Electromagnetism

Maxwell's eqns

Recall vector calc identities

div curl = 0

curl grad = 0

$$\vec{\nabla}\cdot E = \frac{\rho}{\epsilon_0}$$

$$\vec{\nabla}\times B - \mu_0\epsilon_0\frac{\partial E}{\partial t} = \mu_0 J$$

$$\vec{\nabla}\times E + \frac{\partial B}{\partial t} = 0$$

$$\nabla\cdot B = 0$$

$$\frac{1}{\sqrt{\mu_0\epsilon_0}} = c = 1$$

Choose Heaviside - ~~Lorentz~~ Lorentz units $\epsilon_0 = \mu_0 = 1$

Fine struct constant

$$\alpha = \frac{ke^2}{\hbar c}$$

$$= \frac{e^2}{4\pi\epsilon_0\hbar c} \longrightarrow \frac{e^2}{4\pi}$$

$$\nabla\cdot E = \rho \quad (1)$$

$$\nabla\times B - \frac{\partial E}{\partial t} = J \quad (2)$$

$$\nabla\times E + \frac{\partial B}{\partial t} = 0 \quad (3)$$

$$\nabla\cdot B = 0 \quad (4)$$

These eqns are self consistent iff current is conserved

$$\frac{\partial\rho}{\partial t} = \nabla\cdot\frac{\partial E}{\partial t} = \nabla\cdot\left(\nabla\times B - \vec{J}\right) = -\vec{\nabla}\cdot\vec{J}$$

~~Recall div curl = 0 curl grad = 0~~

so $\underline{\dfrac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0}$

(4) $\nabla \cdot B = 0 \quad \Rightarrow \quad \boxed{\vec{B} = \vec{\nabla} \times \vec{A}}$

$\vec{A}$ = vector potential

(3) $\nabla \times E + \frac{\partial}{\partial t}(\nabla \times A) = \nabla \times \left(E + \frac{\partial \vec{A}}{\partial t}\right) = 0$

choice

$$\Rightarrow \quad \vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla}\phi$$

ϕ = scalar potential

$$\boxed{\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}}$$

If we write B + E in terms of $\vec{A}$ + ϕ,

last 2. Max. eqns <u>automatically</u> satisfied.

<u>Gauge transformation</u> = change in $\vec{A}$ or ϕ
that does not alter $\vec{E}$ + $\vec{B}$
+ ∴ has no physical consequence

e.g. $\phi \to \phi +$ const (change in potential zero.)

More generally: $\vec{A} \to \vec{A} + \vec{\nabla}\chi$ leaves $\vec{B}$ invariant

What happens to ϕ?

$$\nabla\phi = -\vec{E} - \frac{\partial A}{\partial t} \longrightarrow \underbrace{-\vec{E} - \frac{\partial A}{\partial t}}_{\vec{\nabla}\phi} + \vec{\nabla}\frac{\partial \chi}{\partial t}$$

$$\phi \to \phi + \frac{\partial \chi}{\partial t}$$

Electromagnetism in covariant notation

First consider charge density ρ & current $\vec{j}$

Recall $\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0$ $\qquad j^\mu = (\rho, \vec{j})$

4-current

Next consider scalar & vector potential $\phi, \vec{A}$

under a gauge transformation

$$\phi \to \phi + \dot{\chi} = \phi + \partial_0 \chi = \phi + \partial^0 \chi$$

$$A^i \to A^i \cancel{-} \partial_i \chi = A^i + \partial^i \chi$$

This suggests $A^\mu = (\phi, A^i)$

$$A^\mu \to A^\mu + \partial^\mu \chi$$

Thus the electromagnetic potential is a <u>4-vector field</u> A^μ

$$E^i = -\partial_i \phi - \partial_0 A^i = \partial^i A^0 - \partial^0 A^i$$

Looks like a 2nd rank tensor

Define $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ $\Rightarrow$ zero on diagonal

$$F^{i0} = \partial^i A^0 - \partial^0 A^i = E^i$$

row ↓ col. ↓

$$F^{\mu\nu} = \begin{pmatrix} 0 & & & \\ E^1 & 0 & & \\ E^2 & B^3 & 0 & \\ E^3 & & & 0 \end{pmatrix}$$

$$F^{21} = \partial^2 A^1 - \partial^1 A^2 = -\partial_2 A^1 + \partial_1 A^2$$

$$= (\nabla \times A)^3 = B^3$$

[HW: Fill in the rest] The $\vec{E}$ & $\vec{B}$ fields correspond to a 2nd rank antisymmetric tensor field

Maxwell's 1st 2 eqns $\Rightarrow$ $\partial_\mu F^{\mu\nu} = J^\nu$

2nd 2 eqs $\Rightarrow$ $\partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} + \partial^\lambda F^{\mu\nu} \equiv 0$

~~Note that $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ automatically obeys~~

Show: $\partial_\mu F^{\mu\nu} = J^\nu$ gives 1st 2 M.e.

