Time evolution of states

t.d. Schrödinger eqn

$$i\hbar \frac{\partial \psi}{\partial t} = H\,\psi(x,t) \quad \xRightarrow{\text{in Dirac notation}} \quad i\hbar \frac{d}{dt}|\psi(t)\rangle = H\,|\psi(t)\rangle$$

Solution: $|\psi(t)\rangle = U(t)\,|\psi(0)\rangle$

where $U(t)$ is the "unitary time evolution operator"

$$U(t) = e^{-\frac{iHt}{\hbar}} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{iHt}{\hbar}\right)^n$$

Proof: $$\frac{dU}{dt} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!}\left(-\frac{iHt}{\hbar}\right)^{n-1}\left(-\frac{iH}{\hbar}\right)$$ [H commutes w/ itself]

$$= -\frac{iH}{\hbar}\sum_{n'=0}^{\infty} \frac{1}{n'!}\left(-\frac{iHt}{\hbar}\right)^{n'}$$

$$= -\frac{iH}{\hbar}\,U(t)$$ [can also see this formally from $e^{-\frac{iHt}{\hbar}}$]

Then

$$i\hbar\,|\psi(t)\rangle = i\hbar\,\frac{dU(t)}{dt}\,|\psi(0)\rangle$$

$$= i\hbar\left(-\frac{iH}{\hbar}\right)U(t)\,|\psi(0)\rangle$$

$$= H\,|\psi(t)\rangle$$

QED

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle \Rightarrow \langle\psi(t)| = \langle\psi(0)|U^\dagger(t)$$

$$U^\dagger(t) = \sum \frac{1}{n!}\left(\frac{-iHt}{\hbar}\right)^{n\,\dagger}$$

$$= \sum \frac{1}{n!}\left(\frac{iH^\dagger t}{\hbar}\right)^n$$

$$= \sum \frac{1}{n!}\left(\frac{iHt}{\hbar}\right)^n$$

$$= e^{\frac{iHt}{\hbar}}$$ [can also see this formally $e^{-\frac{iHt}{\hbar}}$]

Observe $U^\dagger(t)\,U(t) = e^{\frac{iHt}{\hbar}} e^{-\frac{iHt}{\hbar}} = e^0 = 1$

An operator that satisfies $U^\dagger U = 1$ is called unitary.

The energy operator (Hamiltonian) generates time evolution of the quantum mechanical state $|\psi(t)\rangle$ through the unitary operator $U(t) = e^{-\frac{iHt}{\hbar}}$

Suppose we begin in a normalized state $|\psi(0)\rangle$

$$\langle\psi(0)|\psi(0)\rangle = 1$$

Then $\langle\psi(t)|\psi(t)\rangle = \langle\psi(0)|\underbrace{U^\dagger(t)\,U(t)}_{1}|\psi(0)\rangle = 1$

the state remains normalized.

Time evolution of an energy eigenstate

Suppose $H|\psi(0)\rangle = E|\psi(0)\rangle$

then $|\psi(t)\rangle = e^{-\frac{iHt}{\hbar}}|\psi(0)\rangle$

$$= \sum \frac{1}{n!}\left(-\frac{it}{\hbar}\right)^n \underbrace{H^n|\psi(0)\rangle}_{E^n|\psi(0)\rangle}$$

$$= e^{-\frac{iEt}{\hbar}}|\psi(0)\rangle$$

State picks up a time-dependent phase factor

$$\langle\psi(t)| = \langle\psi(0)|\, e^{\frac{iEt}{\hbar}}$$

$$\Rightarrow \langle\psi(t)|A|\psi(t)\rangle = \langle\psi(0)|A|\psi(0)\rangle$$

all expectation values are time independent

$\Rightarrow$ Energy eigenstates are "stationary states"

