Interactions

How does emc field A^μ interact w/ other fields?

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \mathcal{L}_{int}(A^\mu, \text{other fields})$$

not ∂A

Recall euler-lagrange

$$\partial_\mu F^{\mu\nu} = j^\nu \quad \text{where} \quad j^\mu = -\frac{\delta \mathcal{L}_{int}}{\delta A^\mu}$$

$$0 = \partial_\nu \partial_\mu F^{\mu\nu} = \partial_\nu j^\nu \quad \Rightarrow \quad \frac{\delta \mathcal{L}_{int}}{\delta A_\mu} \text{ must be a conserved current}$$

$\Rightarrow$ constraint on $\mathcal{L}_{int}$

Recall that $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ is gauge invariant

under $A^\mu \to A^\mu + \partial^\mu \chi$
$\delta A^\mu = +\partial^\mu \chi$ } gauge tx of the 2nd kind

What about the full lagrangian?

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}_{int}}{\partial A^\mu} \delta A^\mu = -j^\mu \partial_\mu \chi \neq 0.$$

not gauge invariant

Consider interaction between A^μ and a complex scalar

$$\mathcal{L} = \underbrace{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{\mathcal{L}_{free}(A)} + \mathcal{L}_{free}(\phi) + \mathcal{L}_{int}(A^\mu, \phi)$$

$$\mathcal{L}_{free}(\phi) = |\partial\phi|^2 - \frac{1}{2}m^2|\phi|^2$$

We know that ~~$\mathcal{L}(\phi)$~~ is invariant under phase transformation $\phi \to e^{-i\alpha}\phi$ $\Rightarrow$ conserved current

$$J^\mu = i(\phi^* \partial^\mu \phi - (\partial^\mu\phi)^* \phi)$$

Perhaps this $= -\dfrac{\delta \mathcal{L}_{int}}{\delta A^\mu}$?

Consider a <u>local</u> phase transformation of the first kind

a.k.a gauge transformation $\boxed{\text{Gell Mann Levy}}$

$$\phi \to e^{-iq\chi(x)}\phi$$

$$\phi^* \to e^{iq\chi(x)}\phi$$

$\phi^\dagger\phi$ is <u>still</u> invariant

$$\partial_\mu\phi \to e^{-iq\chi}\partial_\mu\phi - iq(\partial_\mu\chi)\, e^{-iq\chi}\phi$$

$$|\partial\phi|^2 \to \left|e^{-iq\chi}(\partial_\mu\phi - iq\partial_\mu\chi\, \phi)\right|^2$$

$$= |\partial\phi|^2 - iq\partial_\mu\chi\,(\partial_\mu\phi^*)\phi + iq\partial_\mu\chi\, \phi^*(\partial^\mu\phi) + \dots$$

$$= |\partial\phi|^2 + \underline{q(\partial_\mu\chi)J^\mu} + O(\partial\chi^2)$$

Also not invariant, in the opposite way

Consider a gauge transformation that acts simultaneously on A^μ and ϕ

$$A^\mu \to A^\mu + \partial^\mu \chi$$
$$\phi \to e^{-i\chi q}\phi$$

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observe that $\partial_\mu\phi + iqA_\mu\phi$

$$\to \partial_\mu(e^{-iq\chi}\phi) + iq(A_\mu + \partial_\mu\chi)\, e^{-iq\chi}\phi$$
$$= e^{-iq\chi}\left(\partial_\mu\phi - iq\partial_\mu\chi\,\phi + iqA_\mu\phi + iq\partial_\mu\chi\,\phi\right)$$
$$= e^{-iq\chi}(\partial_\mu\phi + iqA_\mu\phi)$$

Thus $|\partial_\mu\phi + iqA_\mu\phi|^2$ is invariant...

