A systematic way of deriving 1^{st} order e.o.m. is provided by

$\boxed{\text{Hamiltonian mechanics}}$

Given a Lagrangian $L(q_i, \dot{q}_i)$, define

canonical (or generalized) momenta $\quad p_i = \frac{\partial L}{\partial \dot{q}_i} \quad (i = 1, \ldots N)$

Then define the Hamiltonian H via Legendre transformation

$$H = \sum p_i \dot{q}_i - L$$

Claim 1: If q and p are complete and independent one can eliminate $\dot{q}_i$ and write H as a fn of q_i & p_i

$H(q_i, p_i)$

$\{q_i, p_i\}$ parametrize phase space (2N-dimensional)

Claim 2: The e.o.m. are 2N 1^{st} order eqns

$$\left.\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} \end{aligned}\right\} \text{ Hamilton's eqns}$$

Usual (slick) proof: vary H wrt. all variables

$$\delta H = \sum \delta p_i \dot{q}_i + \underbrace{\sum p_i \delta \dot{q}_i - \sum \underbrace{\frac{\partial L}{\partial \dot{q}_i}}_{p_i} \delta \dot{q}_i}_{0} - \sum \underbrace{\frac{\partial L}{\partial q_i}}_{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \dot{p}_i} \delta q_i$$

$$= \sum \frac{\partial H}{\partial p_i} \delta p_i + \sum \frac{\partial H}{\partial q_i} \delta q_i$$

But what $\underline{\text{is}}$ $H(q, p)$?

[Let's spell this out more explicitly]

Consider Lagrangian of the form

$$L(q_i, \dot{q}_i) = \Sigma \ \tfrac{1}{2} A_i(q) \ \dot{q}_i^2 - V(q)$$

all the q_j

$$p_j = \frac{\partial L}{\partial \dot{q}_j} = A_j(q)\,\dot{q}_j$$

Initially, however, regard p_j as set of independent variables w/ no a priori relation to $\dot{q}_j$

$\boxed{\text{Step I}}$ Define an alternative Lagrangian w/ gen. coords q_i & p_i

$$L'(q_i, p_i, \dot{q}_i, \dot{p}_i) = L(q_i, \dot{q}_i) - \Sigma \frac{1}{2A_i}(p_i - A_i \dot{q}_i)^2$$

E-L eqn for p_j:

$$0 = \frac{\partial L'}{\partial p_j} - \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{p}_j}\right) = -\frac{1}{A_j}(p_j - A_j \dot{q}_j) = 0 \quad \Rightarrow p_j - A_j \dot{q}_j = 0$$

E-L eqn for q_j:

$$0 = \frac{\partial L'}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_j}\right) = \frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)$$

$$- \Sigma \frac{1}{A_i}(p_i - A_i \dot{q}_i)\left(-\frac{\partial A_i}{\partial q_j}\dot{q}_i\right) + \Sigma \frac{1}{2A_i^2}(p_i - A_i\dot{q}_i)^2 \frac{\partial A_i}{\partial q_j} + \frac{d}{dt}\left[\frac{1}{A_j}(p_j - A_j \dot{q}_j)(-A_j)\right]$$

Last three terms vanish by E-L eqn for p_j so

$$0 = \frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) \Rightarrow L \text{ and } L' \text{ give same e.o.m. for } q_j$$

$\Rightarrow$ they are "equivalent"

$\boxed{\text{Step II}}$ Expand L'

$$L' = \sum \cancel{\tfrac{1}{2} A_i \dot{q}_i^2} - V(q_i) - \sum\left[\frac{1}{2}\frac{p_i^2}{A_i} - p_i\dot{q}_i + \cancel{\tfrac{1}{2} A_i \dot{q}_i^2}\right]$$

$$= \sum p_i \dot{q}_i - \underbrace{\left[\sum \frac{1}{2}\frac{p_i^2}{A_i} + V(q)\right]}$$

- depends only on q & p, not $\dot{q}_i$
- <u>define</u> this to be H

$$\Rightarrow H = \sum p_i \dot{q}_i - L'$$

(usually written as $\sum p_i \dot{q}_i - L$ but $L = L'$ if $p_i = A_i \dot{q}_i$

$\boxed{\text{Step III}}$ Derive E-L eqns from

$$L' = \sum p_i \dot{q}_i - H(q,p)$$

$$0 = \frac{\partial L'}{\partial p_j} - \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{p}_j}\right) = \dot{q}_j - \frac{\partial H}{\partial p_j}$$

$$0 = \frac{\partial L'}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_j}\right) = -\frac{\partial H}{\partial q_j} - \frac{d}{dt} p_j$$

ie
$$\begin{cases} \dot{q}_i = \dfrac{\partial H}{\partial p_i} \\[2ex] \dot{p}_i = -\dfrac{\partial H}{\partial q_i} \end{cases} \qquad \text{(Hamilton)}$$

Poisson brackets

q_i and p_i specify the state of the system

Observables A are determined by q + p.