$\partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} + \partial^\lambda F^{\mu\nu} = 0$ gives other 2.

[$0 = \partial_\mu \partial_\nu F^{\mu\nu} = \partial_\mu J^\mu$] ← later

[$A^\mu \to A^\mu + \partial^\mu \chi$ $\Rightarrow$ $F^{\mu\nu}$ invt]

EM field described by a vector field A^α $(\cancel{\alpha} = 0, 1, 2, 3)$

a priori, 4 d.o.f (redundant: EM wave only has 2 d.o.f.)

Lagrangian for EM

- Lorentz-invariant scalar ~~scalar~~
- quadratic in fields $\rightsquigarrow$ $(\partial_\mu A^\alpha)^2 \equiv \eta^{\mu\nu}\eta_{\alpha\beta}\partial_\mu A^\alpha \partial_\nu A^\beta$
- gauge invariant $\rightsquigarrow$ $F_{\mu\nu}{}^2$

$$\mathcal{L} = \underbrace{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{\text{free vector field}} + \mathcal{L}_{int}(A^\mu)$$

$$\downarrow$$

$$= \frac{1}{2}(E^2 - B^2)$$

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \mathcal{L}_{int}(A)$$

$$= -\frac{1}{4} \eta^{\sigma\alpha} \eta^{\tau\beta} F_{\sigma\tau} F_{\alpha\beta} + \mathcal{L}_{int}$$

Let's derive Euler Lag. eqs by varying w.r.t. A_ν

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\frac{1}{2} \eta^{\sigma\alpha} \eta^{\tau\beta} F_{\sigma\tau} \frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)} = -\frac{1}{2} F^{\mu\nu} + \frac{1}{2} F^{\nu\mu}$$

$$= -F^{\mu\nu}$$

$$\frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)} = \delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha$$

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} - \frac{\partial \mathcal{L}_i}{\partial A_\nu} = -\partial_\mu F^{\mu\nu} - \frac{\partial \mathcal{L}_{int}}{\partial A_\nu} = 0$$

$$\partial_\mu F^{\mu\nu} = -\frac{\partial \mathcal{L}_{int}}{\partial A_\nu}$$

These are Maxwell's eqns if $J^\nu = -\frac{\partial \mathcal{L}_{int}}{\partial A_\nu}$

Other Maxwell eqn's follow automatically

$$\partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} + \partial^\lambda F^{\mu\nu}$$
$$= \partial^\mu(\partial^\nu A^\lambda - \partial^\lambda A^\nu) + \partial^\nu(\partial^\lambda A^\mu - \partial^\mu A^\lambda) + \partial^\lambda(\partial^\mu A^\nu - \partial^\nu A^\mu) = 0$$

Energy-momentum tensor

Invariance of action under spacetime translations

$$\delta \mathcal{L} = \mathcal{L}(x+a) - \mathcal{L}(x) = a_\nu \partial^\nu \mathcal{L} = a_\nu \partial_\mu (\eta^{\mu\nu} \mathcal{L})$$

$\mathcal{L}$ depends on spacetime coordinates only thru its dependence on A_α & $\partial_\mu A_\alpha$

$$\delta \mathcal{L} = \underbrace{\frac{\partial \mathcal{L}}{\partial A_\alpha}}_{\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\alpha)} \right)} \delta A_\alpha + \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\alpha)} \underbrace{\delta(\partial_\mu A_\alpha)}_{\partial_\mu \delta A_\alpha}$$

$$= \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\alpha)} \delta A_\alpha \right]$$

But $\delta A_\alpha = A_\alpha(x+a) - A_\alpha(x) = a_\nu \partial^\nu A_\alpha$

$$\delta \mathcal{L} = a_\nu \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\alpha)} \partial^\nu A_\alpha \right]$$

Equating these expressions for $\delta \mathcal{L}$ we get

$$\partial_\mu \Big[\underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\alpha)} \partial^\nu A_\alpha - \eta^{\mu\nu} \mathcal{L}}_{\equiv\, T_c^{\mu\nu}} \Big] = 0$$

$\equiv T_c^{\mu\nu}$ (canonical energy mom. tensor)

$$T_c^{\mu\nu} = -F^{\mu\alpha}\partial^\nu A_\alpha + \frac{1}{4}\eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} \left[\cancel{-\eta^{\mu\nu}\mathcal{L}_{int}}\right]$$

$$-P^\nu = \int d^3x\, T_c^{0\nu}$$

$\partial_\mu T_c^{\mu\nu} = 0 \quad \Rightarrow P^\nu$ is conserved.

