(2-15-17)

Completeness

(cf Peskin + Schroeder, p. 48 for slightly different approach)

Recall $\chi_{p+} = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{i\phi} \end{pmatrix}$ $\qquad \chi_{p-} = \begin{pmatrix} -\sin\frac{\theta}{2} e^{-i\phi} \\ \cos\frac{\theta}{2} \end{pmatrix}$

$$\chi_{p+}\chi_{p+}^\dagger + \chi_{p-}\chi_{p-}^\dagger = \begin{pmatrix} c \\ s e^{i\phi} \end{pmatrix} (c, s e^{-i\phi}) + \begin{pmatrix} -s e^{-i\phi} \\ c \end{pmatrix} (-s e^{i\phi}, c)$$

$$= \begin{pmatrix} c^2 + s^2 & 0 \\ 0 & c^2 + s^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}$$

$$\chi_{p+}\chi_{p+}^\dagger - \chi_{p-}\chi_{p-}^\dagger = \begin{pmatrix} c^2 - s^2 & 2sce^{-i\phi} \\ 2sce^{i\phi} & s^2 - c^2 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta\cos\phi - i\sin\theta\sin\phi \\ \sin\theta\cos\phi + i\sin\theta\sin\phi & -\cos\theta \end{pmatrix}$$

$$= \frac{p_z}{p}\sigma_z + \frac{p_x}{p}\sigma_x + \frac{p_y}{p}\sigma_y$$

$$= \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}$$

Recall $u_{ps} = \begin{pmatrix} \sqrt{E - s|\vec{p}|}\, \chi_{ps} \\ \sqrt{E + s|\vec{p}|}\, \chi_{ps} \end{pmatrix}$

$$\sum u_{ps} u_{ps}^{\dagger} = \sum_s \begin{pmatrix} (E - s|p|)\, \chi_{ps}\chi_{ps}^{\dagger} & \sqrt{(E-s|p|)(E+s|p|)}\, \chi_{ps}\chi_{ps}^{\dagger} \\ \sqrt{(E-s|p|)(E+s|p|)}\, \chi_{ps}\chi_{ps}^{\dagger} & (E + s|p|)\, \chi_{ps}\chi_{ps}^{\dagger} \end{pmatrix}$$

$$= \begin{pmatrix} E\mathbb{1} - |p| \frac{\vec{\sigma}\cdot\vec{p}}{|p|} & m\mathbb{1} \\ m\mathbb{1} & E\mathbb{1} + |p| \frac{\vec{\sigma}\cdot\vec{p}}{|p|} \end{pmatrix}$$

$$\sum u_{ps} u_{ps}^{\dagger} = \begin{pmatrix} E - \vec{\sigma}\cdot\vec{p} & m \\ m & E + \vec{\sigma}\cdot\vec{p} \end{pmatrix}$$

$$\sum u_{ps}\bar{u}_{ps} = \sum u_{ps}u_{ps}^{\dagger}\gamma^0 = \begin{pmatrix} E - \vec{\sigma}\cdot\vec{p} & m \\ m & E + \vec{\sigma}\cdot\vec{p} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} m & E - \vec{\sigma}\cdot\vec{p} \\ E + \vec{\sigma}\cdot\vec{p} & m \end{pmatrix}$$

But $\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$ so this is $m + E\gamma^0 - \vec{p}\cdot\vec{\gamma} = \not{p} + m$

$$v_{ps} = \begin{pmatrix} \sqrt{E - s|p|}\, \chi_{ps} \\ -\sqrt{E + s|p|}\, \chi_{ps} \end{pmatrix}$$

$$\sum v_{ps} v_{ps}^{\dagger} = \begin{pmatrix} E - \vec{\sigma}\cdot\vec{p} & -m \\ -m & E + \vec{\sigma}\cdot\vec{p} \end{pmatrix} \Rightarrow \sum v_{ps}\bar{v}_{ps} = \begin{pmatrix} -m & E - \vec{\sigma}\cdot\vec{p} \\ E + \vec{\sigma}\cdot\vec{p} & -m \end{pmatrix} = \not{p} - m$$

$$\boxed{\begin{aligned} \sum_s u_{ps}\bar{u}_{ps} &= \not{p} + m \\ \sum_s v_{ps}\bar{v}_{ps} &= \not{p} - m \end{aligned}}$$

$$\mathcal{L} = \bar{\Psi}(i\not{\partial} - m)\psi$$

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} = \bar{\psi}\, i\gamma^0 = i\psi^\dagger$$

If we require $\{\psi(x), \pi(x')\} = i\,\delta(x-x')$

then $\{\psi(x), \psi^\dagger(x')\} = \delta(x-x')$

(Really $\{\psi_\alpha(x), \psi^\dagger_\beta(x')\} = \delta_{\alpha\beta}\,\delta(x-x')$.)

