Fermions

Recall non relativistic Schrodinger eqn for a free particle

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi = -\frac{\hbar^2}{2m}\nabla^2\psi$$

Set $\hbar = 1$:

$$i\frac{\partial \psi}{\partial t} = -\frac{1}{2m}\nabla^2\psi$$

Analogous to K.G eqn, except $\underline{\underline{1^{st}\text{ order}}}$ in time

Ansatz: $\psi(x,t) = e^{-iEt + i\vec{p}\cdot\vec{x}}$ (we used ω, k before)

Satisfies eqn for $\begin{cases} \vec{p} \text{ arbitrary} \\ E = \frac{\vec{p}^2}{2m} \end{cases}$ (N.B only one solution for E not 2, because 1^{st} order)

most general solution

$$\psi(x,t) = \int d^3\vec{p}\, A_p\, e^{-i\left(\frac{p^2}{2m}\right)t + i\vec{p}\cdot\vec{x}}$$

(NB while ϕ had to be real, ψ is complex $\Rightarrow$ no conditions on $A_{\vec{p}}$)

To find A_p in terms of initial conditions $\psi(x,0)$ consider

$$\int d^3\vec{x}\, e^{-i\vec{p}\cdot\vec{x}}\, \psi(\vec{x},0) = \int d^3\vec{x}\, d^3\vec{p}'\, A_{p'}\, e^{i(\vec{p}'-\vec{p})\cdot x}$$

$$= \int d^3\vec{p}'\, A_{p'}\, (2\pi)^3\, \delta(\vec{p}'-\vec{p})$$

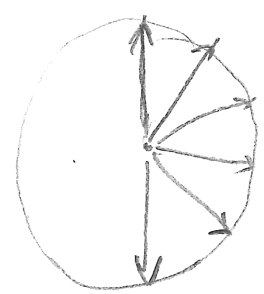
$$= (2\pi)^3 A_p$$

$$\Rightarrow A_p = \frac{1}{(2\pi)^3}\int d^3x\, e^{-i\vec{p}\cdot\vec{x}}\, \psi(\vec{x},0)$$

Particles have intrinsic angular momentum called "spin"

which is quantized: $S = j\hbar$ where $j = \begin{cases} \text{integer (bosons)} \\ \text{half-integer (fermions)} \end{cases}$

z-component of spin $S_z = m\hbar$ where $m = \underbrace{-j, -j+1, \dots, j}_{(2j+1)\text{ possibilities}}$



For $j = \frac{1}{2}$, $m = \pm\frac{1}{2}$ $\uparrow$ spin up $\downarrow$ spin down

For $j = 1$, $m = +1, 0, 1$ $\uparrow, \downarrow, \rightarrow$

$j = 0 \Rightarrow$ 1 state $\Rightarrow$ scalar field ϕ

$j = \frac{1}{2} \Rightarrow$ 2 states $\Rightarrow$ spinor field $\chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}$

$j = 1 \Rightarrow$ 3 states (for a massive particle)
2 states (for a massless particle)

$\Rightarrow$ vector field A^μ (4 d.o.f)

massive $\Rightarrow \partial \cdot A = 0 \Rightarrow$ 3 d.o.f

massless $\Rightarrow$ gauge invariance $\Rightarrow$ 2 d.o.f

$\uparrow$

2 polarizations

Fermion of momentum $\vec{p}$ and $S_z = \frac{1}{2}\hbar$

described by $\chi_+ = \begin{pmatrix} e^{i\vec{p}\cdot\vec{x} - iEt} \\ 0 \end{pmatrix}$

If $S_z = -\frac{\hbar}{2}$, $\chi_- = \begin{pmatrix} 0 \\ e^{i\vec{p}\cdot\vec{x} - iEt} \end{pmatrix}$

indeterminate S_z

transpose conjugate

$$\chi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} e^{i\vec{p}\cdot\vec{x} - iEt}$$

$\chi^\dagger = (\alpha^* \; \beta^*) e^{-ipx + iEt}$

where $|\alpha|^2 + |\beta|^2 = 1$

Determinate states are eigenstates of the S_z operator

$$S_z = \frac{\hbar}{2} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\sigma_3} =$$

$\sigma_3 \leftarrow$ Pauli spin matrix

$$S_z \chi = \frac{\hbar}{2} \begin{pmatrix} \alpha \\ -\beta \end{pmatrix} e^{i\vec{p}\cdot\vec{x} - iEt} = \lambda \begin{pmatrix} \alpha \\ \beta \end{pmatrix} e^{i\vec{p}\cdot\vec{x} - iEt}$$

eigenvalue eqn

χ is eigenstate iff $\alpha = 1, \beta = 0 \Rightarrow \lambda = \frac{\hbar}{2}$

or $\alpha = 0, \beta = 1 \Rightarrow \lambda = -\frac{\hbar}{2}$

equivalently χ are eigenstates of σ_3 of eigenvalue ± 1

x-component of spin $\Rightarrow$ operator $S_x = \frac{\hbar}{2} \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\sigma_1}$

Eigenstates of S_x $\qquad S_x \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \lambda \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

$$\frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \lambda \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Satisfied iff $\alpha = \beta \quad \Rightarrow \lambda = \frac{\hbar}{2}$

or $\alpha = -\beta \quad \Rightarrow \lambda = -\frac{\hbar}{2}$

$$\left\{ \begin{array}{l} \chi_+ = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} e^{i\vec{p}\cdot\vec{x} - iEt} \\ \chi_- = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} e^{i\vec{p}\cdot\vec{x} - iEt} \end{array} \right.$$

y-component of spin $\Rightarrow$ $S_y = \frac{\hbar}{2} \underbrace{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}_{\sigma_2}$

