Hamiltonian for scalar field

Recall $p_i = \frac{\partial L}{\partial \dot{q}_i}$ $\Rightarrow$ $\pi(x,t) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x,t)} = \dot{\phi}(x,t)$

canonical momentum density

Recall $H = \sum p_i \dot{q}_i - L$ $\Rightarrow$ $H = \int d^3x \, [\underbrace{\pi\dot{\phi} - \mathcal{L}}]$

$\mathcal{H}$ = Hamiltonian density

$$\mathcal{H} = \pi\dot{\phi} - \left[\tfrac{1}{2}\dot{\phi}^2 - \tfrac{1}{2}(\nabla\phi)^2 - V(\phi)\right]$$

Use $\dot{\phi} = \pi$ to write

$$\boxed{\begin{aligned}\mathcal{H} &= \tfrac{1}{2}\pi^2 + \tfrac{1}{2}(\nabla\phi)^2 + V(\phi) \\ &= \tfrac{1}{2}\pi^2 + \tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}\mu^2\phi^2 + V_{int}(\phi)\end{aligned}}$$

Hamilton's eqns: $\begin{cases} \dot{\phi} = \dfrac{\partial H}{\partial \pi} \\ \dot{\pi} = -\dfrac{\partial H}{\partial \phi} \end{cases}$

$$\mathcal{H} = \tfrac{1}{2}\pi^2 + \tfrac{1}{2}(\nabla\phi)^2 + V(\phi)$$

$$\delta H = \pi\,\delta\pi + \underbrace{(\nabla\phi)\cdot\nabla\delta\phi}_{-\nabla^2\phi\;\delta\phi} + \frac{\partial V}{\partial\phi}\,\delta\phi$$

$$\begin{cases} \dfrac{\partial H}{\partial \pi} = \pi \\ \dfrac{\partial H}{\partial \phi} = -\nabla^2\phi + \dfrac{\partial V}{\partial \phi} \end{cases}$$

$$\begin{cases} \dot{\phi} = \pi \\ \dot{\pi} = \nabla^2\phi - \dfrac{\partial V}{\partial \phi} = \nabla^2\phi - \mu^2\phi - \dfrac{\partial V_{int}}{\partial \phi} \end{cases}$$

For a free field ($V_{int} = 0$)

$$\begin{cases} \dot{\phi} = \pi \\ \dot{\pi} = (\nabla^2 - \mu^2)\,\phi \end{cases}$$

Similar to harmonic oscillator e.o.m except for ∇^2

To deal w/ ∇^2, we use Fourier transform

[This works for general case, not just free field]

Fourier transform

really $\vec{k}$

$$\phi(x,t) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{x}}\, \tilde{\phi}(k,t)$$

$$\pi(x,t) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{x}}\, \tilde{\pi}(k,t)$$

The inverse relations are [proven later]

$$\tilde{\phi}(k,t) = \int d^3x\, e^{-i\vec{k}\cdot\vec{x}}\, \phi(x,t)$$

$$\tilde{\pi}(k,t) = \int d^3x\, e^{-i\vec{k}\cdot\vec{x}}\, \pi(x,t)$$

- ϕ and $\tilde{\phi}$ carry the same information
- Similar to Fourier series (Phys 3000) but for $-\infty < x < \infty$
  (cf Griffiths, QM, problem ...)
- Could distribute $(2\pi)^3$ differently.
- Observe that, even if $\phi(x,t)$ is a real field ($\phi(x,t) = \phi^*(x,t)$)
  $\tilde{\phi}(k,t)$ is not real!