Let's see if this follows from the anticommutation relations

$$\{b_{ps}, b^\dagger_{p's'}\} = (2\pi)^3\, 2\omega\, \delta(\vec{p}-\vec{p}\,')\,\delta_{ss'}$$
$$\{d_{ps}, d^\dagger_{p's'}\} = (2\pi)^3\, 2\omega\, \delta(\vec{p}-\vec{p}\,')\,\delta_{ss'}$$

$$\psi_\alpha(x) = \int d\tilde{p} \sum_s \left(b_{ps} (u_{ps})_\alpha e^{-ip\cdot x} + d^\dagger_{ps} (v_{ps})_\alpha e^{ip\cdot x} \right)$$

$$\psi^\dagger_\beta(x') = \int d\tilde{p}' \sum_{s'} \left(b^\dagger_{p's'} (u^\dagger_{p's'})_\beta e^{ip'\cdot x'} + d_{p's'} (v^\dagger_{p's'})_\beta e^{-ip'\cdot x'} \right)$$

$$\{\psi_\alpha(x), \psi^\dagger_\beta(x')\} = \int d\tilde{p}\, d\tilde{p}' \sum_{s,s'} \Big[\{b_{ps}, b^\dagger_{p's'}\} (u_{ps})_\alpha (u^\dagger_{p's'})_\beta e^{+i(p'\cdot x' - p\cdot x)} + \{d^\dagger_{ps}, d_{p's'}\} (v_{ps})_\alpha (v^\dagger_{p's'})_\beta e^{i(p\cdot x - p'\cdot x')} \Big]$$

$d\tilde{p} = \frac{d^3p}{(2\pi)^3 2\omega}$

$\left(\begin{array}{l} \vec{p} = \vec{p}' \Rightarrow \omega = \omega' \\ s = s' \end{array} \right)$

$$= \int d\tilde{p} \Big[\underbrace{\sum_s (u_{ps})_\alpha (u^\dagger_{ps})_\beta}_{\underbrace{(\not{p}+m)\gamma^0}_{E - \vec{p}\cdot\vec{\gamma}\gamma^0 + m\gamma^0}} e^{ip\cdot(x'-x)} + \underbrace{\sum_s (v_{ps})_\alpha (v^\dagger_{ps})_\beta}_{\underbrace{(\not{p}-m)\gamma^0}_{E - \vec{p}\cdot\vec{\gamma}\gamma^0 - m\gamma^0}} e^{ip\cdot(x-x')} \Big]$$

Let $\vec{p} \to -\vec{p}$ in 2nd term

$$= \int d\tilde{p} \Big[(E - \vec{p}\cdot\vec{\gamma}\gamma^0 + m\gamma^0) e^{iE(t'-t)} + (E + \vec{p}\cdot\vec{\gamma}\gamma^0 - m\gamma^0) e^{iE(t-t')} \Big] e^{-i\vec{p}\cdot(\vec{x}'-\vec{x})}$$

Now if $t = t'$ then $\vec{p}$ + m terms drop out

$$= \int d\tilde{p}\, 2E\, e^{-i\vec{p}\cdot(\vec{x}'-x)} = \int \frac{d^3p}{(2\pi)^3(2E)} (2E)\, e^{-i\vec{p}\cdot(\vec{x}'-\vec{x})}$$

$$= \delta(\vec{x}'-\vec{x}) \qquad \text{QED}$$

Why do we need anticommutators? (Peskin/Schroeder p.52)

$$\mathcal{L} = \bar\Psi (i\not\partial - m)\Psi$$

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \Psi)} = i\bar\Psi \gamma^0 = i\Psi^\dagger$$

If $[\Psi(x), \pi(x')]_= = i\delta(\vec{x}-\vec{x}')$ then

$$[\Psi_\alpha(x), \Psi^\dagger_\beta(x')]_= = \delta(\vec{x}-\vec{x}')\,\delta_{\alpha\beta}$$

General solution of Dirac eqn

$$\Psi(x) = \int d\tilde{p} \sum_s \left(a_{ps} u_{ps} e^{-ip\cdot x} + b_{ps} v_{ps} e^{ip\cdot x} \right)$$

$$\Psi^\dagger(x) = \int d\tilde{p} \sum_s \left(a^\dagger_{ps} u^\dagger_{ps} e^{ip\cdot x} + b^\dagger_{ps} v^\dagger_{ps} e^{-ip\cdot x} \right)$$