But $T_c^{\mu\nu}$ is <u>not</u> symmetric ($T_c^{\mu\nu} \neq T_c^{\nu\mu}$)

nor is it gauge invariant (depends on A^μ)

& i.e. not physical (change w/ ga tx.)

On the other hand, we are free to modify $T_c^{\mu\nu}$ provided the modification doesn't change P^ν and doesn't ruin $\partial_\mu T^{\mu\nu} = 0$.

Let's see if we can modify $T^{\mu\nu}$ to make it gauge invariant.

Belinfante energy momentum tensor

$$T_B^{\mu\nu} \equiv T_c^{\mu\nu} + \Delta T^{\mu\nu}$$

$$\Delta T^{\mu\nu} = \partial_\alpha \left(F^{\mu\alpha} A^\nu \right)$$

$\Delta T^{\mu\nu}$ is harmless because also conserved

$$\partial_\mu \Delta T^{\mu\nu} = \underbrace{\partial_\mu \partial_\alpha}_{\text{sym}} (\underbrace{F^{\mu\alpha}}_{\text{anti}} A^\nu) = 0$$

Also it does not change P^ν

$$\int d^3x \, \Delta T^{\mu\nu} = \int d^3x \, \partial_\alpha \left(F^{0\alpha} A^\nu \right)$$

$$= \int d^3x \, \partial_i \left(F^{0i} A^\nu \right)$$

$$= \oint \underbrace{dS^i}_{\text{sphere at } \infty} F^{0i} A^\nu = 0$$

provided F^{0i}, A^ν fall off fast enough.

$$T_B^{\mu\nu} = \Delta T^{\mu\nu} + T_c^{\mu\nu}$$

$$= \underbrace{\partial_\alpha (F^{\mu\alpha} A^\nu)} - F^{\mu\alpha}\partial_\alpha A^\nu + \frac{1}{4}\eta^{\mu\nu} F^{\alpha\beta}F_{\alpha\beta}$$

$$\underbrace{(\partial_\alpha F^{\mu\alpha})}_{\substack{0 \text{ for} \\ \text{free field}}} A^\nu + F^{\mu\alpha}\partial_\alpha A^\nu$$

$$= F^{\mu\alpha}\underbrace{(\partial_\alpha A^\nu - \partial_\alpha A^\nu)}_{F_\alpha{}^\nu} + \frac{1}{4}\eta^{\mu\nu} F^{\alpha\beta}F_{\alpha\beta}$$

$$T_B^{\mu\nu} = F^{\mu\alpha}\eta_{\alpha\beta}F^{\beta\nu} + \frac{1}{4}\eta^{\mu\nu} F^{\alpha\beta}F_{\alpha\beta}$$

This is gauge invariant (only depends on $F_{\mu\nu}$)
and symmetric $T_B^{\mu\nu} = T_B^{\nu\mu}$

energy density [HW]

$$T^{00} = \frac{1}{2}(E^2 + B^2)$$

$$T^{0i} = (\vec{E}\times\vec{B})^i \quad \text{(Poynting vector: energy flux)}$$

In[12]:= `tri = {{0, -EE[1], -EE[2], -EE[3]}, {0, 0, -B[3], B[2]}, {0, 0, 0, -B[1]}, {0, 0, 0, 0}}`

Out[12]=
$$\begin{pmatrix} 0 & -\mathrm{EE}(1) & -\mathrm{EE}(2) & -\mathrm{EE}(3) \\ 0 & 0 & -B(3) & B(2) \\ 0 & 0 & 0 & -B(1) \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In[13]:= `Fupup = tri - Transpose[tri]`

Out[13]=
$$\begin{pmatrix} 0 & -\mathrm{EE}(1) & -\mathrm{EE}(2) & -\mathrm{EE}(3) \\ \mathrm{EE}(1) & 0 & -B(3) & B(2) \\ \mathrm{EE}(2) & B(3) & 0 & -B(1) \\ \mathrm{EE}(3) & -B(2) & B(1) & 0 \end{pmatrix}$$