Eigenstates $\qquad \chi_+ = \begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix} e^{i\vec{p}\cdot\vec{x} - iEt}$

$$\chi_- = \begin{pmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{pmatrix} e^{i\vec{p}\cdot\vec{x} - iEt}$$

Pauli spin matrices obey

$$\cancel{\sigma_i \sigma_j = \delta_{ij} \mathbb{1} + i\epsilon_{ijk}\sigma_k}$$ ← (Einstein summation)

$$\sigma_i \sigma_j = \delta_{ij} \mathbb{1} + i\epsilon_{ijk}\sigma_k$$ → $\begin{cases} \epsilon_{ijk} = \text{Levi Civita} \\ \quad = \text{totally antisym.} \\ \epsilon_{123} = 1 \end{cases}$

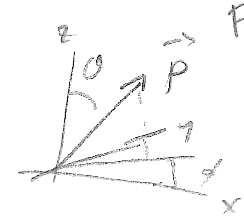
eg: $i=j=1 \Rightarrow \sigma_1^2 = \mathbb{1}$

$i=1, j=2 \Rightarrow \sigma_1\sigma_2 = i\epsilon_{123}\sigma_3$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & \\ & -i \end{pmatrix} = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \checkmark$$

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$$ (ang. mom. comm. rels)

$$\{\sigma_i, \sigma_j\} \equiv \sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}\mathbb{1}$$



Particle momentum $\vec{p} = \begin{pmatrix} p^1 \\ p^2 \\ p^3 \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} p$

Define $\sigma_{\hat{p}} \equiv \hat{p}\cdot\vec{\sigma} = \frac{1}{p}(p^1\sigma_1 + p^2\sigma_2 + p^3\sigma_3)$

$$= \sin\theta\,[\cos\phi\,\sigma_1 + \sin\phi\,\sigma_2] + \cos\theta\,\sigma_3$$

$$= \begin{pmatrix} \cos\theta & \sin\theta(\cos\phi - i\sin\phi) \\ \sin\theta(\cos\phi + i\sin\phi) & -\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{+i\phi} & -\cos\theta \end{pmatrix}$$

Pauli spin <u>matrix in direction $\hat{p}$</u>

Eigenstates: $\chi_{p+} = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\, e^{+i\phi} \end{pmatrix} e^{i\vec{p}\cdot\vec{x} - iEt}$

↳ momentum + 1/2 spin up in direction of $\hat{p}$ ⇒ positive helicity

$\chi_{p-} = \begin{pmatrix} -\sin\frac{\theta}{2}\, e^{-i\phi} \\ \cos\frac{\theta}{2} \end{pmatrix} e^{i\vec{p}\cdot\vec{x} - iEt}$ neg. helicity

Collectively χ_{ps}

verify in HW

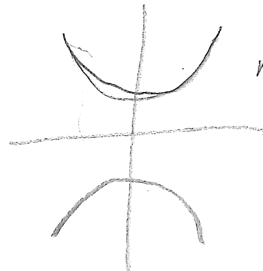
$\underline{\chi^\dagger_{ps}\chi_{ps} = 1}$

$\chi^\dagger_{ps}\chi_{p(-s)} = 0$

so $\underline{\chi^\dagger_{ps}\chi_{ps'} = \delta_{ss'}}$

Relativistic fermions

$E^2 = \vec{p}^{\,2} + m^2$



$E = \pm\sqrt{p^2+m^2}$

only want $+\sqrt{p^2+m^2}$ solution

$\Rightarrow$ KG: $(\partial^2+m^2)\phi = 0$

$\phi = a e^{-ip\cdot x} + a^\dagger e^{ip\cdot x}$

$= a e^{-iEt+ipx} + a^* e^{iEt-ipx}$

pos energy   neg energy

$0 = E^2 - \vec{p}^{\,2} - m^2$

$= (\sqrt{E^2-\vec{p}^{\,2}}+m)\underbrace{(\sqrt{E^2-\vec{p}^{\,2}}-m)}_{\text{want this root only}}$

don't want $\sqrt{\ }$ in diff eqn $\sqrt{\frac{\partial^2}{\partial t^2}-\nabla^2}$

~~How want a 1st order diff eqn?~~

• Dirac proposed $\sqrt{E^2-p^2} \longrightarrow \gamma^0 E - \vec{\gamma}\cdot\vec{p}$

where $\gamma^\mu = \gamma^0, \gamma^1, \gamma^2, \gamma^3$ are 4x4 matrices (Dirac matrices)
→ non commuting

In fact γ^μ transforms as a contravariant 4-vector

$\gamma^0 E - \vec{\gamma}\cdot\vec{p} = \eta_{\mu\nu}\gamma^\mu p^\nu = \underbrace{\gamma_\mu p^\mu}_{\text{"Feynman slash" } \not p} =$ Lorentz invariant

$0 = (\gamma_\mu p^\mu + m\mathbb{1})(\gamma_\nu p^\nu - m\mathbb{1})$

$= (\gamma_\mu p^\mu)(\gamma_\nu p^\nu) - m^2\mathbb{1} \Rightarrow \not p\not p = m^2\mathbb{1}$

γ-matrices must obey this eqn

RF-2

$\not p \not p$ = $\cancel{(\gamma_\mu p^\mu)(\gamma_\nu p^\nu)} = (\gamma_0 p^0 + \gamma_1 p^1 + \gamma_2 p^2 + \gamma_3 p^3)^2$

$= \cancel{(\gamma_\mu p^\mu)(\gamma_\nu p^\nu)}$

$= \gamma_0^2 p^0 + \gamma_1^2 (p^1)^2 + \gamma_2^2 (p^2)^2 + \gamma_3^2 (p^3)^2$

$+ (\gamma_0\gamma_1 + \gamma_1\gamma_0) p^0 p^1 + (\gamma_0\gamma_2 + \gamma_2\gamma_0) p^0 p^2 + \ldots$

$= m^2 = (p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2$

$\Rightarrow \gamma_0^2 = 1, (\gamma^1)^2 = -1, (\gamma^2)^2 = -1, (\gamma^3)^2 = -1$

and $\gamma_0\gamma_1 + \gamma_1\gamma_0 = 0$ etc.

