Quantum mechanics: Schrodinger picture

- review of Phys 2140
- introduction to Dirac bracket notation (cf Phys 3140)

- state $\psi(x,t) \longrightarrow |\psi(t)\rangle$

[where is x? Infinite dimensional vector, $\exists$ a component for each x]

- complex conjugate $\psi^*(x,t) \longrightarrow \langle\psi(t)|$

[dual vector]

- normalization: $\int_{-\infty}^{\infty} dx\ \psi^*(x,t)\,\psi(x,t) = 1 \longrightarrow \langle\psi(t)|\psi(t)\rangle = 1$

[integration is implied. Bracket = inner (dot) product.
sum over components → integral over x]

$$\langle \text{bracket} \rangle = \langle \text{bra} | \text{ket} \rangle$$

- expectation value of observable A

$$\langle A(t)\rangle = \int_{-\infty}^{\infty} dx\ \psi^*(x,t)\ \underset{\substack{\uparrow \\ \text{operator}}}{A}\ \psi(x,t) \longrightarrow \langle\psi(t)|A|\psi(t)\rangle$$

[sometimes use hats, or capital letters. Implied here]

In Schrodinger picture, states are time-dependent but operators are not. Time dependence of expectation values comes from the states.

Operators for observables

position $\quad X \longrightarrow x$

momentum $\quad P \longrightarrow \frac{\hbar}{i}\frac{d}{dx}$

energy $\quad H = \frac{P^2}{2m} + V(X) \longrightarrow -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$

Operators generally do not commute.

Define commutator of two operators

$$[A, B] \equiv AB - BA$$

[measures non commutativity]

Example

$$[X,P]\psi = x\left(\frac{\hbar}{i}\frac{d}{dx}\right)\psi - \underbrace{\frac{\hbar}{i}\frac{d}{dx}(x\psi)}_{\frac{\hbar}{i}\psi + \frac{\hbar}{i}x\frac{d\psi}{dx}}$$

$$= i\hbar\psi$$

Since this holds for any state ψ, we conclude

$$\boxed{[X,P] = i\hbar\,\mathbb{1}}$$

$\mathbb{1}$ = identity operator

Heisenberg commutation relation

3 dimensions: $\quad X_i \to x_i$

$\quad P_i \to \frac{\hbar}{i}\frac{\partial}{\partial x_i}$

$$[X_i, X_j] = 0$$

$$[P_i, P_j] = 0$$

$$\boxed{[X_i, P_j] = i\hbar\,\delta_{ij}}$$

[Note: similar to fundamental Poisson brackets]

Recall $\langle \phi | \chi \rangle = \int_{-\infty}^{\infty} dx \; \phi^*(x,t) \, \chi(x,t)$

then $\langle \phi | \chi \rangle^* = \int_{-\infty}^{\infty} dx \; \phi(x,t) \, \chi^*(x,t) = \int dx \; \chi^*(x,t)\, \phi(x,t) = \langle \chi | \phi \rangle$

$$\boxed{\langle \phi | \chi \rangle^* = \langle \chi | \phi \rangle}$$

Hermitian conjugation

Given A, we define its hermitian conjugate $A^\dagger$ (a.k.a. adjoint)

if $\hat{A} |\psi\rangle = |\chi\rangle$ then $\langle \chi | = \langle \psi | \hat{A}^\dagger$

This may seem somewhat abstract. An operator A is completely specified if we know $\langle \psi | A | \phi \rangle$ for any ψ and ϕ.

The hermitian conjugate can be defined by

$\langle \psi | A^\dagger | \phi \rangle = \langle \chi | \phi \rangle = \langle \phi | \chi \rangle^* = \langle \phi | A | \psi \rangle^*$ for any ψ & ϕ

Since we know the matrix elements of $\hat{A}$, we know those of $A^\dagger$.

Hermitian conjugation is to operators as complex conjugation is to nos.

Show: if $A = a \mathbb{1}$ then $A^\dagger = a^* \mathbb{1}$

show $(A+B)^\dagger = A^\dagger + B^\dagger$

show $(A^\dagger)^\dagger = A$

show $(BA)^\dagger = A^\dagger B^\dagger$ [slightly harder]

Proof of $(BA)^\dagger = A^\dagger B^\dagger$

Consider $\langle \psi | (BA)^\dagger | \phi \rangle$

$$= \langle \phi | BA | \psi \rangle^*$$

Let $\langle \phi | B \equiv \langle \omega |$

$$= \langle \omega | A | \psi \rangle^*$$

$$= \langle \psi | A^\dagger | \omega \rangle$$

Let $\langle \psi | A^\dagger = \langle \chi |$

$$= \langle \chi | \omega \rangle$$

$$= \langle \omega | \chi \rangle^*$$

$$= \langle \phi | B | \chi \rangle^*$$

$$= \langle \chi | B^\dagger | \phi \rangle$$

$$= \langle \psi | A^\dagger B^\dagger | \phi \rangle$$

Hermitian operators

- If $A^\dagger = A$, we say A is hermitian (self-adjoint)

