Lambda \eta$ $\qquad \Longrightarrow \qquad \partial'\phi = (\eta\Lambda\eta)\partial\phi$

$x'_\mu = \eta_{\mu\alpha} x'^\alpha = \eta_{\mu\alpha}\Lambda^\alpha{}_\beta x^\beta = \eta_{\mu\alpha}\Lambda^\alpha{}_\beta \eta^{\beta\nu} x_\nu \Rightarrow \underset{\sim}{x}' = \eta\Lambda\eta\,\underset{\sim}{x}$

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Thomas: uses primes on indices

$$dx^{\mu'} = \Lambda^{\mu'}{}_{\alpha}\, dx^{\alpha}$$

$$\Lambda^{\mu'}{}_{\alpha} = \frac{\partial x^{\mu'}}{\partial x^{\alpha}}$$

Note positions

$$\omega_{\nu'} = \Lambda^{\beta}{}_{\nu'}\,\omega_{\beta} = \omega_{\beta}\,\Lambda^{\beta}{}_{\nu'}$$

$$\Lambda^{\beta}{}_{\nu'} = \frac{\partial x^{\beta}}{\partial x'^{\nu}}$$

$$\omega_{\mu'}\, dx'^{\mu} = \underbrace{\Lambda^{\beta}{}_{\mu'}\,\Lambda^{\mu'}{}_{\alpha}}_{\delta^{\beta}_{\alpha}}\,\omega_{\beta}\, dx^{\alpha}$$