$$0 = \dot{\phi}(x,t) - \pi(x,t)$$
$$= \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \left[\underbrace{\dot{\tilde{\phi}}(k,t) - \tilde{\pi}(k,t)}_{0} \right]$$

$$0 = \dot{\pi}(x,t) - \nabla^2\phi + \mu^2\phi + \underbrace{\frac{\partial V_{int}}{\partial\phi}}_{\text{quadratic \& higher in } \phi}$$

$$= \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \left[\dot{\tilde{\pi}}(k,t) + k^2\tilde{\phi}(k,t) + \mu^2\tilde{\phi}(k,t) \right] + \left(\text{a mess involving } V_{int} \right)$$

"convolutions"

where we Define $\omega_k = \sqrt{k^2+\mu^2}$, then

$$\begin{cases} \dot{\tilde{\phi}}(k,t) = \tilde{\pi}(k,t) \\ \dot{\tilde{\pi}}(k,t) = -\omega_k^2\,\tilde{\phi}(k,t) \quad + \quad (V_{int} \text{ mess}) \end{cases}$$

Recall for a harmonic oscillator that $\begin{cases} \dot{x} = \frac{p}{m} \\ \dot{p} = -m\omega^2 x \end{cases}$

so we see that a free scalar field ($V_{int} = 0$) is just an infinite collection of independent harmonic oscillators, (normal modes) one for each value of $\vec{k}$ (and with different frequencies ω_k)

(If $V_{int} \neq 0$, the oscillators are coupled in a complicated way.)

Caveat: unlike x & p, $\tilde{\phi}$ and $\tilde{\pi}$ are complex nos.

To uncouple Hamilton's eqns, define

$$\tilde{a}(k,t) = \omega_k \tilde{\phi}(k,t) + i\tilde{\pi}(k,t)$$

Then

$$\dot{\tilde{a}}(k,t) = \omega_k \tilde{\pi}(k,t) - i\omega_k^2 \tilde{\phi}(k,t) + (V_{int} \text{ mess})$$

$$= -i\omega_k \tilde{a}(k,t) + (V_{int} \text{ mess})$$

[hard to solve]

For a free field ($V_{int} = 0$):

$$\dot{\tilde{a}} = -i\omega_k \tilde{a} \quad \Rightarrow \quad \tilde{a}(k,t) = \tilde{a}(k,0)\, e^{-i\omega_k t}$$

Similarly define

$$\tilde{c}(k,t) = \omega_k \tilde{\phi}(k,t) - i\tilde{\pi}(k,t)$$

Why not call this $\tilde{a}^*(k,t)$? Because $\tilde{\phi}, \tilde{\pi}$ not real

Then (show this)

$$\dot{\tilde{c}}(k,t) = i\omega_k \tilde{c}(k,t) + (V_{int} \text{ mess})$$

Free field

$$\dot{\tilde{c}} = i\omega_k \tilde{c} \quad \Rightarrow \quad \tilde{c}(k,t) = \tilde{c}(k,0)\, e^{i\omega_k t}$$

Invert eqns above

$$\tilde{\phi}(k,t) = \frac{1}{2\omega_k}\left[\tilde{a}(k,t) + \tilde{c}(k,t)\right]$$

$$\tilde{\pi}(k,t) = \frac{1}{2i}\left[\tilde{a}(k,t) - \tilde{c}(k,t)\right]$$

But $\tilde{a}$ and $\tilde{c}$ are not independent!

Recall $\tilde{\phi}(k,t) = \int d^3x \, e^{-i\vec{k}\cdot\vec{x}} \phi(x,t)$

If $\phi(x,t)$ is a <u>real</u> scalar field, then

$$\tilde{\phi}^*(k,t) = \int d^3x \, e^{i\vec{k}\cdot\vec{x}} \phi(x,t) = \tilde{\phi}(-k,t)$$

Thus

$$\tilde{a}^*(k,t) = \omega_k \tilde{\phi}^*(k,t) - i\tilde{\pi}^*(k,t)$$

$$= \omega_k \tilde{\phi}(-k,t) - i\tilde{\pi}(-k,t)$$

$$= \tilde{c}(-k,t)$$

or equivalently

$$\boxed{\tilde{c}(k,t) = \tilde{a}^*(-k,t)}$$

Note: this is consistent w/ free field solutions

$$\begin{cases} \tilde{a}(k,t) = \tilde{a}(k,0)\, e^{-i\omega_k t} \\ \tilde{c}(k,t) = \tilde{c}(k,0)\, e^{i\omega_k t} \end{cases}$$