Call this combination the gauge covariant derivative $D_\mu\phi \equiv \partial_\mu\phi + iqA_\mu\phi$

Then

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + |D_\mu\phi|^2 - \tfrac{1}{2}m^2|\phi|^2$$

is gauge invariant

$$= -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \underbrace{(\partial_\mu\phi^* - iqA_\mu\phi^*)(\partial^\mu\phi + iqA^\mu\phi)}_{|\partial\phi|^2 - \underbrace{iqA_\mu\phi^*\partial^\mu\phi + iqA_\mu(\partial^\mu\phi^*)\phi + q^2A_\mu A^\mu|\phi|^2}_{\mathcal{L}_{int}}} - \tfrac{1}{2}m^2|\phi|^2$$

Let's check that both defs of j^μ agree

LG4

$\mathcal{L}$ is still also invariant under a global phase transformation

$$\boxed{\begin{aligned} \phi &\to e^{-iq\alpha}\phi \\ A_\mu &\to A_\mu \end{aligned}}$$

what is the conserved current?

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\frac{\partial \phi}{\partial \alpha} + \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)}}_{(\partial^\mu \phi + iqA^\mu)}\frac{\partial \phi^*}{\partial \alpha}$$

$\partial_\mu \phi^* - iA_\mu q \phi^*$

$$= \cancel{(D_\mu \phi)^* (-iq\phi) + (D^\mu \phi)(iq\phi^*)}$$

$$= \cancel{iq(\phi^* D^\mu \phi - \phi D^\mu \phi^*)}$$

$$= iq(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*) - 2q^2 A^\mu \phi^* \phi$$

what is $\frac{\delta \mathcal{L}_{int}}{\delta A^\mu}$?

$$\frac{\delta \mathcal{L}_{int}}{\delta A^\mu} = -iq\phi^* \partial_\mu \phi + iq(\partial^\mu \phi^*)\phi + 2q^2 A^\mu |\phi|^2$$

$= -j^\mu$ so it is indeed conserved

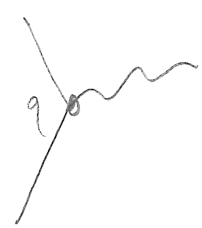
So symmetry $\Rightarrow$ form of interactions

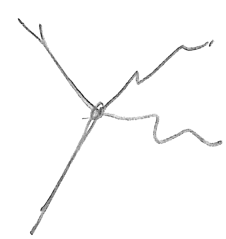
Quadratic terms $\Rightarrow$ free field...

Propagators

Interactions : cubic or above

$$\mathcal{L}_{int} = i(q\phi^\dagger \partial_\mu \phi\, A^\mu - q(\partial\phi^\dagger) A_\mu \phi) + q^2 A_\mu A^\mu \phi^\dagger \phi$$

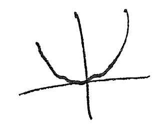




$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - V(\phi) \quad \Rightarrow \quad \partial^2\phi + \frac{\partial V}{\partial \phi} = 0$$

$V = \frac{1}{2}m^2\phi^2 \Rightarrow$ free fd $\Rightarrow$ particle of mass m

↳ $(\partial^2 + m^2)\phi = 0$

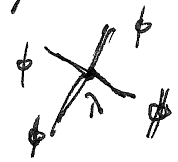


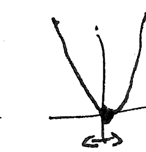
$V = \frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda\phi^4 \Rightarrow$ self interacting scalar field

$(\partial^2 + m^2)\phi + \lambda\phi^3 = 0$

non linearities in eom

$\Rightarrow$ interactions





If ϕ is small, interaction is treated as a perturbation. So

eom $(\partial^2 + m^2)\phi \approx 0$

$$\phi \cong a e^{-ip\cdot x} + a^* e^{+ip\cdot x}$$ provided a is small

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Consider $V = -\frac{1}{2}m^2\phi^2 \Rightarrow$ particle of imaginary mass?