Let us assume

$$[a_{ps}, a^\dagger_{p's'}] = (2\pi)^3 (2\omega)\, \delta(p-p')\, \delta_{ss'}$$

$$[b_{ps}, b^\dagger_{p's'}] = (2\pi)^3 (2\omega)\, \delta(p-p')\, \delta_{ss'}$$

$$[a_{ps}, b^\dagger_{p's'}] = 0$$

$$[b_{ps}, a^\dagger_{p's'}] = 0$$

$a_p(t) = a_p e^{-iEt}$ $\qquad$ $b_p(t) = b_p e^{iEt}$

But $[H, \phi] = -i\dot\phi$ so

$[H, a] = -Ea$ $\qquad$ $[H, b] = Eb$

a = lowering $\qquad$ b = raising

as we calculate below

Then

$$[\psi(x)_\alpha, \psi^\dagger_\beta(x')] = \int d\tilde{p}\, d\tilde{p}' \sum_{s,s'} \Big[[a_{ps}, a^\dagger_{p's'}]\, u_{ps} u^\dagger_{p's'}\, e^{i(p'\cdot x' - p\cdot x)}$$

$$+ [b_{ps}, b^\dagger_{p's'}]\, v_{ps} v^\dagger_{p's'}\, e^{i(p\cdot x - p'\cdot x')} \Big]$$

Now $\begin{cases} \vec{p}\,' = \vec{p} \\ s' = s \end{cases} \Rightarrow \omega' = \omega$

$$[\psi(x)_\alpha, \psi^\dagger_\beta(x')] = \int d\tilde{p} \Big[\underbrace{\sum_s (u_{ps})_\alpha (u^\dagger_{ps})_\beta}_{(\not{p}+m)\gamma^0} e^{ip\cdot(x'-x)} + \underbrace{\sum_s (v_{ps})_\alpha (v^\dagger_{ps})_\beta}_{(\not{p}-m)\gamma^0} e^{ip\cdot(x-x')} \Big]$$

$$(\mathbb{1}E - \vec{p}\cdot\vec{\gamma}\,\gamma^0 + m\gamma^0)_{\alpha\beta} \qquad (\mathbb{1}E - \vec{p}\cdot\vec{\gamma}\,\gamma^0 - m\gamma^0)_{\alpha\beta}$$

$$\Big\downarrow \text{ let } \vec{p} \to -\vec{p}$$

$$\mathbb{1}E + \vec{p}\cdot\vec{\gamma}\,\gamma^0 - m\gamma^0$$

$$= \int d\tilde{p} \Big[(E - \vec{p}\cdot\vec{\gamma}\,\gamma^0 + m\gamma^0)\, e^{iE(t'-t)} + (E + \vec{p}\cdot\vec{\gamma}\,\gamma^0 - m\gamma^0)\, e^{iE(t-t')} \Big] e^{i\vec{p}\cdot(\vec{x}-\vec{x}')}$$

Equal time $\Rightarrow$ γ^0 terms drop out

$$\int d\tilde{p}\, \delta_{\alpha\beta} 2E\, e^{i\vec{p}\cdot(\vec{x}-\vec{x}')} = \int \frac{d^3\vec{p}}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{x}')} = \delta_{\alpha\beta}\, \delta(\vec{x}-\vec{x}')$$

so this gives the expected commutator

Now

$$H = \int d^3x \; i\psi^\dagger \partial^0 \psi$$

$$\psi^\dagger(x) = \int d\tilde{p}' \sum_{s'} \left(a^\dagger_{p's'} u^\dagger_{p's'} e^{ip'\cdot x} + b^\dagger_{p's'} v^\dagger_{p's'} e^{-ip'\cdot x} \right)$$

$$i\partial^0 \psi(x) = \int d\tilde{p} \sum_s \left(E\, a_{ps} u_{ps} e^{-ip\cdot x} - E\, b_{ps} v_{ps} e^{ip\cdot x} \right)$$