Thus

$$\tilde{\phi}(k,t) = \frac{1}{2\omega_k}\left[\tilde{a}(k,t) + \tilde{a}^*(-k,t)\right]$$

$$\tilde{\pi}(k,t) = \frac{1}{2i}\left[\tilde{a}(k,t) - \tilde{a}^*(-k,t)\right]$$

$$\phi(x,t) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \tilde{\phi}(k,t)$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{i\vec{k}\cdot\vec{x}} \left[\tilde{a}(k,t) + \tilde{a}^\dagger(-k,t) \right]$$

Let $k' = -k$ so the second term is

$$\int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{i\vec{k}\cdot\vec{x}} \tilde{a}^\dagger(-k,t) = \int \frac{d^3k'}{(2\pi)^3 2\omega_{k'}} e^{-i\vec{k}'\cdot\vec{x}} \tilde{a}^\dagger(k',t)$$

We have used $\omega_k = \sqrt{k^2+\mu^2} = \sqrt{k'^2+\mu^2} = \omega_{k'}$

Also $\int_{k_x=-\infty}^{\infty} dk_x = \int_{k_x'=\infty}^{-\infty} (-dk_x') = \int_{k_x'=-\infty}^{\infty} dk_x'$ so $\int d^3k = \int d^3k'$

Now rename the dummy variable $k' \to k$ so

$$\phi(x,t) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[\tilde{a}(k,t) e^{i\vec{k}\cdot\vec{x}} + \tilde{a}^\dagger(k,t) e^{-i\vec{k}\cdot\vec{x}} \right]$$

This is called the mode expansion of $\phi(x,t)$

Similarly

$$\Pi(x,t) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \underbrace{\left(\frac{2\omega_k}{2i}\right)}_{-i\omega_k} \left[\tilde{a}(k,t) e^{i\vec{k}\cdot\vec{x}} - \tilde{a}^\dagger(k,t) e^{-i\vec{k}\cdot\vec{x}} \right]$$

A convenient & natural notation is

$$\widetilde{dk} = \frac{d^3k}{(2\pi)^3\, 2\omega_k}$$

Natural because $\widetilde{dk}$ is a Lorentz-invariant measure (see later)

$$\boxed{\begin{aligned} \phi(x,t) &= \int \widetilde{dk}\, \left[\tilde{a}(k,t)\, e^{i\vec{k}\cdot\vec{x}} + \tilde{a}^*(k,t)\, e^{-i\vec{k}\cdot\vec{x}} \right] \\ \pi(x,t) &= \int \widetilde{dk}\, (-i\omega_k) \left[\tilde{a}(k,t)\, e^{i\vec{k}\cdot\vec{x}} - \tilde{a}^*(k,t)\, e^{-i\vec{k}\cdot\vec{x}} \right] \end{aligned}}$$

(we'll need these later when we quantize)

For a free field, $\tilde{a}(k,t) = \tilde{a}(k,0)\, e^{-i\omega_k t}$
$\tilde{a}^*(k,t) = \tilde{a}^*(k,0)\, e^{i\omega_k t}$

$$\phi_{free}(x,t) = \int \widetilde{dk}\, \left[\tilde{a}(k,0)\, e^{-i(\omega_k t - \vec{k}\cdot\vec{x})} + \tilde{a}^*(k,0)\, e^{i(\omega_k t - \vec{k}\cdot\vec{x})} \right]$$

and similarly for $\pi_{free}(x,t)$

initial conditions that determine $\phi_{free}(x,t)$ at all time

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How find $\tilde{a}(k,t)$ in terms of $\phi(x,t)$?

