Classical mechanics (non relativistic)

1) Newtonian
2) Lagrangian
3) Hamiltonian

Newton's 2nd law: $m \frac{d^2\vec{x}}{dt^2} = \vec{F}$

2nd order so need initial conditions $\vec{x}(t_0)$ and $\dot{\vec{x}}(t_0)$

From these, obtain $\vec{x}(t)$ for all t.

Define $\vec{p} = m\dot{\vec{x}}$ then $\dot{\vec{p}} = \vec{F}$

Thus we can replace 3 2nd order eqns $\ddot{\vec{x}} = \frac{\vec{F}}{m}$

w/ 6 coupled 1st order eqns $\begin{cases} \dot{\vec{x}} = \frac{\vec{p}}{m} \\ \dot{\vec{p}} = \vec{F} \end{cases}$

2nd order → Lagrangian
1st order → Hamiltonian

Susskind, Theoretical Minimum
Feynman Lectures, II-19

Lagrangian mechanics

[In the beginning was the Lagrangian.] Joseph-Louis Lagrange 1736-1813 (Italian)

For a nonrelativistic, conservative system, we define

$$L = T - V = \text{Lagrangian}$$

T = nonrelativistic kinetic energy

V = potential energy [not U; standard notation in QM]

Some observations:

- L is a scalar quantity, easy to work with
- L has dimensions of energy
- but L is not the energy due to minus sign
    what does it mean?
- why nonrelativistic? can we use relativistic KE instead? no
- real issue is that potential energy (eg Coulomb, grav.)
    is a problematic concept in relativity (action at a distance)
- to do relativity, we should use fields, as we will!

not for class

Nonrelativistic generalization for a free particle

$$L = -mc^2\sqrt{1-\frac{\dot{x}^2}{c^2}} \approx -mc^2 + \frac{1}{2}m\dot{x}^2 + \ldots$$

$$S = \int dt\, L = -mc^2 \int \sqrt{dt^2 - dx^2} \qquad \text{(invariant)}$$

$$p = \frac{\partial L}{\partial \dot{x}} = \frac{m\dot{x}}{\sqrt{1-\frac{\dot{x}^2}{c^2}}} = \text{relativistic momentum}$$

E-L eqns $\Rightarrow$ p is constant in time $\Rightarrow$ $\dot{x} = \text{const}$

Legendre transformation

$$H = p\dot{x} - L$$

$$= \frac{m\dot{x}^2}{\sqrt{1-\frac{\dot{x}^2}{c^2}}} + mc^2 \cdot \frac{\left(1-\frac{\dot{x}^2}{c^2}\right)}{\sqrt{1-\frac{\dot{x}^2}{c^2}}}$$

$$= \frac{mc^2}{\sqrt{1-\frac{\dot{x}^2}{c^2}}} = \text{relativistic energy,}$$

also constant in time if $\dot{x} = \text{constant}$

[Susskind does this in vol III of Theoretical Minimum] cf Susskind, vol 3

Set $c = 1$

$$\boxed{S = -m\int d\tau - e\int A_\mu \, dx^\mu}$$

Lorentz invariant

Gauge invariant $\Rightarrow$ $\delta S = -e\int \partial_\mu \chi \, dx^\mu = -e\chi\Big| = 0$

$$S = \int dt \, [\underbrace{-m\sqrt{1-\dot{x}^i\dot{x}^i} - e(A_0 + A_i\dot{x}^i)}_{L}]$$

$$\frac{\partial L}{\partial \dot{x}^i} = +\frac{m\dot{x}^i}{\sqrt{1-\dot{x}^i\dot{x}^i}} - eA_i$$

$$\frac{\partial L}{\partial x^i} = -e(\partial_i A_0 + \partial_i A_j \dot{x}^j)$$