$$\begin{aligned} H = \sum_{s,s'} \int d\tilde{p}'\, d\tilde{p} \Big[& E\, a^\dagger_{p's'} a_{ps} u^\dagger_{p's'} u_{ps} \int d^3x\; e^{i(p'-p)\cdot x} \rightarrow (2\pi)^3 \delta(\vec{p}'-\vec{p}) \\ & + E\, b^\dagger_{p's'} a_{ps} v^\dagger_{p's'} u_{ps} \int d^3x\; e^{-i(p'+p)\cdot x} \rightarrow (2\pi)^3 \delta(\vec{p}+\vec{p}') e^{-2iEt} \\ & - E\, a^\dagger_{p's'} b_{ps} u^\dagger_{p's'} v_{ps} \int d^3x\; e^{i(p'+p)\cdot x} \rightarrow (2\pi)^3 \delta(\vec{p}+\vec{p}') e^{2iEt} \\ & - E\, b^\dagger_{p's'} b_{ps} v^\dagger_{p's'} v_{ps} \int d^3x\; e^{i(p-p')\cdot x} \Big] \rightarrow (2\pi)^3 \delta(\vec{p}-\vec{p}') \end{aligned}$$

$=$ (see p. L4-4,5)

$$H = \sum_s \int d\tilde{p} \left[E\, a^\dagger_{ps} a_{ps} - E\, b^\dagger_{ps} b_{ps} \right]$$

But then

$[H, b] = [-Eb^\dagger b, b] = -E[b^\dagger, b]b = Eb$

rather than $-Eb$ as expected for lowering op

$[H, b^\dagger] = [-Eb^\dagger b, b^\dagger] = -Eb^\dagger [b, b^\dagger] = -Eb^\dagger$

rather than $Eb^\dagger$ as expected for raising op

whereas

$$\left.\begin{aligned} [H, a] &= [Ea^\dagger a, a] = E[a^\dagger, a]a = -Ea \\ [H, a^\dagger] &= [Ea^\dagger a, a^\dagger] = Ea^\dagger[a, a^\dagger] = Ea^\dagger \end{aligned}\right\} \text{ as expected}$$

At this point, consider switching to anticommutators: w/ $[a, a^+]_\mp = 1$, $[b, b^+]_\mp = 1$

let's keep both possibilities

$$[AB, C]_- = ABC - CAB = A[B,C]_\mp \pm (ACB \mp CAB)$$
$$= A[B,C]_\mp \pm [A,C]_\mp B$$

Then

$$[H, a]_- = E[a^+a, a]_- = \pm E \underbrace{[a^+, a]_\mp}_{\mp [a, a^+]} a = -Ea$$

$$[H, a^+]_- = E[a^+a, a^+]_- = Ea^+ [a, a^+]_\mp = Ea^+$$

correct for either!

$$[H, b]_- = -E[b^+b, b] = \mp E \underbrace{[b^+, b]_\mp}_{\mp [b, b^+]_\mp} b = Eb$$

$$[H, b^+]_- = -E[b^+b, b^+] = -Eb^+ [b, b^+]_\mp = -Eb^+$$

incorrect either way!

i.e. It looks like b is a raising operator & b^+ a lowering op

Let's try relabelling $b \leftrightarrow b^\dagger$ throughout

Then $\psi(x) = \int d\tilde{p} \sum_s [\, a_{ps} u_{ps} e^{-ip\cdot x} + b^\dagger_{ps} v_{ps} e^{ip\cdot x} \,]$

$$[\psi_\alpha, \psi^\dagger_\beta]_\mp = \delta(x-x')\,\delta_{\alpha\beta} \Rightarrow [b^\dagger_{ps}, b_{p's'}]_\mp = (2\pi)^3 (2\omega)\,\delta(p-p')\,\delta_{ss'}$$

$$\text{or} \quad [b_{ps}, b^\dagger_{ps}]_\mp = \mp (2\pi)^3 (2\omega)\,\delta(p-p')\,\delta_{ss'}$$

Also $H = \sum_s \int d\tilde{p} \, [\, E\, a^\dagger_{ps} a_{ps} - E\, b_{ps} b^\dagger_{ps} \,]$

Finally
$$[H, b^\dagger]_- = E b^\dagger$$
$$[H, b]_- = -E b$$

so this looks good either way!

as we can easily verify:

$$[H, b^\dagger]_- = -E\,[b b^\dagger, b^\dagger]_- = \mp E [b, b^\dagger]_\mp b^\dagger = E b^\dagger$$
$$[H, b]_- = -E\,[b b^\dagger, b] = -E\, b\, [b^\dagger, b]_\mp = -E b$$

$a_p(t) = a_p e^{-iEt}$ $\quad b^\dagger_p(t) = b_p e^{iEt}$

$[H, \phi] = -i\dot{\phi}$

$[H, a] = -E a$ $\quad [H, b^\dagger] = E b^\dagger$

why must we choose anticommutators then?