- All observables correspond to hermitian operators

  Show this for: X, P, H

  i.e. show $\langle\psi|A|\phi\rangle = \langle\phi|A|\psi\rangle^*$ for $A = X, P, H$

- Recall eigenvalue equation

$$H|E\rangle = E|E\rangle$$

$|E\rangle$ is an eigenstate of H

E is the eigenvalue associated w/ $|E\rangle$

- Hermitian operators have real eigenvalues

[Makes sense. Try to prove it.]

Sidney Coleman: "the career of a young theoretical physicist consists of treating the harmonic oscillator in ever-increasing levels of abstraction"

Quantum harmonic oscillator

$$H = \frac{P^2}{2m} + \frac{1}{2} m \omega^2 X^2$$

[what is significant is that both terms are quadratic. ⇒ phase space = circle]

Earlier definition becomes an operator equation

$$a = \sqrt{\frac{m\omega}{2}} \left(X + i \frac{P}{m\omega} \right)$$

not Hermitian [due to i]

observe

$$a^\dagger = \sqrt{\frac{m\omega}{2}} \left(X - i \frac{P}{m\omega} \right)$$

because X, P are hermitian.

As before we can solve for X and P:

$$X = \frac{1}{\sqrt{2m\omega}} (a + a^\dagger)$$

$$P = -i \sqrt{\frac{m\omega}{2}} (a - a^\dagger)$$

Then

$$H = -\frac{\omega}{4} (a - a^\dagger)^2 + \frac{\omega}{4} (a + a^\dagger)^2$$

$$= \frac{\omega}{2} (a^\dagger a + a a^\dagger)$$

[be careful w/ ordering]

Observe

$$[a, a^\dagger] = aa^\dagger - a^\dagger a$$

$$= \frac{i}{2}(PX - XP) - \frac{i}{2}(XP - PX)$$

$$= -i \underbrace{[X, P]}_{i\hbar \mathbb{1}}$$

$$\boxed{[a, a^\dagger] = \hbar}$$

Thus $aa^\dagger = a^\dagger a + \hbar$

$$\boxed{H = \omega\left(a^\dagger a + \frac{\hbar}{2}\right)}$$

Define number operator $N \equiv a^\dagger a \Rightarrow H = \omega\left(N + \frac{\hbar}{2}\right)$

N is hermitian [Proof?] & ∴ has real eigenvalues

If $N|\lambda\rangle = \lambda|\lambda\rangle$

then $|\lambda\rangle$ is also an eigenvalue of H w/ eigenvalue $\omega(\lambda + \frac{\hbar}{2})$

Proof: $H|\lambda\rangle = \omega(N + \frac{\hbar}{2})|\lambda\rangle = \omega(\lambda + \frac{\hbar}{2})|\lambda\rangle$

Vital commutator identity

$$[A, BCD] = [A,B]\,CD + B\,[A,C]\,D + BC\,[A,D]$$

Show this [easy; just write it out]

Consider

$$[a, N] = [a, a^\dagger a]$$

$$= \underbrace{[a, a^\dagger]}_{\hbar} a + a^\dagger \underbrace{[a, a]}_{0}$$

$$\boxed{[a, N] = \hbar a}$$

Similarly [check]

$$\boxed{[a^\dagger, N] = -\hbar a^\dagger}$$

Rewrite these in a more useful form:

$$Na - aN = -\hbar a \Rightarrow \boxed{Na = a(N - \hbar)}$$

$$Na^\dagger - a^\dagger N = \hbar a^\dagger \Rightarrow \boxed{Na^\dagger = a^\dagger (N + \hbar)}$$

[obviously $[A,B] = -[B,A]$]

Define the state $|0\rangle$ such that $a|0\rangle = 0$ and $\langle 0|0\rangle = 1$,

$\Rightarrow N|0\rangle = 0$

$\Rightarrow H|0\rangle = \omega(N + \frac{\hbar}{2})|0\rangle = \frac{\hbar\omega}{2}|0\rangle$

$\Rightarrow |0\rangle$ is an eigenstate of H w/ eigenvalue $E_0 = \frac{\hbar\omega}{2}$