In[14]:= `eta = {{1, 0, 0, 0}, {0, -1, 0, 0}, {0, 0, -1, 0}, {0, 0, 0, -1}}`

Out[14]=
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

In[15]:= `Fdowndown = eta.Fupup.eta`

Out[15]=
$$\begin{pmatrix} 0 & \mathrm{EE}(1) & \mathrm{EE}(2) & \mathrm{EE}(3) \\ -\mathrm{EE}(1) & 0 & -B(3) & B(2) \\ -\mathrm{EE}(2) & B(3) & 0 & -B(1) \\ -\mathrm{EE}(3) & -B(2) & B(1) & 0 \end{pmatrix}$$

In[16]:= `Fupdown = Fupup.eta`

Out[16]=
$$\begin{pmatrix} 0 & \mathrm{EE}(1) & \mathrm{EE}(2) & \mathrm{EE}(3) \\ \mathrm{EE}(1) & 0 & B(3) & -B(2) \\ \mathrm{EE}(2) & -B(3) & 0 & B(1) \\ \mathrm{EE}(3) & B(2) & -B(1) & 0 \end{pmatrix}$$

In[17]:= `Fdownup = eta.Fupup`

Out[17]=
$$\begin{pmatrix} 0 & -\mathrm{EE}(1) & -\mathrm{EE}(2) & -\mathrm{EE}(3) \\ -\mathrm{EE}(1) & 0 & B(3) & -B(2) \\ -\mathrm{EE}(2) & -B(3) & 0 & B(1) \\ -\mathrm{EE}(3) & B(2) & -B(1) & 0 \end{pmatrix}$$

In[22]:= `first = Fupdown.Fupup`

Out[22]=
$$\begin{pmatrix} \mathrm{EE}(1)^2+\mathrm{EE}(2)^2+\mathrm{EE}(3)^2 & B(3)\,\mathrm{EE}(2)-B(2)\,\mathrm{EE}(3) & B(1)\,\mathrm{EE}(3)-B(3)\,\mathrm{EE}(1) & B(2)\,\mathrm{EE}(1)-B(1)\,\mathrm{EE}(2) \\ B(3)\,\mathrm{EE}(2)-B(2)\,\mathrm{EE}(3) & B(2)^2+B(3)^2-\mathrm{EE}(1)^2 & -B(1)\,B(2)-\mathrm{EE}(1)\,\mathrm{EE}(2) & -B(1)\,B(3)-\mathrm{EE}(1)\,\mathrm{EE}(3) \\ B(1)\,\mathrm{EE}(3)-B(3)\,\mathrm{EE}(1) & -B(1)\,B(2)-\mathrm{EE}(1)\,\mathrm{EE}(2) & B(1)^2+B(3)^2-\mathrm{EE}(2)^2 & -B(2)\,B(3)-\mathrm{EE}(2)\,\mathrm{EE}(3) \\ B(2)\,\mathrm{EE}(1)-B(1)\,\mathrm{EE}(2) & -B(1)\,B(3)-\mathrm{EE}(1)\,\mathrm{EE}(3) & -B(2)\,B(3)-\mathrm{EE}(2)\,\mathrm{EE}(3) & B(1)^2+B(2)^2-\mathrm{EE}(3)^2 \end{pmatrix}$$

In[23]:= `trace = Sum[(Fupup.Fdowndown)[[i, i]], {i, 1, 4}]`

Out[23]= $-2\,B(1)^2-2\,B(2)^2-2\,B(3)^2+2\,\mathrm{EE}(1)^2+2\,\mathrm{EE}(2)^2+2\,\mathrm{EE}(3)^2$

$$B^{\mu\rho\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\rho \Phi)} S^{\nu\rho}\Phi + \frac{\partial \mathcal{L}}{\partial(\partial_\rho \Phi)} S^{\mu\nu}\Phi + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \Phi)} S^{\mu\rho}\Phi$$

$$T_B = T_c + \partial B$$

$$j^{\mu\nu\rho} = T_c^{\mu\nu} x^\rho - T^{\mu\rho} x^\nu + \frac{\partial \mathcal{L}}{\partial} S$$

$$= T_B^{\mu\nu} x^\rho - T^{\mu\rho} x^\nu$$

$$j^\mu \partial^\nu \chi - \frac{\partial \mathcal{L}}{\partial A_\mu} \partial^\mu \chi$$

Recall $\vec{E} = -\nabla\Phi - \frac{\partial \vec{A}}{\partial t}$

Gauss's law $\vec{\nabla}\cdot\vec{E} = -\nabla^2\Phi - \frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A} = 0$ QA-1

Maxwell's eqns are invariant under ga. txs.