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i}$$

$$= \frac{d}{dt}\left(\frac{m\dot{x}^i}{\sqrt{1-\dot{x}^i\dot{x}^i}}\right) - e\frac{\partial A_i}{\partial x^j}\dot{x}^j - e\frac{\partial A_i}{\partial t} + e(\partial_i A_0 + \partial_i A_j \dot{x}^j)$$

$$= \frac{d}{dt}\left(\frac{m\dot{x}^i}{\sqrt{1-\dot{x}^i\dot{x}^i}}\right) - e\underbrace{(\partial_0 A_i - \partial_i A_0)}_{F_{0i} = E^i} + e\underbrace{(\partial_i A_j - \partial_j A_i)}_{F_{ij} = -\epsilon_{ijk}B^k}\dot{x}^j$$

$$= \frac{d}{dt}\left(\frac{m\dot{x}^i}{\sqrt{1-\dot{x}^i\dot{x}^i}}\right) - eE^i - e\,\epsilon_{ijk}\dot{x}^j B^k$$

Lorentz force

The Lagrangian $L(q_i, \dot{q}_i)$ is expressed in terms of

generalized coordinates q_i and

generalized velocities $\dot{q}_i$ $\qquad i = 1, \ldots N$

[could be Cartesian coords of a single particle (3)
or of many particles ($3 \cdot N_{particles}$)

or spherical coordinates
or constrained coordinates (eg bead on a wire)]

space of generalized coordinates = configuration space.
(N-dimensional)

optimally generalized coordinates are
"complete and independent"

Define <u>action</u>

$$S[q_i(t)] = \int_{t_0}^{t_1} dt \; L(q_i, \dot{q}_i)$$

written in square brackets $\Rightarrow$ functional
(function of a function)

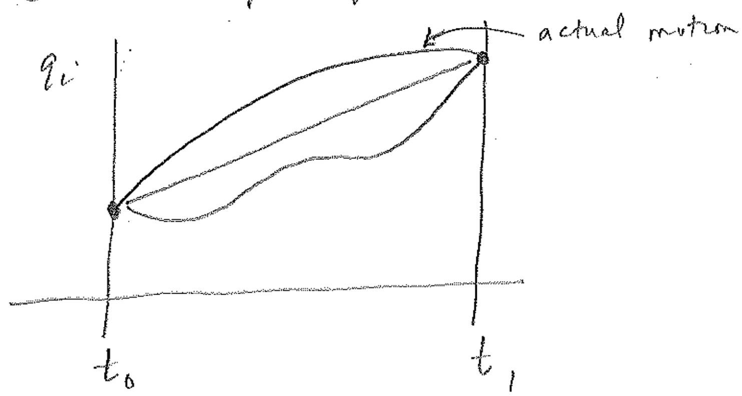
Boundary conditions [are dealt w/ differently in Newtonian + Lagrangian mechanics]

Newton: initial position + velocity $q_i(t_0)$, $\dot{q}_i(t_0)$

Lagrange: initial + final positions $q_i(t_0)$, $q_i(t_1)$

[in solving problems we can ~~do~~ it both ways. e.g. throw ball upward. When does it pass through same point?]

Consider space of all motions $q_i(t)$ w/ fixed initial + final positions



Hamilton's principle of least action (1834)

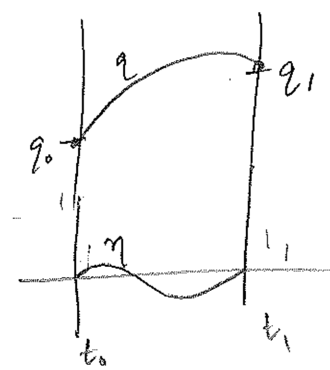
William Rowan Hamilton [cf Tim Blais]

The actual motion of a system
(subject to fixed boundary conditions)
is the one that minimizes (extremizes) the action

[Let's make this more precise.
For now drop the index i; can be re-instated later]

$$S[q(t)] = \int_{t_0}^{t_1} dt\, L(q, \dot{q})$$

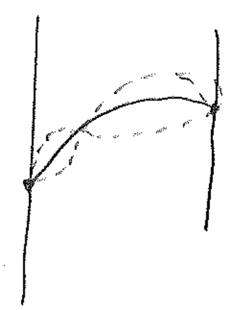
Let $q(t)$ denote the actual motion that minimizes the action
w/ $q(t_0) = q_0$ and $q(t_1) = q_1$



Let $\eta(t)$ be an arbitrary function
w/ $\eta(t_0) = \eta(t_1) = 0$

Define a family of motions w/ same boundary condition

$$q_\epsilon(t) = q(t) + \epsilon\,\eta(t)$$



$q_\epsilon(t_0) = q_0$
$q_\epsilon(t_1) = q_1$

Then $S[q_\epsilon(t)] \geq S[q(t)]$ for all ϵ and all η.

$$\left.\frac{dS}{d\epsilon}\right|_{\epsilon=0} = 0$$, ie actual motion is an extremum of S

[classic calculation. We'll end up doing variants of this calculation 4 or 5 times in this course. The first time we'll do it carefully, so be sure you understand it completely.]