$\Rightarrow \boxed{\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}}$ Clifford algebra!

$\therefore$ $\cancel{(\gamma_\mu a^\mu)(\gamma_\nu b^\nu) = a_\mu b^\mu}$ ← $\boxed{\text{HW} \Rightarrow \not a \not b = a \cdot b}$

~~Then~~ $\cancel{\gamma_\mu p^\mu = m}$

$0 = \cancel{(\gamma_\mu p^\mu + m)}\cancel{(\gamma_\nu p^\nu - m)}$

~~choose this~~

~~Now $p^\mu \to i\partial^\mu$ so posit~~ $\cancel{(i\not\partial - m)\psi = 0}$

~~Dirac eqn~~ $\cancel{(i\gamma^\mu \partial_\mu - m)\psi = 0}$

$$p^2 = E^2 - \vec{p}^{\,2} = m^2$$

$$0 = p^2 - m^2 = (\sqrt{p^2} + m)(\sqrt{p^2} - m^2)$$

we have factored this eqn a different way!

$$= (\not{p} + m)(\not{p} - m)$$

QM: $p^\mu \to i\partial^\mu$ differential op acts on a fcn or fld

γ^μ 4x4 matrix act on a column vect

Let ψ = <u>Dirac spinor</u> = 4-component complex function $\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$

$$0 = (i\not{\partial} + m)(i\not{\partial} - m)\psi$$

But we want a 1st order eqn in time (& space!)

$(i\not{\partial} - m)\psi = 0$ Dirac eqn!

If we then act w/ $i\not{\partial} + m$ we get $(-\not{\partial}\not{\partial} - m^2)\psi = 0$

$(-\partial^2 - m^2)\psi = 0$

Klein Gordon

if ψ obeys Dirac, it also obeys KG

w/ solutions $e^{\pm ip\cdot x} \Rightarrow (p^2 - m^2)\psi = 0$

ie $p^2 = m^2$

Dirac eqn = $\sqrt{\text{Klein Gordon}}$

$0 = E^2 - p^2 - m^2$

There are several ways to factor this

$$0 = \left(E + \sqrt{p^2-m^2}\right)\left(E - \sqrt{p^2-m^2}\right) \quad \text{or} \quad 0 = \left(\sqrt{E^2-p^2} + m\right)\left(\sqrt{E^2-p^2} - m\right)$$

Instead we will try

$$\sqrt{p^2+m^2} \to \vec{\alpha}\cdot\vec{p} + \beta m$$

then

$$0 = (E + \vec{\alpha}\vec{p} + \beta m)(E - \vec{\alpha}\cdot\vec{p} - \beta m)$$

where $\beta^2 = 1$, $\vec{\alpha}^2 = 1$

$\beta\vec{\alpha} + \vec{\alpha}\beta = 0$

$$= (\beta E\beta - \beta\vec{\alpha}\cdot\vec{p}\beta + m\beta)(E - \vec{\alpha}\cdot\vec{p} - \beta m)$$

$$= (\beta E - \beta\vec{\alpha}\cdot\vec{p} + m)(\beta - \vec{\beta}\vec{\alpha}\cdot\vec{p} - m)$$

Instead we will try

$$\sqrt{E^2-p^2} \to \gamma_\mu p^\mu$$

$$0 = (\gamma_\mu p^\mu + m)(\gamma_\mu p^\mu - m)$$

$$= (\gamma^0 E - \vec{\gamma}\cdot\vec{p} + m)(\gamma^0 E - \vec{\gamma}\cdot\vec{p} - m)$$

$\longleftrightarrow$ agree if $\beta = \gamma^0$, $\beta\vec{\alpha} = \vec{\gamma}$

There are many representations of the Dirac matrices

Here we will use the Weyl basis.

show this for HW

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

↑ position important

$$\{\gamma^0, \gamma^0\} = 2(\gamma^0)^2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 2\eta^{00}$$

$$\gamma^0\gamma^i + \gamma^i\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\sigma & \\ & \sigma \end{pmatrix} + \begin{pmatrix} \sigma & \\ & -\sigma \end{pmatrix} = 0$$

$$(\gamma^i\gamma^j + \gamma^j\gamma^i) = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\sigma^i\sigma^j - \sigma^j\sigma^i & 0 \\ 0 & -\sigma^i\sigma^j - \sigma^j\sigma^i \end{pmatrix}$$

$$= \begin{pmatrix} -2\delta^{ij} & \\ & -2\delta^{ij} \end{pmatrix} = -2\delta^{ij} = 2\eta^{ij}$$

HW: prove $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$ $\qquad (\gamma^0)^\dagger = \gamma^0$ ✓

$$(\gamma^i)^\dagger = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix} = -\gamma^i \qquad \gamma^0\gamma^i\gamma^0 = -\gamma^i\gamma^0\gamma^0$$

$$\gamma^\nu\gamma_\nu = \tfrac{1}{2}\{\gamma^\nu, \gamma_\nu\} = \eta^\mu{}_\mu = d$$

HW: $\gamma^\nu\gamma^\mu\gamma_\nu = \{\gamma^\nu, \gamma^\mu\}\gamma_\nu - \gamma^\mu\gamma^\nu\gamma_\nu$

$$= 2\gamma^\mu - d\gamma^\mu = (2-d)\gamma^\mu = -2\gamma^\mu$$

Let's solve Dirac eqn

$$(i\not{\partial} - m)\psi = 0$$

Ansatz $\psi = u(p)\, e^{-ip\cdot x} = u(p)\, e^{-iEt + i\vec{p}\cdot\vec{x}}$

where $\vec{p}$ is arbitrary and $E = \sqrt{\vec{p}^2 + m^2}$ from K-G eqn!