$A^\mu \to A^\mu + \partial^\mu \chi$

$$\begin{cases} \Phi \to \Phi + \frac{\partial \chi}{\partial t} \\ \vec{A} \to \vec{A} - \vec{\nabla}\chi \end{cases}$$

$\Phi = \frac{Q}{4\pi\epsilon_0 r}$

$\epsilon_0 = 1$

$\Phi = \int \frac{\rho(x)}{4\pi|x-x'|}$

$\nabla\cdot E = \rho$

$-\nabla^2\Phi = \rho$

$\nabla^2\left(\frac{-1}{4\pi|x-x'|}\right) = \delta$

There exists a gauge in which $\Phi = 0$.

Suppose $\Phi \neq 0$.

Define $\chi(\vec{x},t) = -\int_{t'=0}^{t'=t} \Phi(\vec{x},t')\,dt'$

then $\frac{\partial \chi}{\partial t} = -\Phi(\vec{x},t)$

$\Phi' = \Phi + \frac{\partial \chi}{\partial t} = 0$

We can do a further gauge transformation to set $\vec{\nabla}\cdot\vec{A} = 0$.

Suppose $\vec{\nabla}\cdot\vec{A}' \neq 0$

Define $\chi'(\vec{x},t) = -\int \frac{d^3x'}{4\pi} \frac{\vec{\nabla}\cdot\vec{A}'(x',t)}{|x-x'|} \Rightarrow \nabla^2\chi' = +\vec{\nabla}\cdot\vec{A}'$

Define $\begin{cases} \vec{A}'' = \vec{A}' - \vec{\nabla}\chi' \Rightarrow \vec{\nabla}\cdot\vec{A}'' = \vec{\nabla}\cdot\vec{A}' - \nabla^2\chi' = 0 \\ \Phi'' = \Phi' + \frac{\partial \chi'}{\partial t} \quad \text{where } \Phi' = 0 \end{cases}$

$$\Phi'' = \int \frac{d^3x'}{4\pi} \frac{\left(-\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A}'\right)}{|x-x'|}$$

$$= \int \frac{d^3x'}{4\pi} \frac{(+\nabla^2\Phi')}{|x-x'|} = 0 \quad \text{because } \Phi' = 0$$

So we can choose a gauge $\ni$ $\Phi'' = 0$ and $\vec{\nabla}\cdot\vec{A}'' = 0$

Since $A^0 = 0$ and $\vec\nabla\cdot\vec A$, we have $\partial\cdot A = 0$

Let's solve Maxwell's eqns for a free field ($J^\mu = 0$)

$$0 = \partial_\mu F^{\mu\nu}$$

$$= \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

$$= \partial^2 A^\nu - \partial^\nu \underbrace{(\partial\cdot A)}_{0} \quad \text{in our gauge}$$

$\nu = 0$ eqn is trivial

$\nu = i$ $\quad \partial^2 \vec A(\vec x, t) = 0 \quad$ (wave eqn

$\quad\quad \vec\nabla\cdot\vec A = 0 \quad$ gauge condition)

Ansatz $\quad \vec A\, e^{-ik\cdot x} = \vec A\, e^{-i\omega t + i\vec k\cdot\vec x}$

$$\partial^2\vec A = \eta^{\mu\nu}\partial_\mu\partial_\nu\left(\vec A\, e^{-ik\cdot x}\right)$$

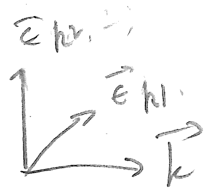
$$= \eta^{\mu\nu}(-ik_\mu)(-ik_\nu)\,\vec A\, e^{ik\cdot x}$$

$$= -k^2\vec A\, e^{ik\cdot x} = 0 \quad \text{if } k^2 = 0$$

ie $\omega = |\vec k|$

$$\vec\nabla\cdot\vec A = i\vec k\cdot\vec A\, e^{ik\cdot x} = 0 \quad \text{if } \vec k\cdot\vec A = 0$$

Given $\vec{k}$, choose 2 unit vectors perpendicular to $\vec{k}$ + to each other, $\vec{\epsilon}_{\vec{k}\lambda}$. $\lambda = 1, 2$



$$\left\{\begin{array}{l} \text{Let } \vec{\epsilon}_{k\lambda} \cdot \vec{\epsilon}_{k\lambda'} = \delta_{\lambda\lambda'} \\ \vec{\epsilon}_{k\lambda} \cdot \vec{k} = 0 \\ \vec{\epsilon}_{k1} \times \vec{\epsilon}_{k2} = \hat{k} \end{array}\right.$$

$$\text{Let } \vec{A} = \frac{1}{(2\pi)^3 (2\omega)} \Big[\underbrace{\vec{\epsilon}_{k1} \, a_{k1} + \vec{\epsilon}_{k2} \, a_{k2}}_{\sum_{\lambda=1}^{2} \vec{\epsilon}_{k\lambda} \, a_{k\lambda}} \Big]$$