How does A change in time?

$$\frac{d}{dt} A(q,p) = \sum_i \left[\frac{\partial A}{\partial q_i} \dot{q}_i + \frac{\partial A}{\partial p_i} \dot{p}_i \right]$$

$$= \sum \left[\frac{\partial A}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial H}{\partial q_i} \right]$$

$$= \{A, H\}$$

where Poisson bracket is defined as $\{A,B\} = \sum \left[\frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} \right]$.

We say "the energy (Hamiltonian) generates time translation of A through the Poisson bracket of A + H"

[other symmetries are generated by Poisson brackets of conserved quant.]

Hamilton's eqn can be written

$$\dot{q}_j = \{q_j, H\} = \sum \underbrace{\frac{\partial q_j}{\partial q_i}}_{\delta_{ij}} \frac{\partial H}{\partial p_i} = \frac{\partial H}{\partial p_k}$$

Recall Kronecker delta $\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

$$\dot{p}_j = \{p_j, H\} = -\frac{\partial H}{\partial q_k}$$

"Fundamental" Poisson brackets

$$\{q_j, p_k\} = \delta_{jk}$$
$$\{q_j, q_k\} = 0$$
$$\{p_j, p_k\} = 0$$

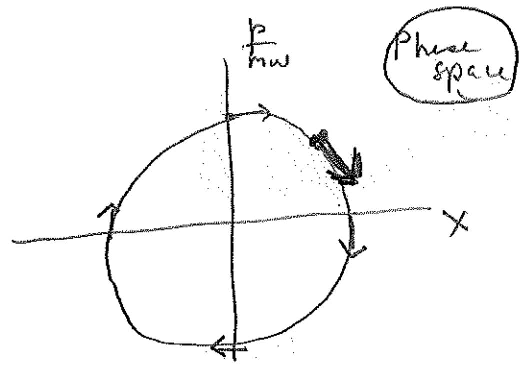
[echoed in QM commutation relations]

Harmonic oscillator in Hamiltonian formalism

$$L = \tfrac{1}{2} m \dot{x}^2 - \tfrac{1}{2} m \omega^2 x^2$$

$$\Rightarrow H = \frac{p^2}{2m} + \tfrac{1}{2} m \omega^2 x^2$$

$$\Rightarrow \begin{cases} \dot{x} = \dfrac{\partial H}{\partial p} = \dfrac{p}{m} \\[2ex] \dot{p} = -\dfrac{\partial H}{\partial x} = -m\omega^2 x \end{cases}$$



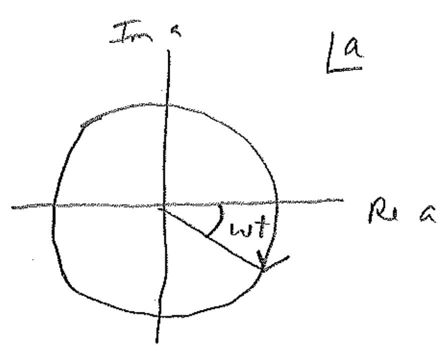
To uncouple these eqn's define

$$a = \sqrt{\frac{m\omega}{2}} \left(x + i \frac{p}{m\omega} \right)$$

($\sqrt{\frac{m\omega}{2}}$ not strictly necessary but useful later)

Consider $\dot{a} = \sqrt{\frac{m\omega}{2}} \left(\dot{x} + i \frac{\dot{p}}{m\omega} \right) = \sqrt{\frac{m\omega}{2}} \left(\frac{p}{m} - i\omega x \right) = -i\omega a$

$$\Rightarrow a(t) = a(0)\, e^{-i\omega t}$$



How do we find $x(t)$?

Observe $\quad a^* = \sqrt{\frac{m\omega}{2}}\left(x - i\frac{p}{m\omega}\right)$

Show $\quad x = \frac{1}{\sqrt{2m\omega}}(a + a^*)$

$$p = -i\sqrt{\frac{m\omega}{2}}(a - a^*)$$

Then $\quad \boxed{x(t) = \frac{1}{\sqrt{2m\omega}}\left[a(0)e^{-i\omega t} + a^*(0)e^{i\omega t}\right]}$

ie Form III

Hamiltonian $\quad H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2$

$$= -\frac{\omega}{4}(a - a^*)^2 + \frac{\omega}{4}(a + a^*)^2$$

$$H = \omega a^* a$$

Note $\quad H(t) = \omega\, a^*(0)e^{i\omega t}\, a(0)e^{-i\omega t} = \omega a^*(0)a(0)$

ie H is constant in time