$$0 = (\not{p} - m)u = (\gamma^0 p^0 - \gamma^i p^i - m)u$$

2 component spinors

$$0 = \begin{pmatrix} -m & E - \vec{\sigma}\cdot\vec{p} \\ E + \vec{\sigma}\cdot\vec{p} & -m \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

Lower eqn

$$(E + \vec{\sigma}\cdot\vec{p})u_1 - m u_2 = 0 \quad \Rightarrow \quad u_2 = \frac{E + \vec{\sigma}\cdot\vec{p}}{m} u_1$$

Choose $u_1 = N\chi_{ps}$ where $s = \pm$ (spin up/down along $\hat{p}$)

$$\vec{\sigma}\cdot\vec{p}\,\chi_{ps} = s|\vec{p}|\chi_{ps}$$

$$u_2 = N\left(\frac{E + s|\vec{p}|}{m}\right)\chi_{ps}$$

Now $m = \sqrt{E^2 - |\vec{p}|^2} = \sqrt{(E + s|p|)(E - s|\vec{p}|)}$

$$\Rightarrow \quad u_2 = N\sqrt{\frac{E + s|\vec{p}|}{E - s|\vec{p}|}}\,\chi_{ps}$$

$u_{ps} = N\begin{pmatrix} \chi_{ps} \\ \sqrt{\frac{E+s|p|}{E-s|p|}}\,\chi_{ps} \end{pmatrix}$: choose $N = \sqrt{E - s|p|}$ so

$$u_{ps} = \begin{pmatrix} \sqrt{E - s|\vec{p}|}\,\chi_{ps} \\ \sqrt{E + s|\vec{p}|}\,\chi_{ps} \end{pmatrix}$$

$$\psi = \begin{pmatrix} \sqrt{E - s|\vec{p}|}\; \chi_{ps} \\ \sqrt{E + s|\vec{p}|}\; \chi_{ps} \end{pmatrix} e^{-ip\cdot x}$$

2 solutions, 4 mom $\vec{p}$, $\underline{\underline{\text{pos en}}}$ E, + helicity $s = \pm$

$$\psi^\dagger = \left(\sqrt{E - s|p|}\; \chi^\dagger_{ps} \quad \sqrt{E + s|p|}\; \chi_{ps} \right) e^{ip\cdot x}$$

$$\psi^\dagger \psi = (E - s|\vec{p}|) + (E - s|p|) = 2E$$ for later...

$$\boxed{u^\dagger_{ps} u_{ps'} = 2E\, \delta_{ss'}}$$

However there are $\underline{2}$ more solutions to $(i\not\partial - m)\psi = 0$

Ansatz $\psi = v(p)\, e^{ip\cdot x} = v(p)\, e^{iEt - i\vec{p}\cdot\vec{x}}$

$$= v(p)\, e^{-i(-E)t + i(-\vec{p})\cdot\vec{x}}$$

corresp to wave fn w/ mom. $-\vec{p}$ and energy $-E$

(where $E = \sqrt{p^2+m^2}$)

$$0 = (\not p + m)\, v(p)$$

$$= \begin{pmatrix} m & E - \vec{\sigma}\cdot\vec{p} \\ E + \vec{\sigma}\cdot\vec{p} & m \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

lower eqn $(E + \vec{\sigma}\cdot\vec{p})\, v_1 + m v_2 = 0 \Rightarrow v_2 = \dfrac{E + \vec{\sigma}\cdot\vec{p}}{-m}\, v_1$

Let $v_1 = N \chi_{ps}$

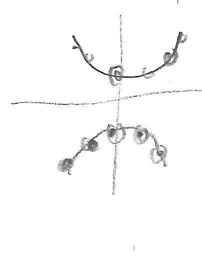
$$v_2 = -N \frac{(E + s|\vec{p}|)}{m} \chi_{ps} = -N \sqrt{\frac{E + s|\vec{p}|}{E - s|\vec{p}|}}\, \chi_{ps}$$

$$v_p = \begin{pmatrix} \sqrt{E - s|p|}\, \chi_{ps} \\ -\sqrt{E + s|p|}\, \chi_{ps} \end{pmatrix}$$

$\underline{v_{ps}^\dagger v_{ps'} = 2E\, \delta_{ss'}}$

$$\psi = \begin{pmatrix} \sqrt{E - s|p|}\, \chi_{ps} \\ -\sqrt{E + s|p|}\, \chi_{ps} \end{pmatrix} e^{ip\cdot x}$$

. We have not $\underline{\text{banished}}$ the negative energy solutions



← fermions: fill the Dirac sea!

← holes

~~(11-2-11)~~

New approach:

Treat ψ as a "classical field" obeying euler-lagrange eqn.

Let $\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi$

where ψ and $\bar{\psi}$ are independent.

[later we'll see what $\bar{\psi}$ is]

Vary $\bar{\psi}$:

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} = (i\not{\partial} - m)\psi$$

Vary $\partial_\mu \bar{\psi}$:

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} = 0$$

so $\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})}\right) - \frac{\partial \mathcal{L}}{\partial \bar{\psi}} = 0 \implies (i\not{\partial} - m)\psi = 0$ Dirac!