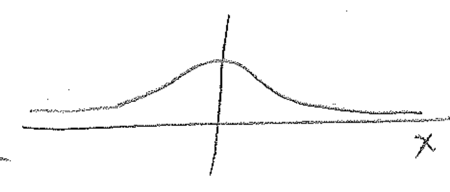
Since $a = \sqrt{\frac{m\omega}{2}}\left(X + i\frac{P}{m\omega}\right) = \sqrt{\frac{m\omega}{2}}\left(x + \frac{\hbar}{m\omega}\frac{d}{dx}\right)$

the wavefunction of the state $|0\rangle$ obeys

$$\frac{du_0}{dx} + \frac{m\omega}{\hbar} x\, u_0 = 0$$

The solution is a gaussian centered at $x = 0$

$$u_0 = C e^{-\frac{m\omega}{2\hbar}x^2}$$



C determined by normalization

We'll see that this is the ground state (lowest energy state)

$\frac{\hbar\omega}{2}$ is called the "zero point energy"

$$H|0\rangle = \frac{\hbar\omega}{2}|0\rangle$$

Claim: $a^\dagger|0\rangle$ is also an eigenstate of H

$$N\, a^\dagger|0\rangle = a^\dagger(N+\hbar)|0\rangle = \hbar\, a^\dagger|0\rangle$$

$$H\, a^\dagger|0\rangle = \omega\left(N+\frac{\hbar}{2}\right) a^\dagger|0\rangle = \frac{3}{2}\hbar\omega\, a^\dagger|0\rangle$$

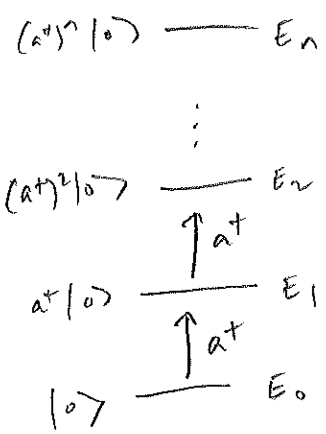
$a^\dagger|0\rangle$ is the first excited state w/ $E_1 = \frac{3}{2}\hbar\omega$

Consider $(a^\dagger)^n|0\rangle$ for any integer n

$$N(a^\dagger)^n|0\rangle = a^\dagger(N+\hbar)(a^\dagger)^{n-1}|0\rangle = \dots = (a^\dagger)^n(N+n\hbar)|0\rangle$$

$$= n\hbar\,(a^\dagger)^n|0\rangle$$

$$H(a^\dagger)^n|0\rangle = \omega\left(N+\frac{\hbar}{2}\right)(a^\dagger)^n|0\rangle = \underbrace{\hbar\omega\left(n+\frac{1}{2}\right)}_{E_n}(a^\dagger)^n|0\rangle$$



$a^\dagger$ = raising operator

$(a^\dagger)^n|0\rangle$ is an eigenstate of N w/ eigenvalue $n\hbar$

but it is not normalized.

Define $|n\rangle$ to be the normalized eigenstate $N|n\rangle = \hbar n|n\rangle$.
$\langle n|n\rangle = 1$

Since $a^\dagger$ is a raising operator: $a^\dagger|n-1\rangle = K_n|n\rangle$ (*)

(HW: Verify this using $Na^\dagger = a^\dagger(N+\hbar)$)

Show that a is lowering operator: $a|n\rangle = L_n|n-1\rangle$

(HW. Verify this using $Na = a(N-\hbar)$)

Since $a|0\rangle$, can't lower past $|0\rangle$ so $|0\rangle$ = ground state

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Recall that if $|\chi\rangle = A|\psi\rangle \Rightarrow \langle\chi| = \langle\psi|A^\dagger$.

(HW: Use this to show that $K_n = \sqrt{\hbar n}$. Also compute L_n)

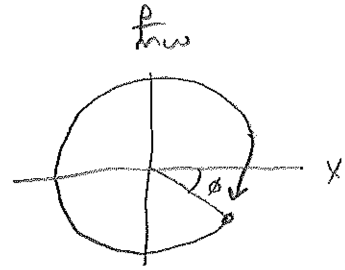
Iterate (*) to show

$$|n\rangle = \frac{1}{\sqrt{\hbar n}}a^\dagger|n-1\rangle = \frac{1}{\sqrt{\hbar n}}\frac{1}{\sqrt{\hbar(n-1)}}(a^\dagger)^2|n-2\rangle = \dots = \frac{(a^\dagger)^n}{\sqrt{\hbar^n\, n!}}|0\rangle$$

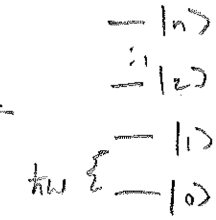
$|0\rangle \rightarrow$ gaussian wavefunction