Then most general solution to the wave eqn that also obeys the Coulomb gauge condition.

$$\vec{A}(\vec{x}, t) = \sum_{\lambda=1}^{2} \int \vec{dk} \left[\vec{\epsilon}_{k\lambda} \, a_{k\lambda} e^{-ik\cdot x} + \vec{\epsilon}_{k\lambda} \, a^*_{k\lambda} e^{ik\cdot x} \right]$$

[cf Bjorken Drell, p. 75: $\epsilon(-k,1) = -\epsilon(k,1)$; $\epsilon(-k,2) = \epsilon(k,2)$]

$$A^\dagger = \int \vec{dk} \left[\epsilon_{k\lambda} \, a^\dagger_{k\lambda} e^{ik\cdot x} + \vec{\epsilon}_{k\lambda} \, a_{k\lambda} e^{-ik\cdot x} \right]$$

$$\vec{\epsilon}_{-k1} = -\vec{\epsilon}_{k,1}$$

$$\vec{\epsilon}_{-k,2} = \vec{\epsilon}_{k,2}$$

so that $\vec{\epsilon}_{-k,1} \times \vec{\epsilon}_{-k,2} = -\hat{k}$

Thus $\vec{\epsilon}_{k\lambda} \cdot \vec{\epsilon}_{-k,\lambda'} = (-)^\lambda \delta_{\lambda\lambda'}$

quantize:

$$\vec{A}(\vec{x},t) = \sum_{\lambda=1}^{2} \int \widetilde{dk}\; \vec{\epsilon}(k,\lambda) \left[a_{k\lambda} e^{-ik\cdot x} + a^{\dagger}_{k\lambda} e^{ik\cdot x} \right]$$

Commutation relations

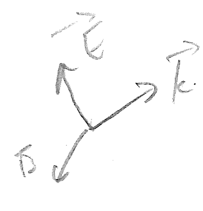
$$[a_{k\lambda}, a^{\dagger}_{k'\lambda'}] = (2\pi)^3 (2\omega)\, \delta(\vec{k}-\vec{k}')\, \delta_{\lambda\lambda'}$$

all others vanish...

[We could invert these and calculate

$$[\dot{A}^i, A^j]_{ETCR} \sim \int d^3k \underbrace{\sum_{\lambda=1}^{2} \epsilon^i(k,\lambda)\, \epsilon^j(k,\lambda)}_{\delta^{ij} - \frac{k^i k^j}{|k|^2}} \left[e^{i\vec{k}\cdot\vec{x}} + e^{-i\vec{k}\cdot\vec{x}} \right]$$

$$\sim \delta^{tr}_{ij}(\vec{x})$$



but it is prob. not necessary]

$$\vec{E} = -\cancel{\vec{\nabla}\phi}^{\,0} - \frac{\partial \vec{A}}{\partial t} = \sum_{\lambda} \int \widetilde{dk}\; \vec{\epsilon}(k,\lambda) \left[i\omega\, a_{k\lambda} e^{-ik\cdot x} - i\omega\, a^{\dagger}_{k} e^{ik\cdot x} \right]$$

= polarization vector

$$\vec{B} = \vec{\nabla} \times \vec{A} = \sum_{\lambda} \int \widetilde{dk} \left[+i\vec{k} \times \vec{\epsilon}(k,\lambda)\, a_{k\lambda} e^{-ik\cdot x} - i\vec{k} \times \vec{\epsilon}(k,\lambda)\, a^{\dagger}_{k\lambda} e^{ik\cdot x} \right]$$

observe $\vec{k} \times \vec{E} \parallel \vec{B}$

$$H = \int d^3x \; T_B^{00}$$

You will show

$$=$$

$$= \sum_\lambda \int \widetilde{dk} \;\; \omega \, a^\dagger_{k\lambda} a_{k\lambda}$$

where $\omega = |\vec{k}|$

Also $\vec{P} = \int d^3x \; T_B^{0i} = \sum_\lambda \int \widetilde{dk} \; \vec{k} \, a^\dagger_{k\lambda} a_{k\lambda}$

Define $|k,\lambda\rangle \equiv a^\dagger_{k\lambda}|0\rangle$

Then $H\,|k,\lambda\rangle = \omega\,|k,\lambda\rangle$

$|k,\lambda\rangle$ represents a photon of mom $\vec{k}$, energy ω
and polarization λ.