Vary ψ:

$$\frac{\partial \mathcal{L}}{\partial \psi} = -\bar{\psi} m$$

Vary $\partial_\mu \psi$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = \bar{\psi} i \gamma^\mu$$

so $\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\right) - \frac{\partial \mathcal{L}}{\partial \psi} = 0 \implies$

$$\partial_\mu(\bar{\psi} i \gamma^\mu) + \bar{\psi} m = 0$$

$$\bar{\psi}(i\overleftarrow{\not{\partial}} + m) = 0$$

Quicker: $\int d^3x\, \mathcal{L} = \int d^3x\, \bar{\psi}(i\partial\!\!\!/ - m)\psi$

$= \int d^3x\, \bar{\psi}(-i\overleftarrow{\partial\!\!\!/} - m)\psi$ $\underline{\underline{\text{IBP}}}$

vary wrt $\psi \Rightarrow$ $\bar{\psi}(-i\overleftarrow{\partial\!\!\!/} - m) = 0$ ✓

How is $\bar{\psi}$ related to ψ? Is $\bar{\psi} = \psi^\dagger$? Not quite

$(i\partial\!\!\!/ - m)\psi = i\gamma^\mu \partial_\mu \psi - m\psi$

take herm. conjugate

$-i(\partial_\mu \psi^\dagger)(\gamma^\mu)^\dagger - m\psi^\dagger$

Now $(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0$ so

$-i(\partial_\mu \psi^\dagger)\gamma^0\gamma^\mu\gamma^0 - m\psi^\dagger = 0$

$-i(\partial_\mu \psi^\dagger)\gamma^0\gamma^\mu - m\psi^\dagger\gamma^0$

$-i\psi^\dagger\gamma^0\gamma^\mu\overleftarrow{\partial}_\mu - m\psi^\dagger\gamma^0$

$\psi^\dagger\gamma^0(-i\overleftarrow{\partial\!\!\!/} - m) = 0$

Suggests $\bar{\psi} = \psi^\dagger\gamma^0$

In fact this is required to make $\mathcal{L}$ invariant under Lorentz transformations general soln of Dirac eqn . . .

arbitrary coefficients

$$\psi(x) = \int d\vec{p} \sum_s \left(b_{ps} u_{ps} e^{-ip\cdot x} + d^*_{ps} v_{ps} e^{ip\cdot x} \right)$$

Apparently too many d.o.f. (4 : s = ±1, b + d)

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$$

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\partial^\nu \psi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})}\partial^\nu \bar{\psi} - \eta^{\mu\nu}\mathcal{L}$$

$$= \bar{\psi} i \gamma^\mu \partial^\nu \psi + 0 - \eta^{\mu\nu}\bar{\psi}\underbrace{(i\partial\!\!\!/ - m)\psi}_{0 \text{ using}}$$

euler lagrange

$$T^{00} = \bar{\psi} i \gamma^0 \partial^0 \psi = i\psi^\dagger \partial^0 \psi$$

$$H = \int d^3x\, T^{00}$$

$$\psi = \sum_s \int d\tilde{p}\,(b_{ps} u_{ps} e^{-ip\cdot x} - d^\dagger_{ps} v_{ps} e^{ip\cdot x})$$

$$i\psi^\dagger = \sum_s \int d\tilde{p}\,(i b^\dagger_{ps} u^\dagger_{ps} e^{ip\cdot x} + i d_{ps} v^\dagger_{ps} e^{-ip\cdot x})$$

$$\partial^0\psi = \sum_s \int d\tilde{p}\,(-iE\, b_{ps} u_{ps} e^{-ip\cdot x} + iE\, d^\dagger_{ps} v_{ps} e^{+ip\cdot x})$$

$$H = \sum_{s,s'} \int d\tilde{p} \frac{d^3p'}{(2\pi)^3(2E')}\Big[+ E'\, b^\dagger_{ps} b_{p's'}\, u^\dagger_{ps} u_{p's'} \int d^3x\, e^{ip\cdot x - ip'\cdot x'}$$
$$+ E'\, d_{ps} b_{p's'}\, v^\dagger_{ps} u_{p's'} \int d^3x\, e^{-ip\cdot x - ip'\cdot x'}$$
$$- E'\, b^\dagger_{ps} d^\dagger_{p's'}\, u^\dagger_{ps} v_{p's'} \int d^3x\, e^{+ip\cdot x + ip'\cdot x'}$$
$$- E'\, d_{ps} d^\dagger_{p's'}\, v^\dagger_{ps} v_{p's'} \int d^3x\, e^{-ip\cdot x + ip'\cdot x'} \Big]$$

(see next page)

$$H = \sum_s \int d\tilde{p}\,[\,E\, b^\dagger_{ps} b_{ps} - E\, d_{ps} d^\dagger_{ps}\,]$$

Integrals give either $(2\pi)^3\,\delta(\vec{p}-\vec{p}\,')$ $\Rightarrow E' = E$

or $(2\pi)^3\, e^{\pm 2iEt}\,\delta(\vec{p}+\vec{p}\,')$ $\Rightarrow E' = E$

$$H = \sum_{s,s'} \int d\tilde{p}\,\Big[\tfrac{1}{2}\, b^\dagger_{ps}\, b_{ps'}\, u^\dagger_{ps}\, u_{ps'}$$
$$+ \tfrac{1}{2}\, d_{ps} b_{-ps'}\, v^\dagger_{ps}\, u_{-ps'}\, e^{-2iEt}$$
$$- \tfrac{1}{2}\, b^\dagger_{ps}\, d^\dagger_{-ps'}\, u^\dagger_{ps}\, v_{-ps'}\, e^{2iEt}$$
$$- \tfrac{1}{2}\, d_{ps}\, d^\dagger_{ps'}\, v^\dagger_{ps} v_{ps'}\Big]$$

Now $v^\dagger_{ps}\, u_{-p,s'} = \left(\sqrt{E-s|p|}\,\chi^\dagger_{ps},\, -\sqrt{E+s|p|}\,\chi^\dagger_{ps}\right)\begin{pmatrix}\sqrt{E-s'|p|}\,\chi_{-p,s'}\\ \sqrt{E+s'|p|}\,\chi_{-p,s'}\end{pmatrix}$