Time evolution of a coherent state

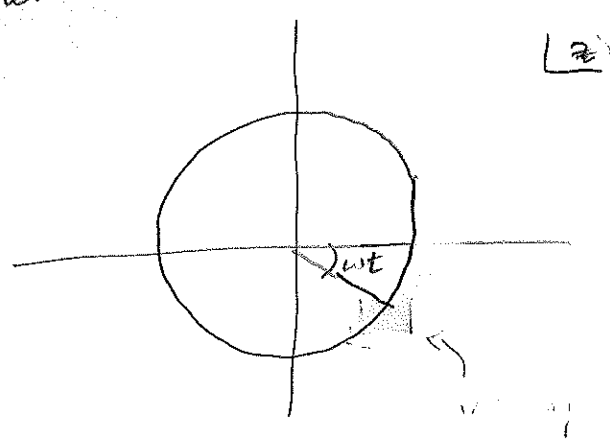
$$|z\rangle = C \sum \frac{z^n}{\sqrt{n!}} |n\rangle$$

$$|z(t)\rangle = e^{-\frac{iHt}{\hbar}} |z\rangle = C \sum \frac{z^n}{\sqrt{n!}} e^{-i\omega(n+\frac{1}{2})t} |n\rangle$$

$$= e^{-\frac{i\omega t}{2}} C \sum \frac{1}{\sqrt{n!}} \left(z e^{-i\omega t}\right)^n |n\rangle$$

$$= e^{-\frac{i\omega t}{2}} \left| z e^{-i\omega t} \right\rangle$$

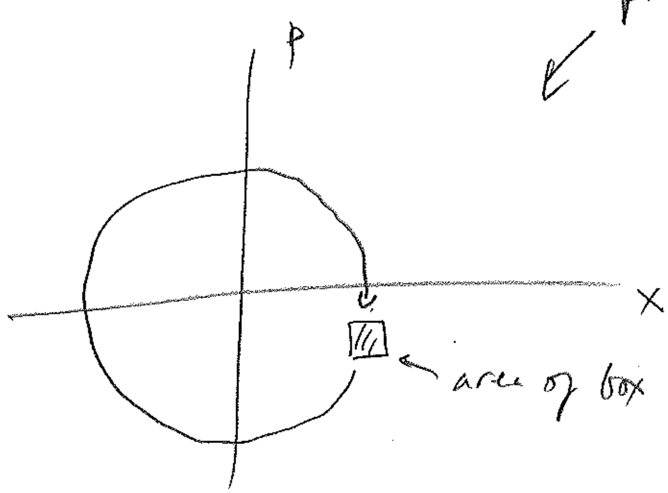
remains a coherent state of a "rotating z"



$$\langle x\rangle = \sqrt{\frac{2\hbar}{m\omega}}\, \mathrm{Re}(z e^{-i\omega t})$$

$$\langle p\rangle = \sqrt{2\hbar m\omega}\, \mathrm{Im}(z e^{-i\omega t})$$

phase space



area of box $\Delta x \Delta p = \frac{\hbar}{2}$

→ oscillating gaussian

Quantum mechanics: Heisenberg picture

Recall expectation value

$$\langle A(t)\rangle = \langle \psi(t)| \; A \; |\psi(t)\rangle$$

$|\psi(t)\rangle$ and A are the state and operator in the Schrodinger picture.

$$\langle A(t)\rangle = \langle \psi(0)| \; U^\dagger(t) \, A \, U(t) \; |\psi(0)\rangle$$

Define Heisenberg picture state and operator to be

$$|\psi\rangle_H = |\psi(0)\rangle$$ ie the initial state in the Schrodinger picture

$$A_H(t) = U^\dagger(t) \, A \, U(t)$$

Then

$$\langle A(t)\rangle = {}_H\langle \psi| \; A_H(t) \; |\psi\rangle_H$$ [Same expression]

Since only $\langle A(t)\rangle$ is physically measurable, we may express it equally validly in either picture

Note: in Heisenberg picture, the operator carries all the time dependence; the state merely conveys the initial conditions.

[Closer to CM, where observable $x(t)$, $p(t)$ vary in time.]

Important: the Hamiltonian is the same in both pictures

$$H_H(t) = U^\dagger(t)\, H\, U(t)$$

$$= e^{\frac{iHt}{\hbar}}\, H\, e^{-\frac{iHt}{\hbar}}$$

$$= H\, e^{\frac{iHt}{\hbar}}\, e^{-\frac{iHt}{\hbar}}$$

$$= H$$

That is, in Heisenberg picture, H is constant in time even though other observables are not.