Fourier's trick

Proof of inverse Fourier transform

Recall Fourier transform

$$\phi(x,t) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{x}}\, \tilde{\phi}(k,t)$$

How do we invert this? "Fourier's trick"

Consider $\quad I_k = \int d^3x\; e^{-i\vec{k}\cdot\vec{x}}\; \phi(x,t)$

$$= \int d^3x\; e^{-i\vec{k}\cdot\vec{x}} \int \frac{d^3k'}{(2\pi)^3}\, e^{i\vec{k}'\cdot\vec{x}}\, \tilde{\phi}(k',t)$$

$$= \int d^3k'\; \tilde{\phi}(k',t)\; \underbrace{\frac{1}{(2\pi)^3}\int d^3x\; e^{i(\vec{k}'-\vec{k})\cdot\vec{x}}}_{\text{call this } \delta^{(3)}(\vec{k}'-\vec{k})}$$

$$\delta^{(3)}(\vec{k}'-\vec{k}) = \left(\frac{1}{2\pi}\int dx\; e^{i(k_x'-k_x)x}\right)\left(\frac{1}{2\pi}\int dy\; e^{i(k_y'-k_y)y}\right)\left(\frac{1}{2\pi}\int dz\; e^{i(k_z'-k_z)z}\right)$$

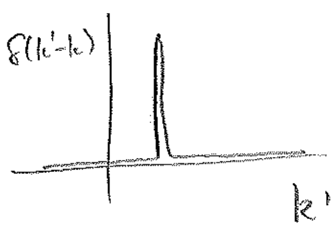
$$= \delta(k_x'-k_x)\;\delta(k_y'-k_y)\;\delta(k_z'-k_z)$$

Dirac delta function

$$\delta(k'-k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx \; e^{i(k'-k)\cdot x} \qquad (*)$$

Observe: if $k' = k \Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} dx = \infty$

if $k' = k \Rightarrow 0$ (phases cancel out)



Key property: $\int_{-\infty}^{\infty} dk' \, \delta(k'-k) = 1$

[we prove this for (*) in Phys 3140]

Furthermore, for any $f(k)$, $\int_{-\infty}^{\infty} dk' \, f(k')\,\delta(k'-k) = f(k)$

Back to 3 dimensions

$$\int_{-\infty}^{\infty} d^3k' \; f(\vec{k}')\,\delta^{(3)}(\vec{k}'-\vec{k})$$

$$= \int_{-\infty}^{\infty} dk_x' \, dk_y' \, dk_z' \; f(k_x', k_y', k_z')\,\delta(k_x'-k_x)\,\delta(k_y'-k_y)\,\delta(k_z'-k_z)$$

$$= f(k_x, k_y, k_z)$$

$$= f(\vec{k})$$

Useful: δ-fcn is even, so $\delta^{(3)}(\vec{k}'-\vec{k}) = \delta^{(3)}(\vec{k}-\vec{k}')$

$$F_k = \int d^3k' \, \tilde{\phi}(k', t) \, \delta^{(3)}(\vec{k}' - \vec{k})$$

$$= \tilde{\phi}(k, t)$$

ie we have established

$$\tilde{\phi}(k, t) = \int d^3x \, e^{-i\vec{k}\cdot\vec{x}} \, \phi(x, t)$$

Similarly

$$\tilde{\pi}(k, t) = \int d^3x \, e^{-i\vec{k}\cdot\vec{x}} \, \pi(x, t)$$

Finally we can express $\tilde{a}(k, t)$ in terms of $\phi(x, t)$ + $\pi(x, t)$

$$\tilde{a}(k, t) = \omega_k \tilde{\phi}(k, t) + i\tilde{\pi}(k, t) \qquad \text{[definition]}$$

$$\boxed{\tilde{a}(k, t) = \int d^3x \, e^{-i\vec{k}\cdot\vec{x}} \left[\omega_k \phi(x, t) + i\pi(x, t)\right]}$$