P&S eq 3.65

$= \sqrt{E-s|p|}\sqrt{E-s'|p|}\;\chi^\dagger_{ps}\chi_{-p,s'} - \sqrt{E+s|p|}\sqrt{E+s'|p|}\;\chi^\dagger_{ps}\chi_{-ps'}$

Now $\chi^\dagger_{ps}\chi_{-p,s'} \neq 0$ iff $s=-s'$, but then $\uparrow$ vanishes

---

$u^\dagger_{ps}u_{ps'} = \left(\sqrt{E-s|p|}\,\chi_{ps},\, \sqrt{E+s|p|}\,\chi_{ps}\right)\begin{pmatrix}\sqrt{E-s'|\vec{p}|}\,\chi_{ps'}\\ \sqrt{E+s'|\vec{p}|}\,\chi_{ps'}\end{pmatrix}$

Now $\chi^\dagger_{ps}\chi_{ps'} = 0$ unless $s=s'$, then

$= (E-s|p|) + (E+s|p|) = 2E$ P S 3.60

$v^\dagger_{ps}v_{ps'} = \left(\sqrt{E-s|p|}\,\chi_{ps},\, -\sqrt{E+s|p|}\,\chi_{ps}\right)\begin{pmatrix}\sqrt{E-s'|p|}\,\chi_{ps'}\\ -\sqrt{E-s'|p|}\,\chi_{ps'}\end{pmatrix}$
$= 2E$

PS 3.63

Recall for a complex scalar field

$$\cancel{[\phi(\vec{x},t), \pi(\vec{x}',t)] = i\delta^{(3)}(\vec{x}-\vec{x}')}$$

$$\cancel{[\phi^\dagger(\vec{x},t), \pi^\dagger(\vec{x}',t)] = i\delta^{(3)}(\vec{x}-\vec{x}')}$$

$$\cancel{[\phi, \pi^\dagger] = [\phi^\dagger, \pi] = 0}$$

$$\Rightarrow \phi = \int d\tilde{k}\, [a_k e^{-ik\cdot x} + b_k^\dagger e^{ik\cdot x}]$$

$$\Rightarrow [a_k, a_{k'}^\dagger] = [b_k, b_k^\dagger] = (2\pi)^3 (2\omega)\, \delta^{(3)}(\vec{k}-\vec{k}')$$

vacuum $a_k|0\rangle = 0 \quad b_k|0\rangle = 0$

$$H = \int d\tilde{k}\ \omega(a_k^\dagger a_k + b_k^\dagger b_k) \quad + \text{(zero point energy)}$$

U(1) symm'y

$$Q = \int d\tilde{k}\ q(a_k^\dagger a_k - b_k^\dagger b_k)$$

$$[H, a_k^\dagger] = \int d\tilde{k}'\, \omega' \underbrace{[a_{k'}^\dagger a_{k'}, a_k^\dagger]}$$

$$a_{k'}^\dagger \underbrace{[a_{k'}, a_k^\dagger]}_{(2\pi)^3 2\omega\, \delta(\vec{k}-\vec{k}')} + \underbrace{[a_{k'}^\dagger, a_k^\dagger]}_{0} a_{k'}$$

$$= \omega a_k^\dagger$$

$$[H, b_k^\dagger] = \omega b_k^\dagger$$

$$H a_k^\dagger = a_k^\dagger (H+\omega)$$

$$H a_k^\dagger |E\rangle = a_k^\dagger (\underbrace{H+\omega}_{E+\omega})|E\rangle = (E+\omega)\, a_k^\dagger |E\rangle$$

so $a_k^\dagger$ raises the energy by ω.

$$H = \sum_s \int d\tilde{p} \, [E \, b^\dagger_{ps} b_{ps} - E \, d_{ps} d^\dagger_{ps}]$$

a $b_{ps}|0\rangle = 0$

$d_{ps}|0\rangle = 0$

~~$b^\dagger_{ps}|0\rangle \Rightarrow$ one fermion state~~

~~$b^\dagger_{ps} b^\dagger_{ps}|0\rangle = 0$ (acc. Pauli exclusion)~~

$\{b_{ps}, b_{p's'}\} = 0$

$\Rightarrow \{b^\dagger_{ps}, b^\dagger_{ps}\} = 0$

Consider $\{b_{ps}, b^\dagger_{p's'}\} = (2\pi)^3 (2E)\, \delta(\vec{p}-\vec{p}')\, \delta_{ss'}$

$$[H, b^\dagger_{ps}] = \sum_{s'} \int d\tilde{k}' \, E' \, [b^\dagger_{p's'} b_{p's'}, b^\dagger_{ps}]$$

$$b^\dagger_{p's'} b_{p's'} b^\dagger_{ps} - \underbrace{b^\dagger_{ps} b^\dagger_{p's'}}_{(-b^\dagger_{p's'} b^\dagger_{ps})} b_{p's'}$$

$$= b^\dagger_{p's'} \, (\underbrace{b_{p's'} b^\dagger_{ps} + b^\dagger_{ps} b_{p's'}}_{(2\pi)^3 \, 2E \, \delta(p-p') \, \delta_{ss'}})$$

$= E \, b^\dagger_{ps}$ $\Rightarrow b^\dagger_{ps}$ creates a state of positive energy E, an electron $(s = \pm 1)$

$$[H, d^\dagger_{ps}] = -\sum_{s'} \int d\tilde{k}' \, E' \, [\underbrace{d_{p's'} d^\dagger_{p's'}, d_{ps}}]$$

$$\underbrace{d_{p's'} \underbrace{d^\dagger_{p's'} d^\dagger_{ps}}_{-d^\dagger_{ps} d^\dagger_{p's'}} - d^\dagger_{ps} d_{p's'} d^\dagger_{p's'}}$$

$$-(\underbrace{d_{p's'} d^\dagger_{ps} + d^\dagger_{ps} d_{p's'}}_{(2\pi)^3 (2E') \, \delta_{\vec{p}-\vec{p}'}}) \, d^\dagger_{p's'}$$