(Energy conservation)

Heisenberg equations

Consider $\frac{d}{dt} A_H(t) = \frac{d}{dt}\left(e^{\frac{iHt}{\hbar}} A e^{-\frac{iHt}{\hbar}} \right)$

$$= \frac{iH}{\hbar} \underbrace{e^{\frac{iHt}{\hbar}} A e^{-\frac{iHt}{\hbar}}}_{A_H(t)} - \underbrace{e^{\frac{iHt}{\hbar}} A e^{-\frac{iHt}{\hbar}}}_{A_H(t)} \frac{iH}{\hbar}$$

$$= \frac{i}{\hbar}[H, A_H(t)]$$

$$\Rightarrow \boxed{i\hbar \frac{dA_H}{dt} = [A_H, H]}$$

Heisenberg equation for Heisenberg picture operator

[first written by Dirac (Sakurai, p. 89)]

N.B. classically $\frac{dA}{dt} = \{A, H\}$ so $\{\ \} \to \frac{[\ ]}{i\hbar}$

Suppose A commutes w/ H

$$[A, H] = 0$$

Then $[A_H(t), H] = 0 \quad \Rightarrow \quad \frac{dA_H}{dt} = 0$

$A_H(t)$ does not depend on time if it commutes w/ H (conservation law)

Trivially: $[H, H] = 0 \quad \Rightarrow$ energy is conserved

Commutation relations in Heisenberg picture

Suppose $[A, B] = C$ in Schrodinger picture.

$\Rightarrow$ $[A_H(0), B_H(0)] = C_H(0)$

Consider $[A_H(t), B_H(t')] = [U^\dagger(t) A U(t), U^\dagger(t') B U(t')]$

$= U^\dagger(t) A U(t) U^\dagger(t') B U(t') - U^\dagger(t') B U(t') U^\dagger(t) A U(t)$

[In general, a mess.]

If $t' = t$ then $U(t) U^\dagger(t') = 1$

$$[A_H(t), B_H(t)] = U^\dagger(t)(\underbrace{AB - BA}_{C}) U(t)$$

$$\boxed{[A_H(t), B_H(t)] = C_H(t)}$$ $\leftarrow$ equal time commutation relations

In particular $[x_H(t), p_H(t)] = i\hbar$

also $[x_H(t), x_H(t)] = 0$ [not worth writing]

but

$[x_H(t), x_H(t')] \neq 0$ for $t \neq t'$

$[x_H(t), p_H(t')] \neq i\hbar$ for $t \neq t'$

Particle in a potential

$$H = \frac{p^2}{2m} + V(x)$$

$$H = H_H = \frac{p_H(t)^2}{2m} + V(x_H(t))$$

Consider $\frac{d}{dt} x_H(t) = \frac{1}{i\hbar}[x_H(t), H]$

$$= \frac{1}{i\hbar} \underbrace{[x_H(t), \frac{p_H(t)^2}{2m}]} + \frac{1}{i\hbar} \underbrace{[x_H(t), V(x_H(t))]}_{0}$$

$$\underbrace{[x_H(t), p_H(t)]}_{i\hbar} \frac{p_H}{2m} + \frac{p_H}{2m} \underbrace{[x_H(t), p_H(t)]}_{i\hbar}$$

$$= \frac{p_H(t)}{m}$$

ie $p_H(t) = m \frac{d}{dt} x_H(t)$

classical relation between p + x holds for Heisenberg picture ops.

Consider $\frac{d}{dt} p_H(t) = \frac{1}{i\hbar} [p_H(t), H]$

$$= \frac{1}{i\hbar} [p_H(t), V(x_H(t))]$$

Consider Schrodinger picture commutator

$$[p, V(x)] \psi$$

$$= \frac{\hbar}{i}\frac{d}{dx}\left(V(x)\psi(x)\right) - V(x) \frac{\hbar}{i}\frac{d\psi}{dx}$$

$$= \frac{\hbar}{i} \frac{dV}{dx} = -i\hbar V'(x)$$

Then $[p_H(t), V(x_H(t))] = -i\hbar V'(x_H(t))$

$$\Rightarrow \frac{d}{dt} p_H(t) = -V'(x_H(t))$$

(Newton's 2nd law holds for Heisenberg picture operators)