Use;

$$\phi(x,t) = \int d\tilde{k} \left[\tilde{a}(k,t)\, e^{i\vec{k}\cdot\vec{x}} + \tilde{a}^* e^{-i\vec{k}\cdot\vec{x}} \right]$$

$$\pi(x,t) = \int d\tilde{k}\, (-i\omega_k) \left[\tilde{a}(k,t) e^{i\vec{k}\cdot\vec{x}} - \tilde{a}^* e^{-i\vec{k}\cdot\vec{x}} \right]$$

to show

$$\int d^3x\; e^{-i\vec{k}\cdot\vec{x}} \left[\omega_k \phi(x,t) + i\pi(x,t) \right]$$

$$= \int d^3x\; e^{-i\vec{k}\cdot\vec{x}} \int d\tilde{k}' \left[2\omega_{k'} \tilde{a}(k',t)\, e^{i\vec{k}'\cdot\vec{x}} \right]$$

$$= \int \frac{d^3k'}{(2\pi)^3}\, \tilde{a}(k',t) \underbrace{\int d^3x\; e^{i(k'-k)\cdot\vec{x}}}_{(2\pi)^3 \delta^{(3)}(k'-k)}$$

$$= \tilde{a}(k,t)$$

Starting from

$$\tilde{a}(k,t) = \int d^3x \quad e^{-ik\cdot x} \, [\omega_k \phi(x,t) + i\pi(x,t)]$$

$$\tilde{a}^*(k,t) = \int d^3x \quad e^{ik\cdot x} \, [\omega_k \phi(x,t) - i\pi(x,t)]$$

Consider

$$\int \tilde{dk} \, [\tilde{a}(k,t) \, e^{ik\cdot x} + \tilde{a}^*(k,t) \, e^{-i\vec{k}\cdot\vec{x}}]$$

$$= \int d^3x' \int \tilde{dk} \, \Big\{ e^{ik\cdot(x-x')} \, [\omega_k \phi(x',t) + i\pi(x',t)]$$

$$+ e^{-ik\cdot(x-x')} \, [\omega_k \phi(x',t) - i\pi(x',t)] \Big\}$$

Let $\vec{k} \to -\vec{k}$ in the 2nd term. Then the π terms cancel.

The ϕ terms give

$$\int d^3x' \int \frac{d^3k}{(2\pi)^3(2\omega_k)} \, e^{ik(\vec{x}-\vec{x}')} \, (2\omega_k) \, \phi(x',t)$$

Then letting $\vec{x} \leftrightarrow \vec{k}$ in result from class $\int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-x')} = \delta^{(3)}(\vec{x}-\vec{x}')$

$$= \int d^3x' \, \delta^{(3)}(\vec{x}-\vec{x}') \, \phi(x',t)$$

$$= \phi(x,t)$$

Similarly

$$\int \tilde{dk} \, [-i\omega_k] [\tilde{a}(k,t) e^{ik\cdot x} - \tilde{a}^*(k,t) \, e^{-ik\cdot x}]$$

$$= \int d^3x' \int \tilde{dk} \, (-i\omega_k) \Big\{ e^{ik\cdot(x-x')} \, [\omega_k \phi(x',t) + i\pi(x',t)]$$

$$- e^{-ik(x-x')} [\omega_k \phi(x',t) - i\pi(x',t)] \Big\}$$

Again letting $k \to -\vec{k}$, the ϕ terms cancel

$$= \int d^3x' \int \frac{d^3k \, (-i\omega_k)}{(2\pi)^3(2\omega_k)} \, e^{ik\cdot(x-\vec{x}')} \, (2i\pi(x',t))$$

$$= \int d^3x' \quad \delta^{(3)}(x-x') \, \pi(x',t)$$

$$= \pi(x,t)$$