$(\partial^2 - m^2)\phi^2 = 0$

$\phi = e^{\pm ipx}$

$p^2 = -m^2$

$E^2 - |\vec{p}|^2 = -m^2$

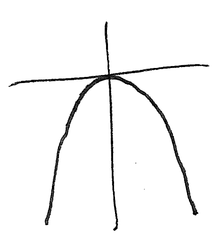
$E^2 = |p|^2 - m^2$

$E < |p|$

$v = \frac{|p|}{E} > 1$! tachyon

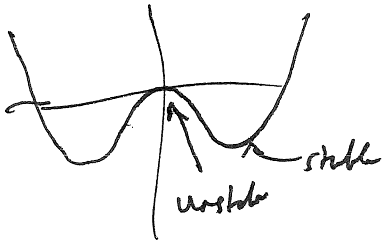
Look at potential:

ϕ is an unstable equilibrium



(tachyon $\Rightarrow$ sign of unstable vacuum)

$V = -\frac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda\phi^4$



$\frac{\partial V}{\partial \phi} = -m^2\phi + \frac{\lambda}{3!}\phi^3 = 0$

$\frac{\lambda}{3!}\phi\left(\phi^2 - \frac{m^2}{\lambda}\right)$ $\quad \phi = \frac{\pm m}{\sqrt{\lambda}} = v$

$v = \frac{m}{\sqrt{\lambda}}$

~~$\phi = \pm\sqrt{\frac{}{\lambda}}\,m = \pm v$~~

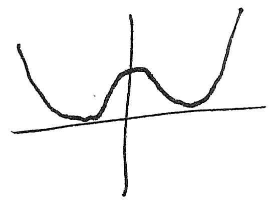
v is called vacuum expectation value

$\langle 0|\phi|0\rangle = v$

~~Let $\phi = v + \eta$~~

~~$(\partial\phi)^2 = (\partial\eta)^2$~~

$\frac{1}{4}\lambda(\phi^2 - v^2)^2 + \text{const}$



$V = \frac{\lambda}{4}(\phi^2 - v^2)^2$

$\frac{\partial V}{\partial \phi} = \lambda(\phi^2 - v^2)\phi$

(agrees w/ above if $\lambda v^2 = m^2$)

$\partial^2\phi + \lambda\phi(\phi^2 - v^2) = 0$

Let $\phi = v + \eta \Rightarrow \phi^2 - v^2 = (\phi + v)(\phi - v) = (2v + \eta)\eta$

$\partial\phi = \partial\eta$

$\partial^2\eta + \lambda(v+\eta)(2v+\eta)\eta = \partial^2\eta + 2\lambda v^2\eta + 3\lambda v\eta^2 + \lambda\eta^3$

For small η $\quad \partial^2\eta + 2\lambda v^2\eta \approx 0$ i.e. particle of mass

~~$\mu = 2v\sqrt{\lambda}$~~

$= \frac{2m}{\sqrt{\lambda}}$

$\mu^2 = 2\lambda v^2$

$\mu = \sqrt{2\lambda}\, v$

$\mu = \sqrt{2}\,m$

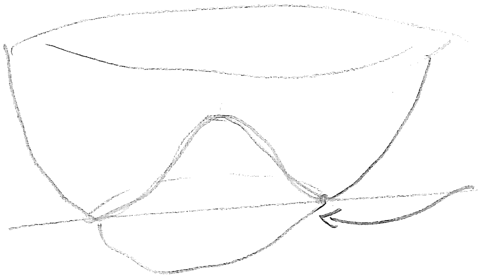
Complex scalar field

$$\mathcal{L} = |\partial\phi|^2 - V(|\phi|) \qquad \phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$$

$$V(|\phi|) = -m^2|\phi|^2 + \lambda|\phi|^4 \qquad \Rightarrow \text{min at } |\phi|^2 = \frac{m^2}{2\lambda} = \frac{v^2}{2}$$

$$\frac{\partial V}{\partial \phi^*} = -2m\phi + 4\lambda|\phi|^2\phi = 0$$

$$= \lambda\left(|\phi|^2 - \frac{v^2}{2}\right)^2 + (\text{const.})$$



$|\phi| = \frac{v}{\sqrt{2}}$ valley

$\phi = \frac{v}{\sqrt{2}}e^{i\theta}$ (all physically equivalent)