$|n\rangle \rightarrow$ Hermite polynomial $\cdot$ gaussian wavefunction

Coherent States



classical oscillator $x(t) = A\cos(\omega t + \phi)$



quantum oscillator: ladder of equally spaced energy eigenstates

[not too similar]

coherent state: state of a quantum oscillator most closely resembling a classical oscillator

"minimum uncertainty state"

Given $z \in \mathbb{C}$, define

$$|z\rangle = C\, e^{\frac{z a^\dagger}{\sqrt{\hbar}}} |0\rangle$$

C = normalization factor $\Rightarrow$ can show $C = e^{-\frac{1}{2}|z|^2}$

$$|z\rangle = C \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{z a^\dagger}{\sqrt{\hbar}}\right)^n |0\rangle = \underbrace{C \sum \frac{z^n}{\sqrt{n!}} |n\rangle}_{\text{linear comb of energy eigenstates} \Rightarrow \text{not stationary}}$$

$$P_n = e^{-|z|^2} \frac{1}{n!} |z|^n$$

called a Poisson distribution

N.B. Fortuitously, $|0\rangle$ denotes both the ground state $|n=0\rangle$ and the coherent state $|z=0\rangle$.

Amazing fact: $|z\rangle$ is an eigenstate of a.

$$a|z\rangle = \sqrt{\hbar}\, z|z\rangle$$

[HW: prove this using $|z\rangle = C \sum \frac{z^n}{\sqrt{n!}} |n\rangle$]

Note: a is not Hermitian, so z is not necessarily real.

$$\langle z| a^\dagger = \langle z| \sqrt{\hbar}\, z^*$$

$$\langle N\rangle = \langle z| a^\dagger a |z\rangle = \langle z| \sqrt{\hbar} z^* \sqrt{\hbar} z |z\rangle = \hbar |z|^2$$

$$\langle E\rangle = \langle \omega (N + \tfrac{\hbar}{2})\rangle = \hbar\omega (|z|^2 + \tfrac{1}{2})$$

[HW: compute ΔN. Answer $\Delta N = \hbar |z|$]

$$\langle x\rangle = \langle z| \frac{1}{\sqrt{2m\omega}} (a + a^\dagger) |z\rangle = \sqrt{\frac{\hbar}{2m\omega}} (z + z^*) = \sqrt{\frac{2\hbar}{m\omega}} (\mathrm{Re}\, z)$$

$$\langle p\rangle = \langle z| -i\sqrt{\frac{m\omega}{2}} (a - a^\dagger) |z\rangle = -i\sqrt{\frac{\hbar m\omega}{2}} (z - z^*) = \sqrt{2\hbar m\omega} (\mathrm{Im}\, z)$$

[HW: compute $\Delta x = \sqrt{\frac{\hbar}{2m\omega}}$ and $\Delta p = \sqrt{\frac{m\omega\hbar}{2}}$. $\Delta x \Delta p = \frac{\hbar}{2}$.]

What is position space wavefunction for coherent state?

$$a|z\rangle = \sqrt{\hbar}\, z\, |z\rangle$$

$$\frac{d\psi}{dx} + \frac{m\omega x}{\hbar}\psi = \sqrt{\frac{2m\omega}{\hbar}}\, z\, \psi$$

$$\frac{d\psi}{dx} = -\frac{m\omega}{\hbar}\left(x - \sqrt{\frac{2\hbar}{m\omega}}\, z\right)\psi$$

$$\frac{d\psi}{\psi} = -\frac{m\omega}{\hbar}\left(x - \sqrt{\frac{2\hbar}{m\omega}}\, z\right)dx$$

$$\ln\psi = -\frac{m\omega}{2\hbar}\left(x - \sqrt{\frac{2\hbar}{m\omega}}\, z\right)^2 + \text{const}$$

$$\psi = C e^{-\frac{m\omega}{2\hbar}\left(x - \sqrt{\frac{2\hbar}{m\omega}}\, z\right)^2}$$

$$= C e^{-\frac{m\omega}{2\hbar}\left(x - \sqrt{\frac{2\hbar}{m\omega}}\,\mathrm{Re}\, z - i\sqrt{\frac{2\hbar}{m\omega}}\,\mathrm{Im}\, z\right)^2}$$

$$= C e^{-\frac{m\omega}{2\hbar}\left(x - \langle x\rangle - \frac{i}{m\omega}\langle p\rangle\right)^2}$$

$$= C' e^{-\frac{m\omega}{2\hbar}\left(x - \langle x\rangle\right)^2} e^{\frac{i\langle p\rangle x}{\hbar}}$$

gaussian displaced from origin by $\langle x\rangle$ times oscillatory function w/ wavelength $\frac{h}{\langle p\rangle}$