"Calculus of variations"

$$S[q_\epsilon(t)] = \int_{t_0}^{t_1} dt \; L(q+\epsilon\eta, \dot q + \epsilon\dot\eta)$$

$$\frac{dS}{d\epsilon} = \int_{t_0}^{t_1} dt \left[\frac{\partial L}{\partial q}(q+\epsilon\eta, \dot q+\epsilon\dot\eta)\,\eta + \frac{\partial L}{\partial \dot q}(q+\epsilon\eta, \dot q+\epsilon\dot\eta)\,\dot\eta \right]$$

$$\left.\frac{dS}{d\epsilon}\right|_{\epsilon=0} = \int_{t_0}^{t_1} dt \left[\frac{\partial L}{\partial q}\eta + \frac{\partial L}{\partial \dot q}\dot\eta \right]$$

$$= \int_{t_0}^{t_1} dt \left[\frac{\partial L}{\partial q}\eta - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right)\eta + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\eta\right) \right]$$

$$= \int_{t_0}^{t_1} dt \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) \right]\eta + \underbrace{\left.\frac{\partial L}{\partial \dot q}\eta\right|_{t_0}^{t_1}}_{\text{vanishes because } \eta(t_0)=\eta(t_1)=0}$$

But $\left.\frac{dS}{d\epsilon}\right|_{\epsilon=0} = 0$ for any $\eta(t)$, so

$$\Rightarrow \boxed{\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right) = 0}$$

[satisfied by the actual motion]

Euler-Lagrange equations

a.k.a. e.o.m.

e.o.m. = eqn of motion

E-L eqns are equivalent to Newton's 2nd law

e.g. let q_i = Cartesian coordinates of a point particle

$$L = \sum_i \frac{1}{2} m \dot{x}_i^2 - V(x_i) \qquad (x_1, x_2, x_3) = (x, y, z)$$

$$\frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i = p_i$$

$$\frac{\partial L}{\partial x_i} = -\frac{\partial V}{\partial x_i} = F_i \quad \text{(conservative force)}$$

$$\text{E-L} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x_i}$$

$$\dot{p}_i = F_i \quad \checkmark$$

Finding Newton's laws in spherical coordinates [problem] is actually much easier using Lagrangian approach

[2022: Athis asks is Lagrangian unique?]

Lagrangian is <u>not unique</u>

E.g. can add a total time derivative of fcn of q_i

$$L' \doteq L + \frac{d}{dt} f(q_i)$$

$$\Rightarrow S' = S + \int_{t_0}^{t_1} dt \, \frac{d}{dt} f(q)$$

$$= S + f(q) \Big|_{t_0}^{t_1}$$

$$= S + \underbrace{f(q_1) - f(q_0)}_{\text{same for any family of functions w/ fixed endpoints}}$$

If $q(t)$ minimizes S, then it also minimizes S'

$\Rightarrow S'$ & S gives same Euler-Lagrange eqns [show this]

Check:

$$L' = L + \sum_i \frac{\partial f}{\partial q_i}(q) \, \dot{q}_i$$

$$\frac{\partial L'}{\partial q_j} = \frac{\partial L}{\partial q_j} + \sum_i \frac{\partial^2 f}{\partial q_i \partial q_j} \dot{q}_i$$

$$\frac{\partial L'}{\partial \dot{q}_j} = \frac{\partial L}{\partial \dot{q}_j} + \frac{\partial f}{\partial q_j}$$

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_j}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) + \sum_i \frac{\partial^2 f}{\partial q_i \partial q_j} \dot{q}_i$$

terms cancel in E-L eqns

[You knew this was coming ...]