$= \cancel{0} + E \, d^\dagger_{ps}$ so $d^\dagger_{ps}$ creates a state of pos. eng E, a posit. $(s = \pm 1)$

$\underbrace{b^\dagger_{ps}|0\rangle}_{\text{1 ferm st}}$ $\quad b^\dagger_{ps} b^\dagger_{ps}|0\rangle = \frac{1}{2}\{b^\dagger_{ps}, b^\dagger_{ps}\}|0\rangle = 0 \Rightarrow$ Pauli ...

L4-8

$$H = \sum_s \int d\tilde{p}\; E\,(b^\dagger_{ps} b_{ps} - d_{ps} d^\dagger_{ps})$$

$$= \sum_s \int d\tilde{p}\; E\,(b^\dagger_{ps} b_{ps} + d^\dagger_{ps} d_{ps} - \underbrace{\{d_{ps}, d^\dagger_{ps}\}}_{(2\pi)^3 (2E)\,\delta(k-k)})$$

$$= (2E) \underbrace{\int d^3x\, e^{i(k-k)\cdot x}}_{V}$$

$$= \sum_s \int d\tilde{p}\, [E\,(b^\dagger_{ps} b_{ps} + d^\dagger_{ps} d_{ps}) - 2E^2 V]$$

$$\downarrow$$

$$-\sum_s \int \frac{d^3p}{(2\pi)^3} E$$

ie $\frac{E}{2}$ per degree of freedom

(electron + positron)

$$\text{cosmo const} = V \int \frac{d^3k}{(2\pi)^3}\; \frac{1}{2}E\,(n_{\text{boson dof}} - n_{\text{ferm dof}})$$

2016 course ended here

$$T^{0i} = \bar{\psi}\, i\gamma^0 \partial^i \psi$$

$$\psi = \sum \int d\vec{p}\, \left[b_{ps} u_{ps} e^{-iEt + i\vec{p}\cdot\vec{x}} + d^\dagger_{ps} v_{ps} e^{iEt - i\vec{p}\cdot\vec{x}} \right]$$

$$\partial^i \psi = -\partial_i \psi = \sum \int d\vec{p}\,' \left[b_{p's} u_{p's} (-ip^i) e^{-ipx} + d^\dagger_{p's} v_{p's} (ip^i) e^{ipx} \right]$$

so same as T^{00} xc $E \to p^i$

Then integrals give $\delta(\vec{p}-\vec{p}\,') \Rightarrow p^{i\prime} \to p^i$

or $\delta(\vec{p}+\vec{p}\,') \Rightarrow p^{i\prime} \to -p^i$

$$P^i = \sum_s \int d\vec{p}\, \left[p^i b^\dagger_{ps} b_{ps} - p^i d_{ps} d^\dagger_{ps} \right]$$

Final exam

$$J^\mu = \bar{\psi}\gamma^\mu \psi$$

$$Q = \int \bar{\psi}\gamma^0\psi = \int \psi^\dagger \psi$$

$$= \sum_{s,s'} \int d\tilde{p}\, \frac{d^3p'}{(2\pi)^3(2E')} \underbrace{\Big[\begin{array}{ll} b^\dagger_{ps}\, b_{p's'} & u^\dagger_{ps}\, u_{p's'} \\ + d_{ps}\, b_{p's'} & \cdots \\ b^\dagger\, d^\dagger & \cdots \\ d\, d^\dagger_{ps} & \cdots \end{array} \Big]}$$

$$= \sum_s \int d\tilde{p}\, \left(b^\dagger_{ps}\, b_{ps} + d_{ps}\, d^\dagger_{ps} \right)$$

$$[Q, b^\dagger_{ps}] = [b^\dagger_{ps} b_{ps}, b^\dagger_{ps}] = \underline{\underline{b^\dagger_{ps}}}$$

$$[Q, d^\dagger_{ps}] = [d_{ps} d^\dagger_{ps}, d^\dagger_{ps}] = \underline{\underline{-d^\dagger_{ps}}}$$

$$\mathcal{L} = \bar{\psi}(i\cancel{\partial} - m)\psi$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} = \bar{\psi} i \gamma^0 = i\psi^\dagger$$

$$\{\psi(x), i\psi^\dagger(y)\} = i\delta(x-y)$$

$$\{\psi(x), \psi^\dagger(y)\} = \delta(x-y)$$

$$\psi = \int d\tilde{p} \sum_s \left[b_{ps} u_{ps} e^{-ipx} + d^\dagger_{ps} v_{ps} e^{ipx} \right]$$

$$\psi^\dagger = \int d\tilde{p} \sum_s \left[b^\dagger_{ps} u^\dagger_{ps} e^{+ipx} + d_{ps} v^\dagger_{ps} e^{-ipx} \right]$$

$$\{\psi, \psi^\dagger\} = \int d\tilde{p}\, d\tilde{p}' \sum_{s,s'} \Big[\{b_{ps}, b^\dagger_{p's'}\} u_{ps} u^\dagger_{p's'} e^{-ipx + ip'x'}$$

$$+ \{d^\dagger_{ps}, d_{p's'}\} v_{ps} v^\dagger_{p's'} e^{ipx - ip'x'} \Big]$$

Let $\{b_{ps}, b^\dagger_{p's'}\} = (2\pi)^3 (2\omega)\, \delta^{(3)}(p-p')\, \delta_{ss'}$