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if not seen above HP-7

Recall $\frac{dA}{dt} = \{A, H\}$ in Hamiltonian mechanics

here $\frac{dA_H}{dt} = \frac{1}{i\hbar}[A, H]$ in quantum mechanics

Thus $\{\,,\,\} \longrightarrow \frac{1}{i\hbar}[\,,\,]$

Harmonic oscillator

$$H = \frac{1}{2m} p_H(t)^2 + \frac{1}{2} m\omega^2 x_H(t)^2$$

$$\Rightarrow \begin{cases} \dfrac{dx_H}{dt} = \dfrac{p_H}{m} \\[2ex] \dfrac{dp_H}{dt} = -m\omega^2 x_H \end{cases}$$

Heisenberg eqns = Hamilton's eqns

[but we got these not from least action principle, but from Schrodinger eqn evolution]

Solve just as before, except replace $*$ w/ $\dagger$.

[If everything linear, needn't worry about noncommutativity]

$$a_H(t) = \sqrt{\frac{m\omega}{2}}\left(x_H(t) + \frac{i}{m\omega} p_H(t)\right)$$

$$\dot{a}_H(t) = -i\omega a_H$$

$$\Rightarrow a_H(t) = a_H(0)\, e^{-i\omega t} = a\, e^{-i\omega t}$$

↖ Schrodinger picture

$$a_H^\dagger(t) = a^\dagger e^{i\omega t}$$

$$x_H(t) = \frac{1}{\sqrt{2m\omega}}\left[a\, e^{-i\omega t} + a^\dagger e^{i\omega t}\right]$$

Alternatively $i\hbar \dfrac{da_H}{dt} = [a_H, H]$

Since $H = \omega(a^\dagger a + \frac{\hbar}{2})$, we found $[a, H] = \hbar\omega a \Rightarrow [a_H(t), H] = \hbar\omega a_H(t)$

$$i\hbar \frac{da_H}{dt} = \hbar\omega a_H \Rightarrow \dot{a}_H = -i\omega a_H \Rightarrow a_H(t) = a_H(0)\, e^{-i\omega t}$$

Coherent states

$$\langle x(t)\rangle = \langle z|\ x_H(t)\ |z\rangle$$

$$= \frac{1}{\sqrt{2m\omega}} \langle z|\ a e^{-i\omega t} + a^\dagger e^{i\omega t} \ |z\rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \left(z e^{-i\omega t} + z^* e^{i\omega t}\right)$$

$$= \sqrt{\frac{2\hbar}{m\omega}} \ \mathrm{Re}.\left(z e^{-i\omega t}\right)$$

This is exactly what we found before in Schrodinger picture

$$\langle x(t)\rangle = \langle z(t)|\ x\ |z(t)\rangle \qquad |z(t)\rangle = e^{-\frac{i\omega t}{2}}|z e^{-i\omega t}\rangle$$

$$= \sqrt{\frac{1}{2m\omega}} \langle z e^{-i\omega t} |(a + a^\dagger)|\ z e^{-i\omega t}\rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \left(z e^{-i\omega t} + z^* e^{-i\omega t}\right)$$

Symmetries

Recall: if L is invariant under an infinitesimal variation δq_i

then $Q = \sum_i \frac{\partial L}{\partial \dot{q}_i} \frac{\delta q_i}{\alpha} = \sum p_i \frac{\delta q_i}{\alpha}$ is conserved

$\left(\frac{dQ}{dt} = 0 \right)$

As a Hermitian operator, Q generates the variation via the equal-time commutator, i.e.

$$[q_i, Q] = i\hbar \frac{\delta q_i}{\alpha}$$

$$[p_i, Q] = i\hbar \frac{\delta p_i}{\alpha}$$

(Since Q is conserved, it is the same at all times)

e.g. consider the spatial shift of Cartesian coordinate

$$\delta x_i = \alpha$$

$$\Rightarrow Q = \sum p_j \quad \text{(total momentum)}$$

Then $[x_i, Q] = \sum_j \underbrace{[x_i, p_j]}_{i\hbar\delta_{ij}} = i\hbar$

$[p_i, Q] = \sum_j [p_i, p_j] = 0$

(Since $p_i = m\dot{x}_i$, we expect $\delta p_i = 0$)

HW: Show that this works for spatial rotations.