HW

$$\vec{E} = \sum_\lambda \int \widetilde{dk} \quad i\omega\, \vec{\epsilon}_{k\lambda} \left[a_{k\lambda} e^{-ik\cdot x} - a^\dagger_{k\lambda} e^{ik\cdot x} \right]$$

$$\vec{B} = \sum_\lambda \int \widetilde{dk} \quad i\vec{k} \times \vec{\epsilon}_{k\lambda} \left[a_{k\lambda} e^{-ik\cdot x} - a^\dagger_{k\lambda} e^{ik\cdot x} \right]$$

$$(\vec{k} \times \vec{\epsilon}_{k\lambda}) \cdot (\vec{k}' \times \vec{\epsilon}_{k'\lambda'}) = (\vec{k}\cdot\vec{k}')(\vec{\epsilon}_{k\lambda} \cdot \vec{\epsilon}_{k'\lambda'}) - (\vec{k}\cdot\vec{\epsilon}_{k'\lambda'})(\vec{k}'\cdot\vec{\epsilon}_{k\lambda})$$

Then

$$H = \frac{1}{2} \int d^3x \,(\vec{E}^2 + \vec{B}^2)$$

$$= -\frac{1}{2} \sum_{\lambda\lambda'} \int \widetilde{dk}\, \widetilde{dk}' \left[\omega\omega' \vec{\epsilon}_{k\lambda}\cdot\vec{\epsilon}_{k'\lambda'} + \vec{k}\cdot\vec{k}'\, \vec{\epsilon}_{k\lambda}\cdot\vec{\epsilon}_{k'\lambda'} - \vec{k}\cdot\vec{\epsilon}_{k'\lambda'}\, \vec{k}'\cdot\vec{\epsilon}_{k\lambda} \right]$$

$$\times \Big[a_{k\lambda} a_{k'\lambda'} \underbrace{\int d^3x\, e^{-i(k+k')\cdot x}}_{(2\pi)^3 e^{-2i\omega t} \delta(\vec{k}+\vec{k}')} + a^\dagger_{k\lambda} a^\dagger_{k'\lambda'} \underbrace{\int d^3x\, e^{i(k+k')\cdot x}}_{(2\pi)^3 e^{2i\omega t} \delta(\vec{k}+\vec{k}')}$$

$$- a_{k\lambda} a^\dagger_{k'\lambda'} \underbrace{\int d^3x\, e^{i(k'-k)\cdot x}}_{(2\pi)^3 \delta(\vec{k}-\vec{k}')} - a^\dagger_{k\lambda} a_{k'\lambda'} \underbrace{\int d^3x\, e^{i(k-k')\cdot x}}_{(2\pi)^3 \delta(\vec{k}-\vec{k}')} \Big]$$

Since $\vec{k}' = \pm\vec{k}$, we have $\vec{k}\cdot\vec{\epsilon}_{k'\lambda'} = \pm\vec{k}'\cdot\vec{\epsilon}_{k'\lambda'} = 0$

$\vec{k}'\cdot\vec{\epsilon}_{k\lambda} = \pm\vec{k}\cdot\vec{\epsilon}_{k\lambda} = 0$ So last term $= 0$

For $\vec{k}' = \vec{k}$, $(\vec{k}\cdot\vec{k}')(\vec{\epsilon}_{k\lambda}\cdot\vec{\epsilon}_{k'\lambda'}) = \vec{k}^2\, \vec{\epsilon}_{k\lambda}^{\,2} = \vec{k}^2$

For $\vec{k}' = -\vec{k}$, $(\vec{k}\cdot\vec{k}')(\vec{\epsilon}_{k\lambda}\cdot\vec{\epsilon}_{k'\lambda'}) = -\vec{k}^2\, \vec{\epsilon}_{k\lambda}\cdot\epsilon_{-k,\lambda'} = \underbrace{-\vec{k}^2 (-1)^\lambda \delta_{\lambda\lambda'}}$

