[our philosophy is to begin, not w/ e.o.m., but w/ Lagrangian]

Lagrangian for scalar field

we mean $\vec{x}$ but omit vectors; boldface in LaTeX

$$\phi(x, t) \longrightarrow q_i(t)$$

x labels generalized coordinates

Recall $L = T - V$ where $T = \sum \frac{1}{2} A_i \dot{q}_i^2$

Suggests $L = \int d^3x \left[\frac{1}{2}\dot{\phi}^2 - V(\phi) \right]$

[But this L gives independent dynamics at each point x whereas for wave motion we need ϕ to influence nearby points → potential energy should depend on $\phi(x+\Delta x) - \phi(x)$ or $\frac{\partial \phi}{\partial x}$]

Special relativity suggests $\dot{\phi}^2$ must be accompanied by $(\nabla\phi)^2$

Lagrangian $L = \int d^3x \, [\underbrace{\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - V(\phi)}_{\equiv \mathcal{L}, \text{ Lagrangian density}}]$

Action $S = \int dt \, L = \int dt \, d^3x \, \mathcal{L} = \int d^4x \, \mathcal{L}$

- Let ϕ = actual motion which satisfies e.o.m.
- Consider other motions $\phi + \delta\phi$, where $\delta\phi$ = variation of ϕ,

but satisfying boundary conditions $\begin{cases} \delta\phi(x, t_0) = 0 \\ \delta\phi(x, t_1) = 0 \end{cases}$

- We'll also require $\delta\phi(x,t) \xrightarrow[|x|\to\infty]{} 0$

, δ-notation: usually use dx or Δx for a single variable, $\delta\phi$ for varying a function

- Principle of least action: actual motion minimizes S
- Varying ϕ causes S to increase, but for small variations, $\delta S = 0$

$$S = \int d^4x \left[\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - V(\phi) \right]$$

Use chain rule to compute δS:

$$\delta S = \int d^4x \left[\dot{\phi}\,\delta\dot{\phi} - \nabla\phi \cdot \delta(\nabla\phi) - \frac{\partial V}{\partial\phi}\,\delta\phi \right]$$

Now $\phi \to \phi + \delta\phi$

$\dot{\phi} \to \dot{\phi} + \frac{d}{dt}(\delta\phi)$ so $\delta\dot{\phi} = \frac{d}{dt}\delta\phi$

$\vec{\nabla}\phi \to \vec{\nabla}\phi + \vec{\nabla}(\delta\phi)$ so $\delta\vec{\nabla}\phi = \vec{\nabla}\delta\phi$

$$\delta S = \int d^4x \left[\dot{\phi}\,\frac{d}{dt}\delta\phi - \vec{\nabla}\phi \cdot \vec{\nabla}\delta\phi - \frac{\partial V}{\partial\phi}\,\delta\phi \right]$$

Integrate by parts

$$\int d^4x \; \dot{\phi} \, \frac{d}{dt} \delta\phi = \int d^3x \underbrace{\int_{t_0}^{t_1} dt \; \frac{d}{dt}(\dot{\phi}\, \delta\phi)}_{\underbrace{\dot{\phi}\,\delta\phi \Big|_{t_0}^{t_1}}_{0}} - \int d^4x \; \ddot{\phi} \; \delta\phi$$

$$\int d^4x \quad \vec{\nabla}\phi \cdot \vec{\nabla}\delta\phi = \int dt \underbrace{\int d^3x \; \vec{\nabla} \cdot (\vec{\nabla}\phi \, \delta\phi)}_{\int_{\text{sphere at } \infty} d\vec{A} \cdot \vec{\nabla}\phi \; \delta\phi} - \int d^4x \; \nabla^2\phi \; \delta\phi$$

$\delta\phi \rightarrow 0$ at ∞

$$\Rightarrow \delta S = \int d^4x \left[-\ddot{\phi} + \nabla^2\phi - \frac{\partial V}{\partial \phi} \right] \delta\phi$$

Since $\delta S = 0$ for all variation $\delta\phi$, we have

$$\boxed{\ddot{\phi} - \nabla^2\phi + \frac{\partial V}{\partial \phi} = 0} \qquad \text{E-L eqn for } \phi$$

Let's assume $\phi = 0$ is a minimum of $V(\phi)$

If not, we have "spontaneous symmetry breaking".

Also, set $V(0) = 0$ by fiat.

Then $$V(\phi) = \underbrace{V(0)}_{0} + \underbrace{V'(0)}_{0}\,\phi + \tfrac{1}{2} V''(0)\,\phi^2 + \ldots$$

$$= \underbrace{\tfrac{1}{2}\mu^2\phi^2}_{\text{"mass term"}} + \underbrace{\tfrac{1}{3!}g\phi^3 + \tfrac{1}{4!}\lambda\phi^4}_{\text{self-interaction terms}} + \ldots$$

self-interaction terms, call these $V_{int}(\phi)$

$$\frac{\partial V}{\partial \phi} = \mu^2\phi + \frac{1}{2!}g\phi^2 + \frac{1}{3!}\lambda\phi^3 + \ldots$$

$$\Rightarrow \ddot{\phi} - \nabla^2\phi + \mu^2\phi + \underbrace{\frac{1}{2!}g\phi^2 + \frac{1}{3!}\lambda\phi^3 + \ldots}_{\text{these terms make e.o.m. nonlinear}} = 0$$

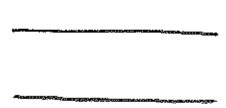
these terms make e.o.m. nonlinear

"Free field" means $V_{int}(\phi) = 0$

$$\Rightarrow V_{free}(\phi) = \tfrac{1}{2}\mu^2\phi^2$$

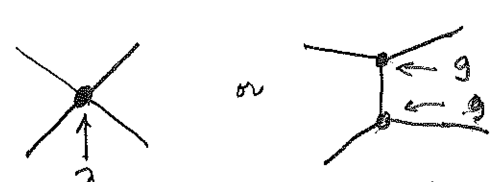
$$\ddot{\phi} - \nabla^2\phi + \mu^2\phi = 0 \qquad \text{Klein Gordon eqn}$$

Non-interacting (free) particles



$\longrightarrow t$

Interacting particles



Feynman vertices correspond to terms in $V_{int}(\phi)$.