Choose $\theta = 0$. $\Rightarrow \phi = \frac{v}{\sqrt{2}}$

Expand fields around value at minimum

$$\phi = \frac{1}{\sqrt{2}}(v + h + i\eta) \qquad \text{ie} \begin{cases} \phi_1 = v + h \\ \phi_2 = \eta \end{cases}$$

$$\partial_\mu\phi = \frac{1}{\sqrt{2}}(\partial_\mu h + i\partial_\mu \eta)$$

$$|\partial\phi|^2 = \frac{1}{2}(\partial h)^2 + \frac{1}{2}(\partial\eta)^2$$

$$m_h = \sqrt{2\lambda}\, v$$

$$|\phi|^2 = \frac{1}{2}\left[(v+h)^2 + \eta^2\right]$$

$$|\phi|^2 - \frac{v^2}{2} = vh + h^2 + \frac{1}{2}\eta^2$$

$$= \frac{1}{2}m_h^2 h^2$$

$$V(|\phi|) = \lambda\left(vh + h^2 + \frac{1}{2}\eta^2\right)^2 = (\lambda v^2)h^2 + \underline{\text{cubic \& quartic}}$$

$$\mathcal{L} = \frac{1}{2}(\partial h)^2 + \frac{1}{2}(\partial \eta)^2 + \underbrace{(\lambda v^2)}_{\frac{\mu^2}{2}} h^2 \quad + \text{cubic + quartic}$$

h is a real scalar w/ mass $\mu = \sqrt{2\lambda}\, v = \sqrt{2}\, m$
(Higgs)

η is a real massless scalar (Goldstone boson)

Abelian Higgs model

$$\mathcal{L} = |D\phi|^2 - V(|\phi|) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$V(|\phi|) = \frac{\lambda}{4}\left(\phi^2 - \frac{v^2}{2}\right)^2 \quad \Rightarrow \quad \phi = v + h + i\eta$$

$$D\phi = \partial\phi + iqA\phi$$

$$= \frac{1}{\sqrt{2}}\left(\partial h + i\partial\eta + iqA(v+h+i\eta)\right)$$

$$= \frac{1}{\sqrt{2}}\left(\partial h - qA\eta + i[\partial\eta + qvA + qhA]\right)$$

$$|D\phi|^2 = \frac{1}{2}(\partial h - qA\eta)^2 + \frac{1}{2}(\partial\eta + qvA + qhA)^2$$

$$= \frac{1}{2}(\partial h)^2$$

$$+ \frac{1}{2}(\partial\eta + qvA)^2 + \text{cubic + quartic}$$

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qvA'

$\underline{m_A = qv}$

$\underline{m_H = \sqrt{2\lambda}\, v}$

masses prop to v

Masses arise from Higgs fld SSB

$$\phi \to \phi e^{-iq\chi}$$

$$(v+h+i\eta) \to (v+h+i\eta)(1 - iq\chi)$$

$$= v + h + i(\eta - qv\chi) + \ldots$$

ie

$$A^\mu \to A^\mu + \partial^\mu\chi = A^{\mu\prime}$$

$$\eta \to \eta - qv\chi = \eta'$$

$\Rightarrow (\partial\eta + qvA)$ is inv't.

gauge ~~choice~~ choice: $\chi = \frac{\eta}{qv} \Rightarrow \eta' = 0$

$$\Rightarrow \frac{1}{2}(qvA')^2 = \frac{1}{2}(qv)^2 A^2 \qquad \underline{m_A = qA}$$