$\{d^\dagger_{ps}, d_{p's'}\} = (2\pi)^3 (2\omega)\, \delta^{(3)}(p-p')\, \delta_{ss'}$

$$= \int d\tilde{p} \sum_s \Big[\underbrace{u_{ps} u^\dagger_{ps}}_{\substack{(\cancel{p} + m)\gamma^0 \\ E + \gamma^0\gamma_i p^i + m\gamma^0}} e^{-ip(x-x')} + \underbrace{v_{ps} v^\dagger_{ps}}_{\substack{(\cancel{p} - m)\gamma^0 \\ E + \gamma^0\gamma_i p^i - m\gamma^0}} e^{ip(x-x')} \Big]$$

Now let $\vec{p} \to -\vec{p}$ in 2nd integral. Also assume equal times so $e^{iE(t-t')}$ term drops out

$$\int d\vec{p}\ \left[E + \gamma^0\gamma_i p^i + m\gamma^0 + (E - \gamma^0\gamma_i p^i - m\gamma^0)\right] e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}$$

cancel

$$= \int \frac{d^3p}{(2\pi)^3 (2E)} (2E)\, e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}$$

$$= \delta(\vec{x}-\vec{y})$$

Why need anticommutators

general soln to Dirac eqn (cf Peskin Schroeder 3.87)

$$\psi(x) = \int d\tilde{p}\, [a_{ps} u_{ps} e^{-ip\cdot x} + b_{ps} v_{ps} e^{ip\cdot x}]$$

$$\psi^\dagger(x) = \int d\tilde{p}\, [a^\dagger_{ps} u^\dagger_{ps} e^{+ip\cdot x} + b^\dagger_{ps} v^\dagger_{ps} e^{-ip\cdot x}]$$

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi \Rightarrow \pi_\psi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} = \bar{\psi} i\gamma^0 = i\psi^\dagger$$

Assume commutators $\}$ $[\psi(x), \pi(x')] = i\delta(x-x') \Rightarrow [\psi(x), \psi^\dagger(x')] = \delta(x-x')$

$$[\psi(x), \psi(x')] = \int d\tilde{p}\, d\tilde{p}' \sum_{s,s'} \Big([a_{ps}, a^\dagger_{p's'}]\, u_{ps} u^\dagger_{p's'} e^{-ip\cdot x + ip'x'} + [b_{ps}, b^\dagger_{p's'}]\, v_{ps} v^\dagger_{p's'} e^{ip\cdot x - ip'\cdot x'} \Big)$$

Assume $[a_{ps}, a^\dagger_{p's'}] = (2\pi)^3 (2\omega)\, \delta(p-p')\, \delta_{ss'}$

$[b_{ps}, b^\dagger_{p's'}] = (2\pi)^3 (2\omega)\, \delta(p-p')\, \delta_{ss'}$

$[a, b^\dagger] = [a^\dagger, b] = 0$

then

$$[\psi(x), \psi(x')] = \int d\tilde{p} \Big(\underbrace{\sum_s u_{ps} u^\dagger_{ps}}_{\underbrace{(\not{p}+m)\gamma^0}_{E + \gamma_i p^i + m\gamma^0}} e^{-ip\cdot(x-x')} + \underbrace{\sum_s v_{ps} v^\dagger_{ps}}_{\underbrace{(\not{p}-m)\gamma^0}_{E + \gamma_i p^i - m\gamma^0}} e^{ip(x-x')} \Big)$$

Now assume <u>equal time</u>:

$$[\psi(x), \psi(x')]_{=} = \int d\tilde{p} \left([E + \gamma_i p^i + m\gamma^0]\, e^{+i\vec{p}(\vec{x}-\vec{x}')} + [E + \gamma_i p^i - m\gamma^0]\, e^{-i\vec{p}(\vec{x}-\vec{x}')} \right)$$

Let $\vec{p} \to -\vec{p}$ in 2nd term

$$= \int d\tilde{p} \left([E + \gamma_i p^i + m\gamma^0]\, e^{i\vec{p}(\vec{x}-\vec{x}')} + [E - \gamma_i p^i - m\gamma^0]\, e^{i\vec{p}(\vec{x}-\vec{x}')} \right)$$

Then we get cancellation between $p^0 \sigma^m$ terms and

$$\int d\tilde{p}\, 2E\, e^{i\vec{p}\cdot(\vec{x}-\vec{x}')}$$

$$= \int \frac{d^3p}{(2\pi)^3}\, e^{i\vec{p}\cdot(\vec{x}-\vec{x}')}$$

$$= \delta(\vec{x}-\vec{x}')$$

so this works

Now $H = \int d^3x\ \bar{\psi} i \gamma^0 \partial^0 \psi = \int d^3x\ i\psi^\dagger \partial_0 \psi$

$$= \int \frac{d^3p}{(2\pi)^3} \sum_s \left[E a_p^\dagger a_p - E b_p^\dagger b_p \right]$$

Then $[H, b] = [-E b_p^\dagger b_p, b] = -E[b_p^\dagger, b] b_p = E b_p$ rather than $-Eb$

Why can't we just let $b_p^\dagger \to \tilde{b}_p$ and $b_p \to \tilde{b}_p^\dagger$

Then we would have $[b_{ps}^\dagger, b_{p's'}] = (2\pi)^3 (2\pi) \delta(p - p') \delta_{ss'}$

and $H \sim -E \tilde{b}_{ps} \tilde{b}_{ps}^\dagger$

and $[H, \tilde{b}] = [-E \tilde{b}_{ps} \tilde{b}_{ps}^\dagger] \tilde{b} = -E\ \tilde{b}_{ps} [\tilde{b}_{ps}^\dagger, \tilde{b}] = -E \tilde{b}_{ps}$ ✓

P+S says this would ruin (3.89) but I can't see that.