<u>one-dimensional harmonic oscillator</u>

$$V(x) = \frac{1}{2} k_s x^2 = \frac{1}{2} m\omega^2 x^2 \qquad \text{where} \quad \omega = \sqrt{\frac{k_s}{m}}$$

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} m\omega^2 x^2$$

E-L eqn $\Rightarrow$ $\ddot{x} + \omega^2 x = 0$

Form III solution from Phys 300: $x(t) = Re\left[c e^{i\omega t} \right]$

$$= \frac{1}{2} c e^{i\omega t} + \frac{1}{2} c^* e^{-i\omega t}$$

i.e. linear comb. of $e^{i\omega t}$ & $e^{-i\omega t}$
[verified by plugging into eqn]

Symmetries

Noether's theorem: continuous symmetry $\Rightarrow$ conserved quantity

Emmy Noether (1882–1935) [brilliant German mathematician
also female → discriminated
Hilbert tried to hire her at Göttingen
philosophical faculty objected
"I don't see why. This is a university not a bathhouse"
Nazis → emigrated to US → Bryn Mawr → died age 53

Time translation invariance

The laws of physics are invariant under a shift in time.

$\Rightarrow$ Lagrangian depends on time only through its dependence on the configuration space variables, not explicitly

$$L\,(q_i(t), \dot{q}_i(t)) \quad \text{not} \quad L(q_i(t), \dot{q}_i(t), t)$$

Consider

$$\frac{dL}{dt} = \sum\left[\frac{\partial L}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial L}{\partial \dot{q}_i}\frac{d\dot{q}_i}{dt}\right] \qquad \text{(chain rule)}$$

$$= \sum\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}(\dot{q}_i)\right] \qquad \text{(E-L)}$$

$$= \frac{d}{dt}\sum \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i \qquad \text{(product rule)}$$

Therefore

$$\frac{d}{dt}\Big[\underbrace{\sum \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L}_{\text{conserved quantity.}}\Big] = 0$$

[We can see what it is by considering Cartesian coords:]

$$\sum \frac{\partial L}{\partial \dot{x}_i}\dot{x}_i = \sum m\dot{x}_i^2 = 2T \quad \text{so} \quad 2T - (T-V) = T+V = E$$

$\boxed{\text{time translation symmetry} \Rightarrow \text{conservation of energy}}$

Suppose there exists an infinitesimal variation of generalized coordinates δq_i & velocities $\delta \dot{q}_i$ that leaves Lagrangian invariant

$$L(q_i + \delta q_i, \dot{q}_i + \delta \dot{q}_i) = L(q_i, \dot{q}_i)$$

→ called a symmetry of the Lagrangian.

The variation of the Lagrangian vanishes

$$0 = \delta L = L(q_i + \delta q_i, \dot{q}_i + \delta \dot{q}_i) - L(q_i, \dot{q}_i)$$

$$= \sum_i \left[\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right] + O(\delta q^2)$$

Use e.o.m. $\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)$ and $\delta \dot{q}_i = \frac{d}{dt} \delta q_i$

$$0 = \sum_i \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt} \delta q_i \right]$$

$$= \frac{d}{dt}\left[\sum_i \frac{\partial L}{\partial \dot{q}_i} \delta q_i \right]$$

α = infinitesimal parameter associated with the variation

Thus $\boxed{Q \equiv \sum \frac{\partial L}{\partial \dot{q}_i} \frac{\delta q_i}{\alpha}}$ is a conserved quantity ("charge")

[slightly different than previous case because under time translation L is not invariant.]

<u>Example: Space translation invariance</u>

Consider an isolated system of particles interacting with each other

$$L = \sum \tfrac{1}{2} m_i \dot{x}_i^2 - \sum_{i<j} V(x_i - x_j)$$

interaction depends on distance between objects not absolute position

Under a uniform spatial shift α of all the coordinates, the Lagrangian is invariant

$$\delta x_i = \alpha \quad \Rightarrow \quad \delta \dot{x}_i = 0,$$
$$\delta x_i - \delta x_j = \alpha - \alpha = 0$$
$$\Rightarrow \delta L = 0$$

The conserved quantity associated with this shift is

$$Q = \sum \frac{\partial L}{\partial \dot{x}_i} \frac{\delta x_i}{\alpha} = \sum \frac{\partial L}{\partial \dot{x}_i} = \underbrace{\sum m_i \dot{x}_i}_{\text{total momentum of the system}}$$

$$\boxed{\text{spatial translation symmetry} \Rightarrow \text{conservation of total momentum}}$$