But this isn't too relevant except in the case of momentum when we need to know that

$$\vec{\epsilon}_{-k,\lambda}\cdot\vec{\epsilon}_{k,\lambda'} = \vec{\epsilon}_{k\lambda}\cdot\vec{\epsilon}_{-k,\lambda'}$$

$$H = -\frac{1}{2} \sum_{\lambda} \sum_{\lambda'} \int d\tilde{k} \; \frac{1}{2\omega} \Big[e^{-2i\omega t} a_{k\lambda} a_{-k\lambda'} \big(\underbrace{\omega^2 - \vec{k}^2}_{0}\big) \vec{\epsilon}_{k\lambda} \cdot \vec{\epsilon}_{-k,\lambda'}$$

$$+ e^{2i\omega t} a^{\dagger}_{k\lambda} a^{\dagger}_{-k,\lambda} \big(\underbrace{\omega^2 - \vec{k}^2}_{0}\big) \vec{\epsilon}_{k\lambda} \cdot \epsilon_{-k,\lambda'}$$

$$- a_{k\lambda} a^{\dagger}_{k\lambda'} \big(\underbrace{\omega^2 + \vec{k}^2}_{2\omega^2}\big) \delta_{\lambda\lambda'}$$

$$- a^{\dagger}_{k\lambda} a_{k\lambda'} \big(\underbrace{\omega^2 + \vec{k}^2}_{2\omega^2}\big) \delta_{\lambda\lambda'} \Big]$$

$$= \sum_{\lambda} \int d\tilde{k} \; \frac{\omega}{2} \left(a_{k\lambda} a^{\dagger}_{k\lambda} + a^{\dagger}_{k\lambda} a_{k\lambda} \right)$$

$$\boxed{H = \sum_{\lambda} \int d\tilde{k} \; \omega \, a^{\dagger}_{k\lambda} a_{k\lambda}}$$

HW

$$\vec{P} = \int d^3x\, (\vec{E}\times\vec{B})$$

$$= -\sum_{\lambda\lambda'} \int \widetilde{dk}\, \widetilde{dk}'\, \omega\, \underbrace{\vec{\epsilon}_{k\lambda} \times (\vec{k}' \times \vec{\epsilon}_{k'\lambda'})}\ [\text{as before}]$$

$$\underbrace{\vec{\epsilon}_{k\lambda}\cdot\vec{\epsilon}_{k'\lambda'}}\,\vec{k}' - \underbrace{\vec{\epsilon}_{k\lambda}\cdot\vec{k}'}_{0 \text{ if } \vec{k}'=\pm\vec{k}}\,\vec{\epsilon}_{k'\lambda'}$$

$$\begin{cases} \delta_{\lambda\lambda'} \text{ if } \vec{k}'=\vec{k} \\ (-)^\lambda \delta_{\lambda\lambda'} \text{ if } \vec{k}=-\vec{k} \end{cases}$$

$$= -\sum_\lambda \int \widetilde{dk}\, \frac{1}{2\omega}\, \omega \vec{k}\, [-a_{k\lambda}a^\dagger_{k\lambda} - a^\dagger_{k\lambda}a_{k\lambda}]$$

$$= \sum_\lambda \int \widetilde{dk}\, \frac{\vec{k}}{2}\, (a_{k\lambda}a^\dagger_{k\lambda} + a^\dagger_{k\lambda}a_{k\lambda})$$

$$\boxed{\vec{P} = \sum_\lambda \int \widetilde{dk}\, \vec{k}\, a^\dagger_{k\lambda}a_{k\lambda}}$$

we need to use

$$\vec{\epsilon}_{k\lambda}\cdot\vec{\epsilon}_{-k,\lambda'} = \vec{\epsilon}_{-k,\lambda}\cdot\vec{\epsilon}_{k,\lambda'}$$

to show this is odd

$$-\sum_\lambda \int \widetilde{dk}\, \frac{1}{2\omega}\, \omega(-\vec{k})(-)^\lambda e^{-2i\omega t}\, \underbrace{a_{k\lambda}a_{-k\lambda}}$$

$$= a_{-k\lambda}a_{k\lambda}$$

so integrand is odd under $\vec{k}\to-\vec{k}$ & $\therefore$ integral vanishes

Quantization

$\Pi^0 \equiv 0$!

$\Longrightarrow$ set $A^0 = 0$

$[\Pi^i, A^j] = \delta^{ij}$

violates

$\nabla \cdot E = 0$

Also $\vec{\nabla} \cdot \vec{A} = 0$

$[\Pi^i, A^j] = \delta^{ij}_\perp$

Soln

$[a_p, a^\dagger_q] = (2\pi)^3 (2\omega)\, \delta$

$\Downarrow$

$[A^i, \Pi^j] = \ldots = \delta_\perp$

check $\nabla \cdot E = 0$

$\nabla \cdot A = 0$

Sidestep all this